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Green's Classifications and Evolutions of Fixed-Order Networks

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Abstract: It is shown that the set of all networks of fixed order n form a semigroup that is isomorphic to the semigroup B_X of binary relations on a set X of cardinality n. Consequently, B_X provides for Green's $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{D}$ equivalence classifications of all networks of fixed order $\boldsymbol{n}$. These classifications reveal that a fixed-order network which evolves within a Green's equivalence class maintains certain structural invariants during its evolution. The "Green's symmetry problem" is introduced and is defined as the determination of all symmetries (i.e., transformations) that produce an evolution between an initial and final network within an $\mathcal{L}$ or an $\mathcal{R}$ class such that each symmetry preserves the required structural invariants. Such symmetries are shown to be solutions to special Boolean equations specific to each class. The satisfiability and computational complexity of the "Green's symmetry problem" are discussed and it is demonstrated that such symmetries encode information about which node neighborhoods in the initial network can be joined to form node neighborhoods in the final network such that the structural invariants required by the evolution are preserved, i.e., the internal dynamics of the evolution. The notion of "propensity" is also introduced. It is a measure of the tendency of node neighborhoods to join to form new neighborhoods during a network evolution and is used to define "energy", which quantifies the complexity of the internal dynamics of a network evolution.

Keywords: network classification; network evolution; network symmetries; Green's symmetry problem; network invariants; network internal dynamics; symmetry ensembles; propensities; energy

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1. Introduction

Symmetry is a principle which has served as a guide for the spectacular advances that have been made in modern science, especially physics. For example, the continuous translational symmetry of ordinary space and time guarantees the invariance of the laws of physics under such translations. Thus, any mathematical expression describing a physical system, whether subatomic or macroscopic, must be invariant under space and time translations.

Group theory is the mathematical language used to describe symmetry and its associated invariant properties (recall that an abstract group is a set S of elements together with a law of composition "$\circ$" such that for $x, y, z \in S$ (*i*) $x \circ y \in S$; (*ii*) $x \circ (y \circ z) = (x \circ y) \circ z$; (*iii*) there is an identity element $e \in S$ such that $x \circ e = e \circ x = x$; and (*iv*) for $x \in S$ there is an inverse $x^{-1} \in S$ such that $x \circ x^{-1} = x^{-1} \circ x = e$). As a simple example, the set S of 0°, 90°, 180°, and 270° rotations in the plane of a square about its fixed center under "composition of rotations" form a symmetry group for the square (0° is the identity element and the inverse of an $X^\circ \in S$ rotation is a $360^\circ - X^\circ$ rotation). Each of these rotations is a symmetry which brings the square into coincidence with itself, i.e., they preserve the invariant shape of the square. A much more complicated example are the so called gauge symmetries of the standard model of physics which classify and describe three fundamental forces of nature (i.e., the electromagnetic, weak, and strong forces) in terms of groups (specifically, the unitary group $U(1)$ of degree 1 and the special unitary groups $SU(2)$ and $SU(3)$ of degree 2 and 3, respectively).

In recent years, the notion of generalized symmetry has been introduced to further describe graph symmetry [1,2]. The generalized symmetries of a graph are a generalization of the notion of the automorphism group of a graph and are derived from the application of Green's equivalence relations to the endomorphism monoid of the graph (the automorphism group is a subgroup of the graph's endomorphism monoid). Since these symmetries and invariant properties are strictly associated with a single graph, they do not address properties that remain fixed when the connection topology of the graph changes.

An important problem in network theory is identifying those properties of networks that remain fixed (invariant) as the network's connection topology changes with time. It was shown in [3] that the set of all networks (i.e., all connection topologies) on a fixed number of nodes also forms a semigroup. There it was also shown that the application of Green's equivalence relations to this semigroup partitions the associated set of networks into equivalence classes, each of which contains many fixed node number networks with various connection topologies, such that all networks within each class share some identifiable invariant connectivity property. If the connection topology of a network changes such that its initial and final configurations are in the same equivalence class, then the initial and final configurations share a common invariant property. It follows that, in this context, Green's equivalence classifications can be useful for identifying invariant properties of networks which evolve within an equivalence class. Such connectivity invariants can be used, for example, to identify important actors in evolving social networks and to select communication network reconfigurations that will retain a desired connectivity between specific node sets.

Transformations between networks within an equivalence class which preserve the associated invariant connectivity properties are called "Green's symmetries". Here, in addition to reviewing the Green's classification of networks [3], the "Green's symmetry problem" is introduced and defined. This problem is to determine (by calculation) the ensemble set of all the Green's symmetries which evolve an initial network configuration into a final configuration within a fixed Green's $\mathcal{R}$ equivalence class or within a fixed Green's $\mathcal{L}$ equivalence class. As discussed below, each such symmetry encodes information about the internal dynamics of the evolution, i.e., how node neighborhoods in the initial network configuration are joined to form node neighborhoods in the final configuration such that the invariant properties are preserved.

Since the cardinality of such ensembles can be large, the statistical notion of propensity is introduced. This quantity provides measures of the overall tendency of node neighborhoods in an initial network configuration to associate and form node neighborhoods in the final network configuration. Propensities are used to define "propensity energies", which quantify the overall complexity of the internal dynamics of a network evolution, and "energies of evolution", which quantify the complexity of internal dynamical activity for an evolution produced by a specific ensemble symmetry.

The objective of this paper is to motivate the application of Green's symmetry principles to network science by demonstrating how Green's equivalence relations can be applied to: classify networks; identify associated structural invariants; determine symmetries that preserve these invariants; and define associated measures that quantify aspects of the internal dynamics of network evolutions. The remainder of this paper is organized as follows: To make this paper reasonably self-contained, the relevant definitions and terminology from semigroup theory are summarized in the next section (for additional depth and clarification the reader is invited to consult such standard references as [4,5]). The semigroup B_X of all binary relations on a finite set X and the semigroup B_n of $n \times n$ Boolean matrices are defined and shown to be isomorphic to one another in Section 3. The semigroup of networks N_V on a fixed set V of nodes is introduced and is shown to be isomorphic to B_V in Section 4. This isomorphism provides for the Green's equivalence classifications of N_V given in Section 5. Green's evolutions of networks and their associated invariant properties are discussed in Section 6. The "Green's symmetry problem" is defined in Section 7 and its satisfiability and computational complexity are discussed in Section 8. The information encoded in symmetries as internal dynamics is detailed in Section 9. Symmetry "ensembles" and their "propensities" and "energies" are introduced in Section 10. A simple example illustrating aspects of the theory is presented in Section 11. Concluding remarks comprise the final section of this paper.

2. Semigroups

A *semigroup* $S \equiv (S, \circ)$ is a set S and an associative binary operation "$\circ$" called multiplication defined upon the set (contrast this with the above definition of a group and note that a group is a semigroup endowed with the additional special properties given by items (*iii*) and (*iv*)). The one-sided right (one-sided left) multiplication of $x \in S$ by $y \in S$ is the product $x \circ y \in S$ ($y \circ x \in S$). An element $e \in S$ is an *identity* if $x \circ e = e \circ x = x$ for $x \in S$. An identity can be adjoined to S by setting $S^1 = S \cup \{e\}$ and defining $x \circ e = e \circ x = x$ for $x \in S^1$. Semigroup $S \equiv (S, \circ)$ and the semigroup $T \equiv (T, *)$ on set T with associative binary operation "$*$" are *isomorphic* (denoted $S \approx T$) when there is a bijective map (i.e., an isomorphism) $\theta : S \to T$ such that $\theta(x \circ y) = \theta(x) * \theta(y)$ for all $x, y \in S$.

The well-known $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{D}$ Green's equivalence relations on a semigroup S partition S into a highly organized "egg box" structure using their relatively simple algebraic properties. In particular, the equivalence relation $\mathcal{L}(\mathcal{R})$ on S is defined by the rule that $x\mathcal{L}y$ ($x\mathcal{R}y$) if and only if $S^1x = S^1y$ $\left(xS^1 = yS^1\right)$ for $x, y \in S$ and the equivalence relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ is similarly defined so that $x\mathcal{H}y$ if and only if $x\mathcal{L}y$ and $x\mathcal{R}y$. The relations $\mathcal{L}$ and $\mathcal{R}$ commute under the composition "$\bullet$" of binary relations and $\mathcal{D} \equiv \mathcal{L} \bullet \mathcal{R} = \mathcal{R} \bullet \mathcal{L}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$.

For $x \in S$ and $\mathbf{X} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}\}$ denote the $\mathbf{X}$ class containing x by $X(x)$ where $X = L, R, H$, or D when $\mathbf{X} = \mathcal{L}, \mathcal{R}, \mathcal{H}$, or $\mathcal{D}$, respectively. Thus, xXy if and only if $X(x) = X(y)$. If $x, y \in S$ and $R(x) = R(y)(L(x) = L(y))$, then there exist elements s (t) in S^1 such that $xs = y$ ($tx = y$) (hereafter the juxtaposition xy will *also* be used for the multiplication $x \circ y$).

3. The Semigroups B_n and B_X

The semigroup B_n of Boolean matrices is the set of all $n \times n$ matrices over $\{0, 1\}$ with Boolean composition $\gamma = \alpha \circ \beta$ defined by

$$\gamma_{ij} = \vee_{k \in J}\left(\alpha_{ik} \wedge \beta_{kj}\right), \tag{1}$$

as the semigroup multiplication operation. Here $J = \{1, 2, \cdots, n\}$, where $n \geq 1$ is a counting number, $\wedge$ denotes Boolean multiplication (i.e., $0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1$), and $\vee$ denotes Boolean addition (i.e., $0 \vee 0 = 0,\ 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$).

The rows (columns) of any $\alpha \in B_n$ are Boolean row (column) n, vectors, i.e., row (column) n, tuples over $\{0, 1\}$, and come from the set $V_n(W_n)$ of all Boolean row (column) n-vectors. These vectors can be added coordinate-wise using Boolean addition. If $u, v \in V_n(W_n)$, then $u \sqsubseteq v$ when the ith coordinate $u_i = 1$ implies the ith coordinate $v_i = 1, 1 \leq i \leq n$ ($\sqsubseteq$ is a partial order).

Let $\mathbf{0}(\mathbf{1})$ be either the zero (unit) row or zero (unit) column vector (the context in which $\mathbf{0}(\mathbf{1})$ is used defines whether it is a row or column vector). The matrix with $\mathbf{0}$ in every row, i.e., the zero matrix, is denoted by "$\mathcal{Z}$" and the matrix with $\mathbf{1}$ in every row is denoted by "ω". For $\alpha \in B_n$, the row space $\Gamma(\alpha)$ of α is the subset of V_n consisting of $\mathbf{0}$ and all possible Boolean sums of (one or more) nonzero rows of α. $\Gamma(\alpha)$ is a lattice $(\Gamma(\alpha), \sqsubseteq)$ under the partial order $\sqsubseteq$. The row (column) *basis* $r(\alpha)$ ($c(\alpha)$) of α is the set of all row (column) vectors in α that are not Boolean sums of other row (column) vectors in α. Please note that each vector in $r(\alpha)$ ($c(\alpha)$) must be a row (column) vector of α. The vector $\mathbf{0}$ is never a basis vector and the empty set $\varnothing$ is the basis for the $\mathcal{Z}$ matrix [6,7].

The semigroup B_X of binary relations on a set X of cardinality n (denoted $|X| = n$) is the power set of $X \times X$ with multiplication $a = bc$ being the "composition of binary relations" defined by

$$a = \{(x, y) \in X \times X : (x, z) \in b,\ (z, y) \in c,\ \text{when } z \in X\}. \tag{2}$$

It is easy to see that a bijective index map $f : X \to J$ induces an isomorphism $\lambda : B_X \to B_n$ defined by $\lambda(a) = \alpha$, where $\alpha_{ij} = 1$ if $\left(f^{-1}(i), f^{-1}(j)\right) \in a$ and is 0 if $\left(f^{-1}(i), f^{-1}(j)\right) \notin a$. B_n is therefore the Boolean matrix representation of B_X [8].

4. The Semigroup N_V

A network E of order n is the pair $E = (V, C)$, where V is a nonempty set of nodes with $|V| = n$, and the binary relation $C \subseteq V \times V$ is the set of directed links connecting the nodes of the network. Thus, E is both a digraph and a binary relation. If $(x, y) \in C$, then node $x(y)$ is an in(out)-neighbor of node $y(x)$. The *in-neighborhood of* $x \in V$ is the set $I(E; x)$ of all in-neighbors of x and the *out-neighborhood of* $x \in V$ is the set $O(E; x)$ of all out-neighbors of x.

Let N_V be the set of networks on V and define "multiplication of networks" by $EF = G \equiv \left(V, C^{\#}\right)$, where $E = (V, C)$, $F = (V, C')$, and

$$C^{\#} = \left\{(x, y) \in V \times V : (x, z) \in C, (z, y) \in C', \text{ when } z \in V\right\}. \tag{3}$$

Lemma 1. *N_V is a semigroup that is isomorphic to B_V.*

Proof. The operation "multiplication of networks" is the same as the operation "composition of binary relations". Since it is clearly an associative binary operation on N_V, then N_V is a semigroup under the operation "multiplication of networks". Also, the bijective map $\varphi : N_V \to B_V$ defined by $\varphi(E) = C$ preserves multiplication. Thus, φ is a semigroup isomorphism and $N_V \approx B_V$. □

Lemma 2. *If $|V| = n$, then $N_V \approx B_n$.*

Proof. This follows from the facts that $N_V \approx B_V$ (Lemma 1) and $B_V \approx B_n$ [8]. □

Thus, B_n is also a Boolean matrix representation of N_V.

5. Green's Equivalence Classifications of N_V

Let $\theta : N_V \to B_n$ be the isomorphism of Lemma 2 and $f : V \to J$ be an associated index bijection. If α_{i*} is the ith Boolean row vector and α_{*j} is the jth Boolean column vector in the matrix $\alpha = \theta(E)$ corresponding to network E, then α_{i*} encodes the out-neighbors of node $f^{-1}(i)$ in E as the set

$$O\left(E; f^{-1}(i)\right) = \left\{f^{-1}(k) : \alpha_{ik} = 1, k \in J\right\} \tag{4}$$

and α_{*j} encodes the in-neighbors of node $f^{-1}(j)$ in E as the set

$$I\left(E; f^{-1}(j)\right) = \left\{f^{-1}(j) : \alpha_{kj} = 1, k \in J\right\}. \tag{5}$$

When $\alpha_{i*} \in r(\alpha)$ and $\alpha_{*j} \in c(\alpha)$, then $O_r\left(E; f^{-1}(i)\right) \equiv O\left(E; f^{-1}(i)\right)$ is a basis out-neighborhood and $I_c\left(E; f^{-1}(j)\right) \equiv I\left(E; f^{-1}(j)\right)$ is a basis in-neighborhood for network E. Thus, a basis neighborhood in E is a nonempty neighborhood in E which is not the set union of other neighborhoods in E.

Let $O_r(E)$ be the set of basis out-neighborhoods and $I_c(E)$ be the set of basis in-neighborhoods in network E. Also, define $P(E)$ as the set whose elements are $\varnothing$ and the sets generated by the closure under set union of the out-neighborhoods in E and let $(P(E), \subseteq)$ be the poset ordered by the set inclusion relation " $\subseteq$ ". Thus, when $\theta(E) = \alpha$, it may be formally stated that:

Lemma 3. *$(P(E), \subseteq)$ is a lattice that is isomorphic to $(\Gamma(\alpha), \sqsubseteq)$.*

Proof. The proof for this Lemma is the same as that given as the proof of Lemma 3.3 in [3]. □

In what follows, $(P(E), \subseteq)$ will be referred to as the *Π lattice* for E.

The following major theorem provides complete $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{D}$ equivalence classifications of all fixed-order networks:

Theorem 1. *Let $E, F \in N_V$. Then*

i. $L(E) = L(F)$ *if and only if* $O_r(E) = O_r(F)$*;*
ii. $R(E) = R(F)$ *if and only if* $I_c(E) = I_c(F)$*;*
iii. $H(\mathrm{E}) = H(\mathrm{F})$ *if and only if* $O_r(E) = O_r(F)$ *and* $I_c(E) = I_c(F)$*;*
iv. $D(\mathrm{E}) = D(\mathrm{F})$ *if and only if* $(P(E), \subseteq)$ *and* $(P(F), \subseteq)$ *are lattice isomorphic.*

Proof. The proof of this result is the same as the proof of Theorem 3.4 in [3]. □

Thus, the Green's $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ equivalence classifications of the networks in N_V depend entirely upon their having (generally distinct) nodes with identical out-neighborhoods, identical in-neighborhoods, and both identical out-neighborhoods and in-neighborhoods, respectively, whereas the $\mathcal{D}$ equivalence classification of networks in N_V depends entirely upon their having isomorphic *Π* lattices which are generated by their out-neighborhoods. As an illustration of this theorem the reader is invited to consult the simple example given in [3] which corresponds to the complete Green's equivalence classification of (and the associated "egg box" structure for) all order two networks.

6. Green's Evolutions of Fixed-Order Networks

For $E, F \in N_V$, let $E \to F$ denote the evolution of a network during a time interval $[t_1, t_2]$, where E is the initial network at t_1 and F is the final network at $t_2 > t_1$. If $L(E) = L(F)(R(E) = R(F))[H(E) = H(F)]\ \{D(E) = D(F)\}$, then the evolution $E \to F$ is a *Green's* $\mathcal{L}(\mathcal{R})[\mathcal{H}]\{\mathcal{D}\}$ *evolution*. It is important to note that since $\mathcal{D} = \mathcal{L} \bullet \mathcal{R} = \mathcal{R} \bullet \mathcal{L}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, then $\mathcal{L}$ and $\mathcal{R}$ evolutions are also $\mathcal{D}$ evolutions, whereas $\mathcal{H}$ evolutions are both $\mathcal{L}$ and $\mathcal{R}$ evolutions, as well as $\mathcal{D}$ evolutions.

Theorem 2. *The following statements are true for network evolutions in N_V:*

i. $\mathcal{L}$ *evolutions preserve basis out-neighborhood sets and Π lattice isomorphism;*
ii. $\mathcal{R}$ *evolutions preserve basis in-neighborhood sets and Π lattice isomorphism;*
iii. $\mathcal{H}$ *evolutions preserve basis out-neighborhood and in-neighborhood sets and Π lattice isomorphism;*
iv. $\mathcal{D}$ *evolutions preserve Π lattice isomorphism.*

Proof. This is a direct and obvious consequence of the definitions of Green's evolutions and Theorem 1. □

To illustrate this theorem, consider the order two networks $\psi \equiv (V, C_\psi)$ and $\mu \equiv (V, C_\mu)$ in the example in [3], where $V = \{a, b\}$, $C_\psi = \{(a, a)\}$, and $C_\mu = \{(a, a), (b, a)\}$. As can be seen from the associated Green's equivalence classification performed there, since $L(\psi) = L(\mu)$ and $D(\psi) = D(\mu)$, the evolution $\psi \to \mu$ is both a Green's $\mathcal{L}$ evolution and a Green's $\mathcal{D}$ evolution. Theorem 2 (*i*) is satisfied, since, from Table 1 and the discussion in [3], it is also seen that $O_r(\psi) = \{\{a\}\} = O_r(\mu)$ and that the *Π* lattices are isomorphic undirected paths of length 1.

7. The Green's Symmetry Problem

In general, a symmetry associated with a "situation" is defined as an "immunity to change" for some aspect of the "situation". For a "situation" to have a symmetry: (a) the aspect of the "situation" remains unchanged when a change is performed; and (b) it must be possible to perform the change, although the change does not actually have to be performed [9].

Recall from Section 2 that for an $\mathcal{R}(\mathcal{L})$ evolution $E \rightarrow F$ in N_V, there exists at least one $A \in N_V$ ($T \in N_V$) such that $EA = F$ ($TE = F$). Although $A(T)$ does not have to be applied to E, it can produce the desired evolution when applied as a right (left) multiplication of E. In so doing, this multiplication not only preserves $I_c(E)$ ($O_r(E)$), but also E's Π lattice structure. Thus, (a) and (b) above are satisfied and both $I_c(E)(O_r(E))$ and the associated Π lattice structure can be considered as the invariant properties associated with the *symmetries* A (T) which produce the evolution. Symmetries such as A (T) are *Green's* $\mathcal{R}(\mathcal{L})$ *symmetries*.

The "*Green's symmetry problem*" is defined here as the determination of all symmetries that produce an evolution from an initial to a final network within an $\mathcal{R}$ or an $\mathcal{L}$ class such that each symmetry preserves the structural invariants required by Theorem 2. As will be discussed below, such symmetries encode information about which node neighborhoods in the initial network can be joined to form neighborhoods in the final network such that the structural invariants required by the evolution are preserved.

8. Satisfiability and Computational Complexity of the Green's Symmetry Problem

The Green's symmetry problem for an evolution is $m-$ *satisfiable* if there are m symmetries which can produce the evolution.

Theorem 3. *The Green's symmetry problem is at least* $1-$ *satisfiable for both Green's* $\mathcal{R}$ *and* $\mathcal{L}$ *evolutions.*

Proof. Semigroup theory guarantees the existence of at least one Green's symmetry in N_V that can produce a Green's $\mathcal{R}$ evolution and at least one Green's symmetry in N_V that can produce a Green's $\mathcal{L}$ evolution. □

8.1. Green's $\mathcal{R}$ Evolutions

The isomorphism established in Lemma 2 provides for computational solutions to the Green's symmetry problem. In particular, if $E \rightarrow F$ is a Green's $\mathcal{R}$ evolution, then, since E and F are known, the equation $EA = F$ can be solved for A for each $i, j \in J$ using the disjunctive normal form logical expression

$$\vee_{k \in J} \left(E_{ik} \wedge A_{kj} \right) = F_{ij}, \tag{6}$$

where use is now made of the Boolean matrix representations of E, F, and A. This expression for fixed j and all $i \in J$ defines a system of $|J|$ equations for node j.

This system of equations is *column-j satisfied* if there exists a column vector $A_{*j} \in W_n$ for which (1) is a true statement for each $i \in J$. For each $j \in J$, let G_{*j} be the set of all A_{*j} for which the associated system of equations is satisfied and define $\gamma \equiv \prod_{j \in J} |G_{*j}|$. Clearly, if $\gamma > 0$, then $EA = F$ is column-j satisfied for each $j \in J$ and *the evolution* $E \rightarrow F$ *is* γ*-satisfiable*. Each instantiation of A is represented by a Boolean matrix in B_n which has an $x \in G_{*j}$ as its jth column.

Let $M_i = \{k \in J : E_{ik} = 1\}$ index the unit valued entries in the row vector $E_{i*} \in V_n$.

Lemma 4. *Let* $F_{ij} = 0$ *for some* $i, j \in J$ *and* $M_i \neq \varnothing$. *If* $A_{*j} \in W_n$ *column-* j *satisfies* $EA = F$, *then* A_{*j} *has* $A_{kj} = 0$ *when* $k \in M_i$.

Proof. Assume for some $j \in J$ that $A_{*j} \in W_n$ column-j satisfies $EA = F$. If $F_{ij} = 0$ and $M_i \neq \varnothing$ for some $i \in J$, then (1) is true and zero valued for A_{*j} and that i value, and the following implication chain is valid: $\vee_{k \in J} \left(E_{ik} \wedge A_{kj} \right) = 0 \Rightarrow \vee_{l \in J - M_i} \left(0 \wedge A_{lj} \right) \vee_{k \in M_i} \left(1 \wedge A_{kj} \right) = 0 \Rightarrow \vee_{k \in M_i} \left(1 \wedge A_{kj} \right) = 0 \Rightarrow A_{kj} = 0$, $k \in M_i$. However, since $A_{*j} \in W_n$ column-j satisfies $EA = F$, it must also satisfy (1) for all $k \in J \Rightarrow A_{*j}$ has $A_{kj} = 0$ when $k \in M_i$. □

Corollary 1. *If* $E = \omega$, *then* $A_{*j} = \mathbf{0}$.

Proof. $E = \omega \Rightarrow M_i = J \Rightarrow \vee_{k \in J} \left(1 \wedge A_{kj} \right) = 0 \Rightarrow A_{kj} = 0, k \in J \Rightarrow A_{*j} = \mathbf{0}$. □

The computational complexity $\mathcal{C}_{\mathcal{R}}$ of the Green's symmetry problem for Green's $\boldsymbol{\mathcal{R}}$ evolutions is the number of remaining combinations of $A_{kj} \in \{0,1\}$ values which must be checked for $EA = F$ satisfiability after the $A_{kj} = 0$ assignments specified by Lemma 4 have been made. Assume that $E \neq \omega, z$ and for each $j \in J$ let $Q(j) = \left\{i \in J : F_{ij} = 0\right\}$ index the zero valued Boolean equations of form (1).

Theorem 4. $\mathcal{C}_{\mathcal{R}} = \sum_{j \in J}\left[2^{n-\left|\cup_{i \in Q(j)}\ M_i\right|}\right]$.

Proof. For each $j \in J$, the set $\cup_{i \in Q(j)} M_i$ (which can possibly be empty) indexes all row locations $k \in J$ in A_{*j} for which $A_{kj} = 0$ in every A_{*j} that column-j satisfies $EA = F$. The set $J - \cup_{i \in Q(j)} M_i$ indexes all $k \in J$ for which A_{kj} must be evaluated to determine the column-j satisfiability of an associated A_{*j}. Since there are $Z_j = 2^{n-\left|\cup_{i \in Q(j)}\ M_i\right|}$ such evaluations for each $j \in J$, then for all $j \in J$ there are a total of $\mathcal{C}_{\mathcal{R}} = \sum_{j \in J} Z_j$ evaluations required to determine all $A_{*j} \in W_n$ which column-j satisfy $EA = F$. □

8.2. Green's $\boldsymbol{\mathcal{L}}$ Evolutions

If $E \rightarrow F$ is a Green's $\boldsymbol{\mathcal{L}}$ evolution, then, since $TE = F$, it can be solved for T for each $i, j \in J$ using the disjunctive normal form logical expression

$$\vee_{k \in J}\left(T_{ik} \wedge E_{kj}\right) = F_{ij}, \tag{7}$$

which, for fixed i and all $j \in J$, defines a system of $|J|$ equations for node i. This system is *row-i satisfied* if there exists a row vector $T_{i*} \in V_n$ for which (2) is a true statement for each $j \in J$. For each $i \in J$, let H_{i*} be the set of all T_{i*} for which the associated system of equations is row-i satisfied and define $\delta \equiv \prod_{i \in J} |H_{i*}|$. If $\delta > 0$, then $TE = F$ is row-i satisfied for each $i \in J$ and *the evolution* $E \rightarrow F$ *is* $\delta-$ *satisfiable*. Each instantiation of T is represented by a Boolean matrix in B_n which has a $y \in H_{i*}$ as its ith row.

Let $K_j = \left\{k \in J : E_{kj} = 1\right\}$ index the unit valued entrees in the column vector $E_{*j} \in W_n$.

Lemma 5. *Let $F_{ij} = 0$ for some $i, j \in J$ and $K_j \neq \varnothing$. If $T_{i*} \in V_n$ row-i satisfies $TE = F$, then T_{i*} has $T_{ik} = 0$ when $k \in K_j$.*

Proof. Assume for some $i \in J$ that $T_{i*} \in V_n$ row, i satisfies $TE = F$. If $F_{ij} = 0$ for some $j \in J$ and $K_j \neq \varnothing$, then (2) is true and zero valued for T_{i*} and that j value, and the following implication chain is valid: $\vee_{k \in J}\left(T_{ik} \wedge E_{kj}\right) = 0 \Rightarrow \vee_{l \in J-K_j}(T_{il} \wedge 0) \vee_{k \in K_j} (T_{ik} \wedge 1) = 0 \Rightarrow \vee_{k \in K_j}(T_{ik} \wedge 1) = 0 \Rightarrow T_{ik} = 0, k \in K_j$. However, since T_{i*} row-i satisfies $TE = F$, it must also satisfy (2) for all $j \in J \Rightarrow T_{i*}$ has $T_{ik} = 0$ when $k \in K_j$. □

Corollary 2. *If $E = \omega$, then $T_{i*} = 0$.*

Proof. $E = \omega \Rightarrow K_j = J \Rightarrow \vee_{k \in J}(T_{ik} \wedge 1) = 0 \Rightarrow T_{ik} = 0,\ k \in J \Rightarrow T_{i*} = 0$. □

The computational complexity $\mathcal{C}_{\mathcal{L}}$ of the Green's symmetry problem for Green's $\boldsymbol{\mathcal{L}}$ evolutions is the number of remaining combinations of $T_{ik} \in \{0,1\}$ values which must be checked for $TE = F$ satisfiability after the $T_{ik} = 0$ assignments specified by Lemma 5 have been made. Assume that $E \neq \omega, z$ and for each $i \in J$ let $Y(i) = \left\{j \in J : F_{ij} = 0\right\}$ index the zero valued Boolean equations of form (2).

Theorem 5. $\mathcal{C}_{\mathcal{L}} = \sum_{i \in J}\left[2^{n-\left|\cup_{j \in Y(i)}\ K_j\right|}\right]$.

Proof. For each $i \in J$, the set $\cup_{j \in Y(i)} K_j$ (which can possibly be empty) indexes all column locations $k \in J$ for which $T_{ik} = 0$ in every T_{i*} that row-i satisfies $TE = F$. The set $J - \cup_{j \in Y(i)} K_j$ indexes all $k \in J$ for which T_{ik} must be evaluated to determine the row-i satisfiability of an associated T_{i*}. Since there are $Z_i = 2^{n-|\cup_{j\in Y(i)} K_j|}$ such evaluations for each $i \in J$, then for all $i \in J$ there are a total of $\mathcal{C}_{\mathcal{L}} = \sum_{i\in J} Z_i$ evaluations required in order to determine all $T_{i*} \in V_n$ which row-i satisfy $TE = F$. □

9. Symmetries: Instantiations of Internal Dynamics

Since Green's symmetries are themselves effectively elements of B_n, they correspond to special binary relations between network nodes that encode aspects of the internal dynamics of a Green's evolution $E \to F$. In particular, they generally identify many-to-one correspondences between neighborhood sets in E that are joined by set union to produce a neighborhood in F. Each of these correspondences occurs in such a way as to preserve the structural invariants required by Theorem 2. These correspondences are the *internal dynamics* of the evolution.

Consider a Green's $\mathcal{R}$ evolution $E \to F$ where each symmetry A satisfies $EA = F$ and is one instantiation of a possible set of symmetries which produce the evolution and preserve the required invariants. If $j \in J$ is a column in A with a 1 in each of the rows in the set $\Psi_j = \{i_1, i_2, \cdots, i_k\}$ and zeros in every other row location (i.e., there are $|\Psi_j| = k$ 1's and $n-k$ 0's), then this column encodes an internal dynamic of the evolution where the in-neighborhoods of nodes $i_1, i_2, \cdots, i_k$ in E are joined together as $\cup_{i\in\Psi_j} I(E_{*i})$ and associated with the in-neighborhood $I(F_{*j})$ in F according to

$$\cup_{i\in\Psi_j} I(E_{*i}) \subseteq I(F_{*j}). \tag{8}$$

This expression is called a Ψ_j *internal* $\mathcal{R}$ *dynamic* of the evolution and the set Ψ_j is *the associated motion of the dynamic*. Clearly, for the special case where $\Psi_j = \{i\}$,

$$I(E_{*i}) = I(F_{*i}).$$

If $E \to F$ is a Green's $\mathcal{L}$ evolution, a symmetry T which produces the invariant preserving evolution satisfies $TE = F$. If i is a row in T with a 1 in each of the column locations in $\Phi_i = \{j_1, j_2, \cdots, j_l\}$, then this row encodes an internal dynamic of the evolution where the out-neighborhoods of nodes $j_1, j_2, \cdots, j_l$ in network E are joined by set union and associated with the out-neighborhood $O(F_{i*})$ in network F according to

$$\cup_{j\in\Phi_i} O(E_{j*}) \subseteq O(F_{i*}). \tag{9}$$

This expression is a Φ_i internal $\mathcal{L}$ dynamic of the evolution and the set Φ_i is the associated motion of the dynamic. When $\Phi_i = \{j\}$, then

$$O(E_{j*}) = O(F_{i*}).$$

These notions will be clarified below using a simple example.

10. Symmetry Ensembles, Propensities, and Energies

Since the symmetry which produces a Green's evolution is not necessarily unique, it can be unclear as to how to assign a specific symmetry to an evolution. However, the collection of symmetries obtained from Green's symmetry problem, i.e., the *symmetry ensembles*, can be used to construct *propensities*. Propensities can be viewed as weighted symmetries which, in some sense, represent their respective ensembles.

Let $I_{\mathcal{R}}$ $(I_{\mathcal{L}}) \neq \varnothing$ index the symmetries which are solutions to the Green's symmetry problem for some Green's $\mathcal{R}(\mathcal{L})$ evolution $E \to F$. The sets

$$\mathcal{E}_{\mathcal{R}} = \{A^{(i)} : i \in I_{\mathcal{R}},\ EA^{(i)} = F\}$$

and

$$\mathcal{E}_{\mathcal{L}} = \left\{T^{(i)} : i \in I_{\mathcal{L}}, T^{(i)}E = F\right\}$$

are the associated symmetry ensembles. The propensities associated with each ensemble are defined as

$$\overline{A} \equiv |I_{\mathcal{R}}|^{-1} \sum_{i \in I_{\mathcal{R}}} A^{(i)}$$

and

$$\overline{T} = |I_{\mathcal{L}}|^{-1} \sum_{i \in I_{\mathcal{L}}} T^{(i)}.$$

Thus, $\overline{A}_{*j}$ is a measure of the tendency of the nodes in column j in network E to form motions Ψ_j that associate in-neighborhoods in E with in-neighborhoods in network F according to the internal dynamic (3). Similarly, $\overline{T}_{i*}$ is a measure of the tendency of nodes in row i in E to form motions Φ_i that associate out-neighborhoods in E with out-neighborhoods in F according to the internal dynamic (4).

Propensities can be used to associate energies with both ensembles and specific symmetries. These energies quantify in a directly proportional manner the complexity level of the internal dynamical activity that is associated with an evolution. The *propensity energies* provide a representative measure of the "overall" complexity of internal dynamical activity for an evolution based upon ensemble propensity. The propensity energies for ensembles $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{L}}$ are defined as

$$\mathfrak{E}_{\mathcal{R}} \equiv \sum_{i,j \in J} \overline{A}_{ij}$$

and

$$\mathfrak{E}_{\mathcal{L}} \equiv \sum_{i,j \in J} \overline{T}_{ij},$$

respectively.

The *energies of evolution* for the specific symmetries in an ensemble measure the complexity of internal dynamical activity for an evolution produced by a specific symmetry in an ensemble. In particular, if $A^{(k)} \in \mathcal{E}_{\mathcal{R}}$ and $B^{(k)} \in \mathcal{E}_{\mathcal{L}}$, then the associated energies of evolution are defined as

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(k)}\right] \equiv \sum_{i,j \in J} A_{ij}^{(k)} \overline{A}_{ij}$$

and

$$\mathfrak{E}_{\mathcal{L}}\left[T^{(k)}\right] \equiv \sum_{i,j \in J} T_{ij}^{(k)} \overline{T}_{ij}.$$

The following Lemma guarantees that the energy of evolution for a symmetry never exceeds the propensity energy for the associated ensemble.

Lemma 6. *For any Green's* $\boldsymbol{\mathcal{R}}$ *or* $\boldsymbol{\mathcal{L}}$ *evolution,* $\mathfrak{E}_x \geq \mathfrak{E}_x[y]$*, where* $y = A^{(k)}$ *or* $T^{(k)}$ *when* $x = \mathcal{R}$ *or* $\mathscr{L}$.

Proof. $\mathrm{A}_{\mathrm{ij}}^{(\mathrm{k})}, \mathrm{T}_{\mathrm{ij}}^{(\mathrm{k})} \in \{0,1\} \Rightarrow \overline{\mathrm{A}_{\mathrm{ij}}} \geq \mathrm{A}_{\mathrm{ij}}^{(\mathrm{k})}\overline{\mathrm{A}_{\mathrm{ij}}}, \overline{\mathrm{T}_{\mathrm{ij}}} \geq \mathrm{T}_{\mathrm{ij}}^{(\mathrm{k})}\overline{\mathrm{T}_{\mathrm{ij}}} \Rightarrow \sum_{\mathrm{i,j} \in \mathrm{J}} \overline{\mathrm{A}_{\mathrm{ij}}} \geq \sum_{\mathrm{i,j} \in \mathrm{J}} \mathrm{A}_{\mathrm{ij}}^{(\mathrm{k})}\overline{\mathrm{A}_{\mathrm{ij}}}, \sum_{\mathrm{i,j} \in \mathrm{J}} \overline{\mathrm{T}_{\mathrm{ij}}} \geq \sum_{\mathrm{i,j} \in \mathrm{J}} \mathrm{T}_{\mathrm{ij}}^{(\mathrm{k})}\overline{\mathrm{T}_{\mathrm{ij}}} \Rightarrow \mathfrak{E}_{\mathcal{R}} \geq \mathfrak{E}_{\mathcal{R}}\left[\mathrm{A}^{(\mathrm{k})}\right], \ \mathfrak{E}_{\mathcal{L}} \geq \mathfrak{E}_{\mathcal{L}}\left[\mathrm{T}^{(\mathrm{k})}\right].$ □

Recall that internal $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{L}}$ dynamics are strictly defined by their motions. These motions also have energies that provide a measure of the level of internal dynamical activity induced by the motion. Since the symmetries A and T encode $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{L}}$ internal dynamics with motions Ψ_j and Φ_i, respectively, then the associated *energies of motion* are the quantities

$$\mathfrak{E}_{\mathcal{R}}\left[A; \Psi_j\right] \equiv \sum_{i \in \Psi_j} A_{ij} \overline{A}_{ij}$$

and

$$\mathfrak{E}_{\mathcal{L}}[T; \Phi_i] \equiv \sum_{j \in \Phi_i} T_{ij} \overline{T}_{ij}.$$

The energies of motion are related to their energies of evolution by the following theorem:

Theorem 6 (Conservation of Energy of Evolution). *The energy of evolution of a Green's symmetry is conserved by the energies of motion of its internal dynamics.*

Proof. Let $A \in \mathcal{E}_{\mathcal{R}}$ and set M index all the Ψ_j internal $\boldsymbol{\mathcal{R}}$ dynamics encoded by A. Then $\sum_{j\in M} \mathfrak{E}_{\mathcal{R}}[A;\Psi_j] = \sum_{j\in M}\sum_{i\in\Psi_j} A_{ij}\overline{A}_{ij} = \sum_{i,j\in J} A_{ij}\overline{A}_{ij} = \mathfrak{E}_{\mathcal{R}}[A]$, where use has been made of the fact that $\sum_{j\in M}\sum_{i\in\Psi_j}$ is equivalent to $\sum_{i,j\in J}$ because $A_{ij} = 0$ when $i \in J - \Psi_j$ and $j \in J - M$. It is similar for the $\boldsymbol{\mathcal{L}}$ dynamics. □

11. Example

Let $E \to F$ be a Green's $\boldsymbol{\mathcal{R}}$ evolution in N_V, $V = \{1,2\}$ (or equivalently in B_2), where (in B_2)

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

with $I_c(E) = \{2\} = I_c(F)$ (note that this evolution corresponds to the $\tau \to \lambda$ Green's $\boldsymbol{\mathcal{R}}$ evolution in [3]). Theorem 3 guarantees the existence of at least one A such that

$$EA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = F.$$

The disjunctive normal form logical expression (1) for this equation yields the following system of equations

$$(0 \wedge a_{11}) \vee (0 \wedge a_{21}) = 0 \ (0 \wedge a_{12}) \vee (0 \wedge a_{22}) = 0$$

$$(1 \wedge a_{11}) \vee (0 \wedge a_{21}) = 1 \ (1 \wedge a_{12}) \vee (0 \wedge a_{22}) = 1$$

which can be used to solve the associated Green's symmetry problem.

For the two equations in the second row of this system to be satisfied requires the assignment $a_{11} = 1 = a_{12}$. By inspection it is seen that the complete system is satisfied when, in addition to these assignments, a_{21} and a_{22} each assume both values from the set $\{0,1\}$. Thus,

$$G_{*1} = G_{*2} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

so that $\gamma = |G_{*1}||G_{*2}| = 2{\cdot}2 = 4 = |I_{\mathcal{R}}|$ and the evolution $E \to F$ is 4-satisfiable. The associated symmetry ensemble is the set

$$\mathcal{E}_{\mathcal{R}} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \equiv \{A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}\}.$$

To calculate the computational complexity of this Green's symmetry problem, refer to Section 8.1 and observe that $M_1 = \varnothing$, $M_2 = \{1\}$, and $Q(1) = \{1\} = Q(2)$. Application of Theorem 4 yields $\mathcal{C}_{\mathcal{R}} = 2^{2-|M_1|} + 2^{2-|M_1|} = 2^2 + 2^2 = 8$, i.e., four combinations of value assignments must be checked for each j since, according to Lemma 4, a_{ij} values cannot be assigned when $F_{ij} = 0$ because $M_i = \varnothing$.

The propensity and propensity energy for the ensemble are

$$\overline{A} = \begin{bmatrix} 1 & 1 \\ ½ & ½ \end{bmatrix}$$

and $\mathfrak{E}_{\mathcal{R}} = 3$, respectively, and the energies of evolution are $\mathfrak{E}_{\mathcal{R}}[A^{(1)}] = 2$, $\mathfrak{E}_{\mathcal{R}}[A^{(2)}] = 2 = \mathfrak{E}_{\mathcal{R}}[A^{(3)}]$, and $\mathfrak{E}_{\mathcal{R}}[A^{(4)}] = 3$. Please note that this validates Lemma 6. These energies also indicate that $A^{(1)}$ produces the least energy of evolution in the sense that the evolution involves simpler internal dynamical activity than evolutions produced by the other symmetries in the ensemble.

To illustrate this further, first observe that $I(E_{*1}) = \{2\}$, $I(E_{*2}) = \varnothing$, and $I(F_{*1}) = \{2\} = I(F_{*2})$ (here the *j*th column vector is set directly equal to the nodes in the in-neighborhood of node *j*). It is also easily determined that the motions of the dynamics for: $A^{(1)}$ are $\Psi_1 = \{1\} = \Psi_2$; $A^{(2)}$ are $\Psi_1 = \{1\}$ and $\Psi_2 = \{1, 2\}$; $A^{(3)}$ are $\Psi_1 = \{1, 2\}$ and $\Psi_2 = \{1\}$; and $A^{(4)}$ are $\Psi_1 = \{1, 2\} = \Psi_2$. By inspection it is found that each of these motions satisfies (3). Using $A^{(4)}$ as an example, it is seen that (3) yields the correct set theoretic relationship $I(E_{*1}) \cup I(E_{*2}) \subseteq I(F_{*1}) \cup I(F_{*2})$ or $\{2\} \cup \varnothing \subseteq \{2\} \cup \{2\}$ or $\{2\} \subseteq \{2\}$ for both Ψ_1 and Ψ_2. Also note that the internal dynamics for $A^{(1)}$ are simpler than those for the other symmetries in the ensemble, in the sense that both of the $A^{(1)}$ motions are singleton sets, whereas at least one of the motions for the other symmetries is a doubleton set. This is consistent with the fact mentioned above that $A^{(1)}$ produces the least energy of evolution.

Now consider the energies of motion for each ensemble symmetry. They are easily calculated from the theory and are found to be:

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(1)}; \Psi_1\right] = 1 = \mathfrak{E}_{\mathcal{R}}\left[A^{(1)}; \Psi_2\right];$$

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(2)}; \Psi_1\right] = 1,\ \mathfrak{E}_{\mathcal{R}}\left[A^{(2)}; \Psi_2\right] = 1;$$

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(3)}; \Psi_1\right] = 1,\ \mathfrak{E}_{\mathcal{R}}\left[A^{(3)}; \Psi_2\right] = 1;$$

and

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(4)}; \Psi_1\right] = 1 = \mathfrak{E}_{\mathcal{R}}\left[A^{(4)}; \Psi_2\right].$$

Thus, the motions associated with an $A^{(1)}$ evolution are the least energetic since

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(1)}; \Psi_j\right] \leq \mathfrak{E}_{\mathcal{R}}\left[A^{(k)}; \Psi_j\right],\ k = 2, 3, 4; j = 1, 2.$$

This is also consistent with the fact that an $A^{(1)}$ induced evolution is the least energetic and involves the least complex internal dynamics.

Finally, observe that these results validate Theorem 6. In particular,

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(1)}; \Psi_1\right] + \mathfrak{E}_{\mathcal{R}}\left[A^{(1)}; \Psi_2\right] = 2 = \mathfrak{E}_{\mathcal{R}}\left[A^{(1)}\right];$$

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(2)}; \Psi_1\right] + \mathfrak{E}_{\mathcal{R}}\left[A^{(2)}; \Psi_2\right] = 2 = \mathfrak{E}_{\mathcal{R}}\left[A^{(2)}\right];$$

$$\mathfrak{E}_{\mathcal{R}}\left[\mathrm{A}^{(3)}; \Psi_1\right] + \mathfrak{E}_{\mathcal{R}}\left[\mathrm{A}^{(3)}; \Psi_2\right] = 2 = \mathfrak{E}_{\mathcal{R}}\left[\mathrm{A}^{(3)}\right];$$

and

$$\mathfrak{E}_{\mathcal{R}}\left[A^{(4)}; \Psi_1\right] + \mathfrak{E}_{\mathcal{R}}\left[A^{(4)}; \Psi_2\right] = 3 = \mathfrak{E}_{\mathcal{R}}\left[A^{(4)}\right].$$

12. Concluding Remarks

The research documented in [3] was inspired by earlier research performed by Konieczny [6] and Plemmons et al. [7]. This paper has reviewed the results developed in [3], i.e., that the set of all networks on a fixed number of nodes can be classified using the Green's equivalence relations of semigroup theory and that all networks within a Green's equivalence class have a common structural invariant (neighborhoods or poset relationships between node sets generated by neighborhoods). By extension, it was deduced in this paper from these results that if a network evolves from an initial network configuration to a final network configuration such that both the initial and final networks are in the same Green's equivalence class, then the structural invariants for the class are preserved by the evolution. In addition, the Green's symmetry problem was also defined in this paper. This problem is to determine by computation all symmetries which produce a network evolution within a Green's $\mathcal{R}$ or a Green's $\mathcal{L}$ equivalence class (i.e., a symmetry ensemble). These symmetries were shown to be solutions to special Boolean equations whose form is dictated by semigroup theory. Each such symmetry encodes information about the internal dynamics of the associated evolution and an ensemble associated with an evolution was used to define propensities and energies which quantify aspects of the internal dynamics of the evolution. However, it should be noted that a practical limitation exists for solving the Green's symmetry problem. This occurs because the cardinality of symmetry ensembles associated with large real networks can be quite large, thereby requiring the use of considerable computational resources to solve such problems (see future research suggestions below).

In conclusion, it is believed that the results of this paper are new and not in general use (perhaps having the closest resemblance to these results are the applications of Green's relations to social networks [10] and automata theory, e.g., [11]). However, the results of this paper are important and should be of general interest to network science researchers and those working in areas of applied network theory. In addition to applications similar to those mentioned in Section 1 (actor identification in social networks and communication network reconfiguration), contemporary areas of frontier research, such as identifying emerging scientific disciplines, e.g., [12], analyzing brain connectivity, e.g., [13–15], and finding symmetries in engineering processes [16], could also benefit from the results of this paper.

Before closing it is worthwhile to mention several directions for related future research. First, because of the computational resources required to solve the Green's symmetry problem, it would be useful to investigate how sampling and statistics can be used to obtain symmetry sub-ensembles that effectively yield the same information about propensities and energies as the associated full ensemble. A second research area involves understanding symmetries and their computation for network evolutions occurring within Green's $\mathcal{H}$ and $\mathcal{D}$ equivalence classes. A third and potentially very interesting research area concerns determining the relationships (if any) between the theory developed in this paper and the relatively new theory of persistence that is used to analyze large data sets, e.g., [17].

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