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Dirac's Method for the Two-Dimensional Damped Harmonic Oscillator in the Extended Phase Space

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Abstract: The system of a two-dimensional damped harmonic oscillator is revisited in the extended phase space. It is an old problem that has already been addressed by many authors that we present here with some fresh points of view and carry on a whole discussion. We show that the system is singular. The classical Hamiltonian is proportional to the first-class constraint. We pursue with the Dirac's canonical quantization procedure by fixing the gauge and provide a reduced phase space description of the system. As a result, the quantum system is simply modeled by the original quantum Hamiltonian.

Keywords: damped harmonic oscillator; extended phase space; constrained system

1. Introduction

The Hamiltonians of most real physical systems are explicitly time-dependent and do not provide directly conserved quantities. Succeeding in isolating an invariant helps to gain information about a fundamental system property. For instance, in the case of an autonomous Hamiltonian system, the Hamiltonian itself represents an invariant. Many approaches have been developed to identify conserved quantities for explicitly time-dependent systems. The first one was developed by Emmy Noether in the context of the Lagrangian formalism [1]. The invariant for the one-dimensional time-dependent harmonic oscillator was derived by H. R. Lewis [2]. It was demonstrated later that the Lewis procedure follows from Noether's theorem [3], and that was extended by Chattopadhyay to derive invariants for certain one-dimensional non-linear systems [4]. Another approach to finding conserved quantities for explicitly time-dependent systems was developed by Leach by performing a finite time-dependent canonical transformation [5]. A third way of finding exact invariants for time-dependent classical Hamiltonians was derived by Lewis and Leach by using direct *Ansätze* with different powers in the canonical momentum [6].

The invariants for time-dependent Hamiltonian systems are still investigated and of interest in the literature. We were first interested in finding the class of invariants for the two-dimensional time-dependent Landau problem and harmonic oscillator in a magnetic field [7], where we considered an isotropic two-dimensional harmonic oscillator with arbitrarily time-dependent mass $M(t)$ and frequency $\Omega(t)$ in an arbitrarily time-dependent magnetic field $B(t)$. Two commuting invariant observables (in the sense of Lewis and Riesenfeld) L, I were derived in terms of some solutions of an auxiliary ordinary differential equation and an orthonormal basis of the Hilbert space consisting of joint eigenvectors φ_λ of L, I.

Recently, we studied a system of two non-interacting damped oscillators with equal time-dependent coefficients of friction and equal time-dependent frequencies [8]. The system is described by the Lagrangian function

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = f^{-1}(t)\left(\frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{m\omega^2(t)}{2}(x_1^2 + x_2^2)\right), \tag{1}$$

where f is an arbitrary function such that $f(t) = e^{-\int_0^t \eta(t')dt'}$, we assume that the function f is twice differentiable, and the canonical coordinates are x_1, x_2. The canonical momenta are respectively given by

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = mf^{-1}(t)\dot{x}_1, \tag{2}$$

$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = mf^{-1}(t)\dot{x}_2. \tag{3}$$

In the canonical formalism, the dynamics of the system is governed by the classical Hamiltonian

$$H(x_1, x_2, p_1, p_2, t) = p_1\dot{x}_1 + p_2\dot{x}_2 - L(x_1, x_2, \dot{x}_1, \dot{x}_2, t), \tag{4}$$

which is equivalent to

$$H(x_1, x_2, p_1, p_2, t) = \frac{f(t)}{2m}(p_1^2 + p_2^2) + f^{-1}(t)\frac{m\omega^2(t)}{2}(x_1^2 + x_2^2). \tag{5}$$

The dynamics of the system are determined by the values of the canonical coordinates and momenta at any given time t. The coordinates and momenta satisfy a set of Poisson Brackets relations

$$\{x_1, p_1\}_{PB} = 1, \quad \{x_2, p_2\}_{PB} = 1, \quad \{x_1, p_2\}_{PB} = \{x_2, p_1\}_{PB} = 0, \tag{6}$$

and

$$\{f, g\}_{PB} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial x_2}, \tag{7}$$

where f and g are any functions of the x_i's and p_i's, $i = 1, 2$. At the quantum level, the dynamic invariant method formulated by Lewis and Riesenfeld [9] was used to construct an exact invariant operator. The exact solutions for the corresponding time-dependent Schrödinger equations are provided. The solutions were used to derive the generators of the $su(1,1)$ Lie algebra that enable the properties of the coherent states to be constructed and studied à la Barut–Girardello and Perelomov.

In this paper we revisit the model in [8] in the extended phase space. The idea is not new in the literature. For instance, we refer to the following works that we met in the literature: the one by Struckmeier on Hamiltonian dynamics on the symplectic extended phase space for autonomous and non-autonomous systems [10], the work done by Baldiotti et al. on the quantization of the damped harmonic oscillator [11], the work by Menouar and al. entitled *the quantization of the time-dependent singular potential systems: non-central potential in three dimension* [12] and the recent paper by Garcia-Chung et al. entitled *Dirac's method for time-dependent Hamiltonian systems in the extended phase space* [13]. In the extended phase space (i.e., considering the time t as a dynamical variable with a corresponding conjugate momentum), the Lagrangian of the system is then singularly characterized by the presence of constraints. We identify the constraints and apply the Dirac method of quantization. This procedure was presented in the paper by A. Garcia-Chung et al. [13]. An advantage of extending the phase space is that the symplectic group of the system is also enlarged, giving place to study the canonical transformation in the extended phase space such that the final dynamical description of the reduced phase space is no longer time-dependent. An invariant of the system can be obtained by applying a finite canonical transformation to the initial Hamiltonian of the system in the extended phase space.

The quantum Hamiltonian given in [8] is straightforward through canonical quantization, as the Lagrangian (1) is regular. Let us briefly recall here the procedure of canonical quantization of the system described in Equations (1), (5), and (6). The Hessian matrix $\mathbf{M}$ of the Lagrangian function is given by

$$\mathbf{M} = \begin{bmatrix} \frac{\partial^2 L}{\partial \dot{x}_1^2} & \frac{\partial}{\partial \dot{x}_1}\left[\frac{\partial L}{\partial \dot{x}_2}\right] \\ \frac{\partial}{\partial \dot{x}_2}\left[\frac{\partial L}{\partial \dot{x}_1}\right] & \frac{\partial^2 L}{\partial \dot{x}_2^2} \end{bmatrix} = \begin{bmatrix} f^{-1}m & 0 \\ 0 & f^{-1}(m) \end{bmatrix}, \tag{8}$$

and the determinant

$$\det \mathbf{M} = \det \begin{Vmatrix} f^{-1}m & 0 \\ 0 & f^{-1}(m) \end{Vmatrix} = m^2 f^{-2}(t) \neq 0\,. \tag{9}$$

The Lagrangian in Equation (1) is called regular or standard since its Hessian matrix satisfies Equation (9). The system described by the Lagrangian in Equation (1) does not involve constraints, and we assume that the phase space is flat and admits the procedure of canonical quantization which consists of demanding that to the classical canonical pairs (x_1, p_1), (x_2, p_2) that satisfy the Poisson brackets in Equation (6) we associate the operators $\hat{x}_1$, $\hat{x}_2$, $\hat{p}_1$, $\hat{p}_2$ acting both on the Hilbert space of the states $\mathcal{H}$ and obey the canonical commutation relations

$$[\hat{x}_1, \hat{p}_1] = i\hbar; \quad [\hat{x}_2, \hat{p}_2] = i\hbar; \quad [\hat{x}_1, \hat{p}_2] = 0; \quad [\hat{x}_2, \hat{p}_1] = 0\,, \tag{10}$$

where the commutator of two operators is given by $[\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}$. We assign to the classical Hamiltonian $H(x_1, x_2, p_1, p_2, t)$ in Equation (5), which is a function of the dynamical variables x_1, x_2, p_1, p_2 and operator $\hat{H}(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2, t)$ which is obtained by replacing the dynamical variables with the corresponding operators. Other classical dynamical quantities in quantum mechanics are similarly associated with quantum operators that act on the Hilbert space of states.

The aim of this paper is to illuminate the Dirac's method of quantization of the system in the extended phase space. The subject may be of interest to some readers in the community of mathematical physics, as it forms some integrity with all needed elements. The organization of the paper is as follows: in Section 2, we apply Dirac's method for constrained systems to the model in the extended phase space. Concluding remarks are given in Section 3.

2. The Model in the Extended Phase Space

In this section we perform the Dirac method of quantization to constrained systems. The reader interested in knowing more about the method may consult, for example, [14–21]. Here, a constrained system is one in which there exists a relationship between the system's degrees of freedom that holds for all times.

We consider the time integral of the Lagrangian in Equation (1) as the action

$$S\left[x_1(t), x_2(t), \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}\right] = \int_{t_1}^{t_2} L(x_1, x_2, \dot{x}_1, \dot{x}_2)dt. \tag{11}$$

In order to extend the phase space, we consider the time parameter t as an additional degree of freedom for the system S described in Equation (11). We consider the arbitrary time scaling transformation $t = t_\tau(\tau)$, where the parameter τ plays the role of the new time parameter. The function $t_\tau(\tau)$ is chosen such that it gives a smooth one-to-one correspondence of the domain τ and t. This transformation also changes the dependency of the coordinates and requires the following redefinitions:

$$x_1(t_\tau(\tau)) = x_{1,\tau}(\tau), \quad x_2(t_\tau(\tau)) = x_{2,\tau}(\tau). \tag{12}$$

Consequently, we have a new functional expression for the action S denoted by

$$S_\tau\left[x_{1,\tau}, x_{2,\tau}, t_\tau, \frac{dx_{1,\tau}}{d\tau}, \frac{dx_{2,\tau}}{d\tau}, \frac{dt_\tau}{d\tau}\right] = \int_{\tau_1}^{\tau_2} \frac{m}{2} f^{-1}(t_\tau)\left(\left(\frac{\dot{x}_{1,\tau}}{\dot{t}_\tau}\right)^2 + \left(\frac{\dot{x}_{2,\tau}}{\dot{t}_\tau}\right)^2 - \frac{m\omega^2(t_\tau)}{2}\left(x_{1,\tau}^2 + x_{2,\tau}^2\right)\right)\dot{t}_\tau d\tau, \tag{13}$$

where the notations $\dot{x}_{i,\tau},\ i = 1,2,$ and $\dot{t}_\tau$ are respectively $\dot{x}_{i,\tau} = \frac{d}{d\tau}x_{i,\tau},\ i = 1,2,$ and $\dot{t}_\tau = \frac{d}{d\tau}t_\tau$. The generalized configuration variables on the extended phase space are given by $x_{1,\tau},\ x_{2,\tau},\ t_\tau$, and their velocities are respectively given by $\dot{x}_{1,\tau}, \dot{x}_{2,\tau}, \dot{t}_\tau$. We consider the boundary conditions $x_{1,\tau}(t_1) = x_1,\ x_{2,\tau}(t_2) = x_2,\ t_\tau(\tau_1) = t_1,\ t_\tau(\tau_2) = t_2$.

The integrand of Equation (13) thus defines the extended Lagrangian L_τ:

$$L_\tau(x_{1,\tau}, x_{2,\tau}, t_\tau, \dot{x}_{1,\tau}, \dot{x}_{2,\tau}, \dot{t}_\tau) = \frac{f^{-1}(t_\tau)m}{2\dot{t}_\tau}\left(\dot{x}_{1,\tau}^2 + \dot{x}_{2,\tau}^2\right) - \frac{m\omega^2(t_\tau)\dot{t}_\tau}{2} f^{-1}(t_\tau)\left(x_{1,\tau}^2 + x_{2,\tau}^2\right). \tag{14}$$

Let us first determine the Hessian matrix $\mathbf{M}_\tau$ of the Lagrangian function L_τ:

$$\mathbf{M}_\tau = \begin{bmatrix} \frac{\partial^2 L}{\partial \dot{x}_{1,\tau}^2} & \frac{\partial^2 L}{\partial \dot{x}_{1,\tau}\partial \dot{x}_{2,\tau}} & \frac{\partial^2 L}{\partial \dot{x}_{1,\tau}\partial \dot{t}_\tau} \\ \frac{\partial^2 L}{\partial \dot{x}_{2,\tau}\partial \dot{x}_{1,\tau}} & \frac{\partial^2 L}{\partial \dot{x}_{2,\tau}^2} & \frac{\partial^2 L}{\partial \dot{x}_{2,\tau}\partial \dot{t}_\tau} \\ \frac{\partial^2 L}{\partial \dot{t}_\tau\partial \dot{x}_{1,\tau}} & \frac{\partial^2 L}{\partial \dot{t}_\tau\partial \dot{x}_{2,\tau}} & \frac{\partial^2 L}{\partial \dot{t}_\tau^2} \end{bmatrix}, \tag{15}$$

which is equivalent to

$$\mathbf{M}_\tau = \begin{bmatrix} \frac{f^{-1}(t_\tau)m}{\dot{t}_\tau} & 0 & -\frac{f^{-1}(t_\tau)m\dot{x}_{1,\tau}}{\dot{t}_\tau^2} \\ 0 & \frac{f^{-1}(t_\tau)m}{\dot{t}_\tau} & -\frac{f^{-1}(t_\tau)m\dot{x}_{2,\tau}}{\dot{t}_\tau^2} \\ -\frac{f^{-1}(t_\tau)m\dot{x}_{1,\tau}}{\dot{t}_\tau^2} & -\frac{f^{-1}(t_\tau)m\dot{x}_{2,\tau}}{\dot{t}_\tau^2} & \frac{f^{-1}(t_\tau)m(\dot{x}_{1,\tau}^2+\dot{x}_{2,\tau}^2)}{\dot{t}_\tau^3} \end{bmatrix}. \tag{16}$$

It is easy to show that the determinant (Hessian) of the matrix $\mathbf{M}_\tau$ is zero, which means that the Lagrangian L_τ is singular, and a singular Lagrangian theory necessarily involves constraints. Let us now determine the corresponding conjugate momenta of the configuration variables $x_{1,\tau},\ x_{2,\tau},\ t_\tau$. They are respectively given by

$$p_{1,\tau} = \frac{\partial L_\tau}{\partial \dot{x}_{1,\tau}} = \frac{f^{-1}(t_\tau)m\dot{x}_{1,\tau}}{\dot{t}_\tau}, \tag{17}$$

$$p_{2,\tau} = \frac{\partial L_\tau}{\partial \dot{x}_{2,\tau}} = \frac{f^{-1}(t_\tau)m\dot{x}_{2,\tau}}{\dot{t}_\tau}, \tag{18}$$

$$p_\tau = \frac{\partial L_\tau}{\partial \dot{t}_\tau} = -\frac{f(t_\tau)}{2m}(p_{1,\tau}^2 + p_{2,\tau}^2) - \frac{m\omega^2(t_\tau)f^{-1}(t_\tau)}{2}(x_{1,\tau}^2 + x_{2,\tau}^2). \tag{19}$$

The momentum p_τ is expressed in terms of the fundamental variables, and a constraint arises as

$$\phi = p_\tau + \frac{f(t_\tau)}{2m}(p_{1,\tau}^2 + p_{2,\tau}^2) + \frac{m\omega^2(t_\tau)f^{-1}(t_\tau)}{2}(x_{1,\tau}^2 + x_{2,\tau}^2) \sim 0. \tag{20}$$

We derive the extended Hamiltonian H_τ as the Legendre transform of the extended Lagrangian L_τ

$$H_\tau(x_{1,\tau}, x_{2,\tau}, t_\tau, p_{1,\tau}, p_{2,\tau}, p_\tau) = p_{1,\tau}\dot{x}_{1,\tau} + p_{2,\tau}\dot{x}_{2,\tau} + p_\tau \dot{t}_\tau - L\,, \tag{21}$$

that is explicitly

$$H_\tau = \left(\frac{f(t_\tau)}{2m}(p_{1,\tau}^2 + p_{2,\tau}^2) + p_\tau + \frac{m\omega^2(t_\tau) f^{-1}(t_\tau)}{2}(x_{1,\tau}^2 + x_{2,\tau}^2) \right) \dot{t}_\tau. \tag{22}$$

A first remark is that

$$H_\tau = \dot{t}_\tau \phi \ \sim 0\,. \tag{23}$$

The use of the $\sim$ sign instead of the $=$ sign is due to Dirac [14], and has a special meaning: two quantities related by a $\sim$ sign are only equal after all constraints have been imposed. Two such quantities are weakly equal to one another. It is important to note that the Poisson brackets in any expression must be worked out before any constraints are set to zero.

We now have an extended phase space determined by $x_{1,\tau},\ x_{2,\tau},\ p_{1,\tau},\ p_{2,\tau},\ p_\tau$. The simplectic structure is determined by the non-vanishing Poisson brackets

$$\{x_{1,\tau}, p_{1,\tau}\}_{PB} = 1, \quad \{x_{2,\tau}, p_{2,\tau}\}_{PB} = 1, \quad \{t_\tau, p_\tau\}_{PB} = 1\,, \tag{24}$$

and the Poisson brackets for two arbitrary smooth functions f and g in this extended phase space take the following form:

$$\{f, g\}_{PB} = \frac{\partial f}{\partial x_{1,\tau}}\frac{\partial g}{\partial p_{1,\tau}} + \frac{\partial f}{\partial x_{2,\tau}}\frac{\partial g}{\partial p_{2,\tau}} + \frac{\partial f}{\partial t_\tau}\frac{\partial g}{\partial p_\tau} - \frac{\partial f}{\partial p_{1,\tau}}\frac{\partial g}{\partial x_{1,\tau}} - \frac{\partial f}{\partial p_{2,\tau}}\frac{\partial g}{\partial x_{2,\tau}} - \frac{\partial f}{\partial p_\tau}\frac{\partial g}{\partial t_\tau}. \tag{25}$$

The constraint ϕ is a primary constraint, and indeed is the only one, as there are no secondary constraints generated. We have the presence of a first-class constraint. Recall that a dynamical variable R is said to be first-class if it has weakly vanishing Poisson brackets with all constraints. The Hamiltonian H_τ is a first-class Hamiltonian. We set the total Hamiltonian to be

$$H_{\tau T} = \lambda \phi\,, \tag{26}$$

where λ is a Lagrange multiplier (note that λ depends only on time). The Hamiltonian equations of motion derived with this Poisson bracket and the Hamiltonian $H_{\tau T}$ are given by:

$$\dot{x}_{1,\tau} = \{x_{1,\tau}, H_{\tau T}\}_{PB} = \lambda \frac{f(t_\tau)}{m} p_{1,\tau}, \tag{27}$$

$$\dot{x}_{2,\tau} = \{x_{2,\tau}, H_{\tau T}\}_{PB} = \lambda \frac{f(t_\tau)}{m} p_{2,\tau}, \tag{28}$$

$$\dot{p}_{1,\tau} = \{p_{1,\tau}, H_{\tau T}\}_{PB} = -\lambda m \omega^2(t_\tau) f^{-1}(t_\tau) x_{1,\tau}, \tag{29}$$

$$\dot{p}_{2,\tau} = \{p_{2,\tau}, H_{\tau T}\}_{PB} = -\lambda m \omega^2(t_\tau) f^{-1}(t_\tau) x_{2,\tau}, \tag{30}$$

$$\dot{t}_\tau = \{t_\tau, H_{\tau T}\}_{PB} = \lambda, \tag{31}$$

$$\begin{aligned} \dot{p}_\tau &= \{p_\tau, H_{\tau T}\}_{PB}, \\ &= \lambda \left(\frac{\dot{f}(t_\tau)}{2m}(p_{1,\tau}^2 + p_{2,\tau}^2) + \frac{m f^{-1}(t_\tau)\omega(t_\tau)}{2}[2\dot{\omega}(t_\tau) - \omega(t_\tau)\dot{f}(t_\tau) f^{-1}(t_\tau)](x_{1,\tau}^2 + x_{2,\tau}^2) \right). \end{aligned} \tag{32}$$

The total Hamiltonian is proportional to the constraint ϕ, and the coefficient of proportionality is a Lagrange multiplier denoted by λ. The Lagrange multiplier is independent of the phase space points. This kind of Lagrange multiplier is referred to as non-canonical gauge. This particular case of constrained system in which the total Hamiltonian is null when the constraint is strongly set to zero is usually called a reparametrization-invariant system [22]. The fact that we have only a first-class constraint implies that all phase space functions will evolve by gauge transformations, and the system at a given time will gauge equivalent to the system at any other time. To quantize such a theory, we need to choose between the Dirac and the canonical quantization procedures. If we choose the canonical quantization, we face the fact that we have no Schrödinger equations because the total Hamiltonian must necessarily annihilate physical states. The solution is to impose a supplementary constraint η that depends on the time variable. The process in which a value for the Lagrange multiplier λ is fixed is usually called *fixing the gauge*. For instance, the most common gauge fixing is the case in which $\lambda = \frac{t_2-t_1}{\tau_2-\tau_1}$. This gauge solves the Equation in (31) $\dot{t}_\tau = \lambda$, which means

$$t_\tau = \frac{(t_2-t_1)(\tau-\tau_1)}{\tau_2-\tau_1} + t_1 \,, \tag{33}$$

with $t_\tau(\tau_i) = t_i,\ i = 1,2$ holds .

The gauge fixing condition leads to an additional constraint surface

$$\eta = t_\tau - \frac{(t_2-t_1)(\tau-\tau_1)}{\tau_2-\tau_1} + t_1 \sim 0 \,, \tag{34}$$

and

$$\{\phi, \eta\}_{PB} \nsim 0 \,. \tag{35}$$

The constraints ϕ and η are second-class constraints. Recall that a dynamical variable R is said be second-class if it has weakly non-vanishing Poisson brackets with all the constraints. Let us now define the Dirac brackets. The matrix of the constraints is given by

$$\Delta = \begin{pmatrix} \{\phi,\phi\}_{PB} & \{\phi,\eta\}_{PB} \\ \{\eta,\phi\}_{PB} & \{\eta,\eta\}_{PB} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{36}$$

The matrix Δ is obviously invertible, and its inverse is given by

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{37}$$

The Dirac brackets of two extended phase space quantities f and g are given by

$$\{f,g\}_{DB} = \{f,g\}_{PB} - [\{f,\phi\}_{PB}\{\eta,g\}_{PB} - \{f,\eta\}_{PB}\{\phi,g\}_{PB}] \,. \tag{38}$$

The Poisson bracket (25) is then replaced by the Dirac bracket (38). With respect to that, let us calculate the Dirac brackets of the fundamental variables in the extended phase space. The non-vanishing Dirac brackets are as follows:

$$\{x_{1,\tau}, p_{1,\tau}\}_{DB} = 1, \tag{39}$$

$$\{x_{2,\tau}, p_{2,\tau}\}_{DB} = 1, \tag{40}$$

$$\{x_{1,\tau}, p_\tau\}_{DB} = -\frac{f(t_\tau)}{m} p_{1,\tau}, \tag{41}$$

$$\{x_{2,\tau}, p_\tau\}_{DB} = -\frac{f(t_\tau)}{m} p_{2,\tau}, \tag{42}$$

$$\{p_{1,\tau}, p_\tau\}_{DB} = m\omega^2(t_\tau) f^{-1}(t_\tau) x_{1,\tau}, \tag{43}$$

$$\{p_{2,\tau}, p_\tau\}_{DB} = m\omega^2(t_\tau) f^{-1}(t_\tau) x_{2,\tau}. \tag{44}$$

Comparing the Poisson brackets in Equation (24) and the Dirac brackets in Equations (39)–(44), we can note the differences that are essentially $\{t_\tau,\ p_\tau\}_{PB} = 1$ while $\{t_\tau,\ p_\tau\}_{DB} = 0$ and $\{p_{1,\tau},\ p_\tau\}_{PB} = \{p_{2,\tau},\ p_\tau\}_{PB} = 0$ while $\{p_{1,\tau},\ p_\tau\}_{DB} = m\omega^2(t_\tau)f^{-1}(t_\tau)x_{1,\tau}$ and $\{p_{2,\tau},\ p_\tau\}_{DB} = m\omega^2(t_\tau)f^{-1}(t_\tau)x_{2,\tau}$. When the constraints are fulfilled, meaning that $\phi = 0$ and $\eta = 0$, we have the coordinates $x_{1,\tau},\ x_{2,\tau},\ p_{1,\tau},\ p_{2,\tau}$ selected as the physical degree of freedom using τ as the time parameter or instead we can use $x_1,\ x_2,\ p_1,\ p_2$ with t as time parameter in accordance with the initial description. In that situation, $t_\tau = \frac{(t_2-t_1)(\tau-\tau_1)}{\tau_2-\tau_1} + t_1$ and $p_\tau = -H(x_{1,\tau}, x_{2,\tau}, p_{1,\tau}, p_{2,\tau}, \tau)$, where H is the Hamiltonian in (5). The dynamic of the system is then generated by the Hamiltonian H and the non-vanishing Dirac brackets $\{x_{1,\tau},\ p_{1,\tau}\}_{DB} = 1$; $\{x_{2,\tau},\ p_{2,\tau}\}_{DB} = 1$.

The canonical quantization procedure as described in Section (1) for an unconstrained system is to promote the phase space variable $x_1,\ x_2,\ p_1,\ p_2$ to operators $\hat{x}_1,\ \hat{x}_2,\ \hat{p}_1,\ \hat{p}_2$ that act on elements of a Hilbert space, which we denote $|\psi\rangle$. The commutator between phase space variables

$$[\hat{f}, \hat{g}] = i\hbar\{f, g\}_{DB}\ , \tag{45}$$

and the quantum level Hamiltonian is given by

$$\hat{H} = \frac{f(t)}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + f^{-1}(t)\frac{m\omega^2(t)}{2}(\hat{x}_1^2 + \hat{x}_2^2), \tag{46}$$

with

$$[\hat{x}_1, \hat{p}_1] = i\hbar, \quad [\hat{x}_2, \hat{p}_2] = i\hbar, \quad [\hat{x}_1, \hat{p}_2] = 0, \quad [\hat{x}_2, \hat{p}_1] = 0. \tag{47}$$

3. Concluding Remarks and Perspectives

As we already mentioned in the Introduction, our aim in this paper was to illuminate Dirac's quantization procedure for the model in the extended phase space. We focused on the problem of Dirac's canonical quantization of a two-dimensional time-dependent harmonic oscillator. As a result, we showed that after performing all necessary steps, the quantum system is simply modeled by the original quantum Hamiltonian.

The system can be studied as in [8] by means of the Levis–Riesenfeld procedure of finding invariant Hermitian operators. The invariant operator in [8] is given by

$$\hat{I}(t) = \frac{1}{2}\left[(mf^{-1}\dot{\rho}\hat{x}_1 - \rho\hat{p}_1)^2 + \frac{\nu^2}{\rho^2}\hat{x}_1^2 + (mf^{-1}\dot{\rho}\hat{x}_2 - \rho\hat{p}_2)^2 + \frac{\nu^2}{\rho^2}\hat{x}_2^2\right], \tag{48}$$

where the function ρ is the solution of the so-called Ermakov–Pinney Equation [23]:

$$\ddot{\rho} + \eta\dot{\rho} + \omega^2\rho = \frac{\nu^2 f^2}{m^2\rho^3}\ . \tag{49}$$

An alternative way of finding invariants of the system described in Equations (1) and (5) is to study the canonical transformation in the extended phase space such that the final dynamical description of the reduced phase space is no longer time-dependent. This method is discussed in [13]. The canonical transformation is a generalization of the Struckmeier transformation [24]. For the present case, we consider a coordinate transformation of the form

$$\begin{bmatrix} x_{1,\tau} \\ x_{2,\tau} \\ t_\tau \\ p_{1,\tau} \\ p_{2,\tau} \\ p_\tau \end{bmatrix} = \begin{bmatrix} A_1(Q_1, T) \\ A_2(Q_2, T) \\ B(T) \\ C_1(Q_1, T)P_1 + D_1(Q_1, T) \\ C_2(Q_2, T)P_2 + D_2(Q_2, T) \\ F(Q_1, Q_2, T, P_1, P_2, P_T) \end{bmatrix}. \tag{50}$$

The canonical transformation matrix resulting from (50) is given by

$$\mathcal{M} = \begin{bmatrix} \frac{\partial x_{1,\tau}}{\partial Q_1} & \frac{\partial x_{1,\tau}}{\partial Q_2} & \frac{\partial x_{1,\tau}}{\partial T} & \frac{\partial x_{1,\tau}}{\partial P_1} & \frac{\partial x_{1,\tau}}{\partial P_2} & \frac{\partial x_{1,\tau}}{\partial P_T} \\ \frac{\partial x_{2,\tau}}{\partial Q_1} & \frac{\partial x_{2,\tau}}{\partial Q_2} & \frac{\partial x_{2,\tau}}{\partial T} & \frac{\partial x_{2,\tau}}{\partial P_1} & \frac{\partial x_{2,\tau}}{\partial P_2} & \frac{\partial x_{2,\tau}}{\partial P_T} \\ \frac{\partial t_{\tau}}{\partial Q_1} & \frac{\partial t_{\tau}}{\partial Q_2} & \frac{\partial t_{\tau}}{\partial T} & \frac{\partial t_{\tau}}{\partial P_1} & \frac{\partial t_{\tau}}{\partial P_2} & \frac{\partial t_{\tau}}{\partial P_T} \\ \frac{\partial p_{1,\tau}}{\partial Q_1} & \frac{\partial p_{1,\tau}}{\partial Q_2} & \frac{\partial p_{1,\tau}}{\partial T} & \frac{\partial p_{1,\tau}}{\partial P_1} & \frac{\partial p_{1,\tau}}{\partial P_2} & \frac{\partial p_{1,\tau}}{\partial P_T} \\ \frac{\partial p_{2,\tau}}{\partial Q_1} & \frac{\partial p_{2,\tau}}{\partial Q_2} & \frac{\partial p_{2,\tau}}{\partial T} & \frac{\partial p_{2,\tau}}{\partial P_1} & \frac{\partial p_{2,\tau}}{\partial P_2} & \frac{\partial p_{2,\tau}}{\partial P_T} \\ \frac{\partial F}{\partial Q_1} & \frac{\partial F}{\partial Q_2} & \frac{\partial F}{\partial T} & \frac{\partial F}{\partial P_1} & \frac{\partial F}{\partial P_2} & \frac{\partial F}{\partial P_T} \end{bmatrix}, \tag{51}$$

which is equivalent to

$$\mathcal{M} = \begin{bmatrix} A_1' & 0 & \dot{A}_1 & 0 & 0 & 0 \\ 0 & A_2' & \dot{A}_2 & 0 & 0 & 0 \\ 0 & 0 & \dot{B} & 0 & 0 & 0 \\ C_1'P_1 + D_1' & 0 & \dot{C}_1P_1 + \dot{D}_1 & C_1 & 0 & 0 \\ 0 & C_2'P_2 + D_2' & \dot{C}_2P_2 + \dot{D}_2 & 0 & C_2 & 0 \\ \frac{\partial F}{\partial Q_1} & \frac{\partial F}{\partial Q_2} & \frac{\partial F}{\partial T} & \frac{\partial F}{\partial P_1} & \frac{\partial F}{\partial P_2} & \frac{\partial F}{\partial P_T} \end{bmatrix}, \tag{52}$$

where $A_i' = \frac{\partial A_i}{\partial Q_i}, \quad C_i' = \frac{\partial C_i}{\partial Q_i}, \quad D_i' = \frac{\partial D_i}{\partial Q_i}, \quad i = 1, 2$, and the *dot* notation is used for the derivative with respect to T. We would like to solve $\mathcal{M}^T J \mathcal{M} = J$, where J is the matrix

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{53}$$

Solving $\mathcal{M}^T J \mathcal{M} = J$ leads to a system of differential equations

$$\dot{A}_1(C_1'P_1 + D_1') + \dot{B}\frac{\partial F}{\partial Q_1} = (\dot{C}_1P_1 + \dot{D}_1)A_1', \tag{54}$$

$$\dot{A}_2(C_2'P_2 + D_2') + \dot{B}\frac{\partial F}{\partial Q_2} = (\dot{C}_2P_2 + \dot{D}_2)A_2', \tag{55}$$

$$C_1\dot{A}_1 + \dot{B}\frac{\partial F}{\partial P_1} = 0, \tag{56}$$

$$C_2\dot{A}_2 + \dot{B}\frac{\partial F}{\partial P_2} = 0, \tag{57}$$

$$\dot{B}\frac{\partial F}{\partial P_T} = 1, \tag{58}$$

$$C_1 A_1' = 1, \tag{59}$$

$$C_2 A_2' = 1\,, \tag{60}$$

whose general solution is given by

$$C_1 = \frac{1}{A_1'}, \quad C_2 = \frac{1}{A_2'}, \quad t_\tau = B(T), \tag{61}$$

$$F = \frac{P_T}{\dot{B}} - \frac{\dot{A}_1}{A_1'\dot{B}}P_1 - \frac{\dot{A}_2}{A_2'\dot{B}}P_2 + \frac{1}{\dot{B}}\left[\int \left(\dot{D}_1 A_1' - \dot{A}_1 D_1'\right) dQ_1 + \int \left(\dot{D}_1 A_2' - \dot{A}_2 D_2'\right) dQ_2\right], \tag{62}$$

where the functions $A_i(Q_i, T)$, $B(T)$, $D_i(Q_i, T)$, $i = 1, 2$ are arbitrary. We can now write $x_{1,\tau}$, $x_{2,\tau}$, t_τ, $p_{1,\tau}$, $p_{2,\tau}$, p_τ in terms of the new coordinates as

$$\begin{bmatrix} x_{1,\tau} \\ x_{2,\tau} \\ t_\tau \\ p_{1,\tau} \\ p_{2,\tau} \\ p_\tau \end{bmatrix} = \begin{bmatrix} A_1(Q,T) \\ A_2(Q,T) \\ B(T) \\ \frac{1}{A_1'}P_1 + D_1 \\ \frac{1}{A_2'}P_2 + D_2 \\ \frac{P_T}{\dot{B}} - \frac{\dot{A}_1}{A_1'\dot{B}}P_1 - \frac{\dot{A}_2}{A_2'\dot{B}}P_2 + \frac{1}{\dot{B}}\left[\int \left(\dot{D}_1 A_1' - \dot{A}_1 D_1'\right) dQ_1 + \int \left(\dot{D}_1 A_2' - \dot{A}_2 D_2'\right) dQ_2\right] \end{bmatrix}. \tag{63}$$

The new variables are time-independent, since the time variable is τ. A new Hamiltonian of the system can be derived in terms of these new variables that is also an invariant of the system since it is autonomous.

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