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System of Extended General Variational Inequalities for Relaxed Cocoercive Mappings in Hilbert Space

Kyung Soo Kim

Graduate School of Education, Kyungnam University, Changwon 51767, Gyeongnam, Korea; kksmj@kyungnam.ac.kr; Tel.: +82-55-249-2842

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Abstract: In this manuscript, we study a system of extended general variational inequalities (SEGVI) with several nonlinear operators, more precisely, six relaxed (α, r)-cocoercive mappings. Using the projection method, we show that a system of extended general variational inequalities is equivalent to the nonlinear projection equations. This alternative equivalent problem is used to consider the existence and convergence (or approximate solvability) of a solution of a system of extended general variational inequalities under suitable conditions.

Keywords: a system of extended general variational inequality (SEGVI); auxiliary system of extended general variational inequality; relaxed (α, r)-cocoercive mapping; projection method; solution; fixed point

1. Introduction

In recent years, many theories of variational inequality types and its special forms have been extended and generalized to research a variety of applications and problems arising from several fields such as applied mathematics, optimization, control theory, equilibrium problems and nonlinear programming problems, etc. In 1964, a variational inequality problem (VIP) was introduced by Stampacchia [1].

In 2016, Noor [2] introduced and researched the existence of solution by using fixed point theory for a system of extended general variational inequalities with six strongly monotone operators.

From the above results, we intend in this manuscript to consider a system of extended general variational inequalities with nonlinear operators, more precisely, relaxed cocoercive operators which are more generalized than strongly monotone operators. We show that a system of extended general variational inequalities include general variational inequality and several other classes of variational inequalities as special cases. Using the projection method, it is shown that a system of extended general variational inequalities (SEGVI) are equivalent to the nonlinear projection equations. This alternative equivalent problem is used to consider the existence and convergence of a solution of a system of extended general variational inequalities under appropriate conditions.

2. Preliminaries

Hereafter, we take that H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let Ω_1, Ω_2 be two closed convex subsets in H.

For given nonlinear operators $T_1, T_2, g_1, g_2, h_1, h_2$: $H \to H$, consider a problem of finding $x, y \in H$ with $h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ such that

$$\begin{cases} \langle T_1 x, g_1(v) - h_1(y)\rangle \geq 0, \quad \forall v \in H, \quad g_1(v) \in \Omega_1, \\ \langle T_2 y, g_2(v) - h_2(x)\rangle \geq 0, \quad \forall v \in H, \quad g_2(v) \in \Omega_2. \end{cases} \tag{1}$$

The problem (1) is said to be a system of extended general variational inequalities (SEGVI) with six nonlinear operators.

We consider some special cases of the (SEGVI) (1).

I. If $g_1 = g_2 = g$, $h_1 = h_2 = h$ and $\Omega_1 = \Omega_2 = \Omega$, a closed convex subset in H, then problem (1) reduces to find $x, y \in H$ with $h(y), h(x) \in \Omega$ such that

$$\begin{cases} \langle T_1 x, g(\nu) - h(y) \rangle \geq 0, \\ \langle T_2 y, g(\nu) - h(x) \rangle \geq 0, \end{cases} \tag{2}$$

for all $\nu \in H$, $g(\nu) \in \Omega$. The system (2) is said to be a system of extended general variational inequalities with four nonlinear operators.

II. If $g_1 = h_1 = g$, $g_2 = h_2 = h$ and $\Omega_1 = \Omega_2 = \Omega$, a closed convex subset in H, then problem (1) reduces to find $x, y \in H$ with $g(y), h(x) \in \Omega$ such that

$$\begin{cases} \langle T_1 x, g(\nu) - g(y) \rangle \geq 0, \\ \langle T_2 y, h(\nu) - h(x) \rangle \geq 0, \end{cases} \tag{3}$$

for all $\nu \in H$, $g(\nu) \in \Omega$ and $h(\nu) \in \Omega$. The system (3) is said to be a system of general variational inequalities with four nonlinear operators.

III. If $T_1 = T_2 = T$, then problem (2) reduces to find $u \in H$ with $h(u) \in \Omega$ such that

$$\langle Tu, g(\nu) - h(u) \rangle \geq 0, \tag{4}$$

for all $\nu \in H$, $g(\nu) \in \Omega$. The problem of type (4) is said to be an extended general variational inequality (EGVI), which was studied by Noor [3].

For adequate and suitable conditions of spaces and operators, we can obtain several new and known classes of variational inequalities. Recent applications, iteration methods, existence problem and convergence theory are related to the above problems (see [4–14] and other references therein).

Now, we digest some definitions and related basic properties which are indispensable in the following discussions.

Lemma 1. ([15]) *Let Ω be a closed and convex subset in H. Then, for a given $h \in H$, $\omega \in \Omega$ satisfies*

$$\langle \omega - h, v - h \rangle \geq 0, \quad \forall v \in \Omega, \tag{5}$$

if and only if

$$\omega = P_\Omega(h),$$

where P_Ω is the projection of H onto Ω in H.

Remark 1. *It is very well known that the projection operator P_Ω is nonexpansive, i.e.,*

$$\|P_\Omega(s) - P_\Omega(t)\| \leq \|s - t\|, \quad \forall s, t \in H. \tag{6}$$

More information on the projection operator P_Ω can be found in Section 3 of [16].

Definition 1. ([17]) Let H be a Hilbert space.

(1) An operator $T{:}H \rightarrow H$ is said to be *α-strongly monotone*, if for each $s, t \in H$, we have

$$\langle T(s) - T(t), s - t \rangle \geq \alpha \|s - t\|^2,$$

for a constant $r > 0$. This implies that

$$\|T(s) - T(t)\| \geq \alpha \|s - t\|,$$

that is, T is α-expansive and when $\alpha = 1$, it is expansive.

(2) An operator $T{:}H \rightarrow H$ is said to be *β-Lipschitz continuous*, if there exists a constant $\beta \geq 0$ such that

$$\|T(s) - T(t)\| \leq \beta \|s - t\|, \quad \forall s, t \in H.$$

(3) An operator $T{:}H \rightarrow H$ is said to be *μ-cocoercive*, if there exists a constant $\mu > 0$ such that

$$\langle T(s) - T(t), s - t \rangle \geq \mu \|T(s) - T(t)\|^2, \quad \forall s, t \in H.$$

Clearly, every μ-cocoercive operator T is $\frac{1}{\mu}$-Lipschitz continuous.

(4) An operator $T{:}H \rightarrow H$ is said to be *relaxed α-cocoercive*, if there exists a constant $\alpha > 0$ such that

$$\langle T(s) - T(t), s - t \rangle \geq (-\alpha) \|T(s) - T(t)\|^2, \quad \forall s, t \in H.$$

(5) An operator $T{:}H \rightarrow H$ is said to be *relaxed (α, r)-cocoercive*, if there exists a constant $\alpha, r > 0$ such that

$$\langle T(s) - T(t), s - t \rangle \geq (-\alpha) \|T(s) - T(t)\|^2 + r \|s - t\|^2, \quad \forall s, t \in H.$$

For $\alpha = 0$, T is r-strongly monotone. This class of operators is more generalized than the class of strongly monotone operators. One can easily show that the following implication:

$$r - \text{strongly monotonicity} \quad \Rightarrow \quad \text{relaxed} \;\; (\alpha, r) - \text{cocoercivity}.$$

Lemma 2. ([18]) *Let $\{s_n\}$ and $\{t_n\}$ be two nonnegative real sequences satisfying the following condition:*

$$s_{n+1} \leq (1 - \lambda_n) s_n + t_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer and $\lambda_n \in [0, 1]$ is a sequence with $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $t_n = o(\lambda_n)$. Then,

$$\lim_{n \to \infty} s_n = 0.$$

From the auxiliary principle method of Glowinski et al. [19], it is easy to show that we have the system (1) equivalent to the following:

Find $x, y \in H$ with $h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ and

$$\begin{cases} \langle \rho_1 T_1 x + h_1(y) - g_1(x), g_1(\nu) - h_1(y) \rangle \geq 0, \\ \langle \rho_2 T_2 y + h_2(x) - g_2(y), g_2(\nu) - h_2(x) \rangle \geq 0, \end{cases} \tag{7}$$

where, for all $\nu \in H$, $g_1(\nu) \in \Omega_1$, $g_2(\nu) \in \Omega_2$, $\rho_1 > 0$ and $\rho_2 > 0$ (see, [3,20]).

We use this equivalent problem to generate some iteration techniques for solving the system of extended general variational inequalities and its other variant kinds.

3. Results

In this section, we study about a system of extended general variational inequalities (SEGVI) (7) being equivalent to a system of fixed point problems. This alternative equivalent problem is used to generate iteration schemes for solving problem (7), by the method of Noor et al. [21].

Lemma 3. ([2]) *The system of extended general variational inequalities* (7) *has a solution,* $x, y \in H$ *with* $h_1(y) \in \Omega_1 \subset g_1(H), h_1(H)$ *and* $h_2(x) \in \Omega_2 \subset g_2(H), h_2(H)$ *if and only if* $x, y \in H$ *with* $h_1(y) \in \Omega_1$, $h_2(x) \in \Omega_2$ *satisfies the relations*

$$h_1(y) = P_{\Omega_1}[g_1(x) - \rho_1 T_1 x], \tag{8}$$

$$h_2(x) = P_{\Omega_2}[g_2(y) - \rho_2 T_2 y], \tag{9}$$

where $\rho_1 > 0$ *and* $\rho_2 > 0$.

Lemma 3 implies that problem (7) is equivalent to the relations of fixed point problems (8) and (9). Using the fixed point problems (8) and (9), we can suggest and analyze some iteration forms:

$$y = (1-\beta_n)y + \beta_n\{y - h_1(y) + P_{\Omega_1}[g_1(x) - \rho_1 T_1 x]\}, \tag{10}$$

$$x = (1-\alpha_n)x + \alpha_n\{x - h_2(x) + P_{\Omega_2}[g_2(y) - \rho_2 T_2 y]\}, \tag{11}$$

where $0 \le \alpha_n, \beta_n \le 1$, $n \ge 0$.

This alternative problem is used to propose the following iteration schemes for solving a system of extended general variational inequalities (SEGVI) (7) and its variant kinds.

Algorithm 1. *For given* $x_0, y_0 \in H$*:* $h_1(y_0) \in \Omega_1$ *and* $h_2(x_0) \in \Omega_2$*, find* x_{n+1} *and* y_{n+1} *by the iterative schemes*

$$\begin{aligned} y_{n+1} &= (1-\beta_n)y_n + \beta_n\{y_n - h_1(y_n) + P_{\Omega_1}[g_1(x_n) - \rho_1 T_1 x_n]\}, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\{x_n - h_2(x_n) + P_{\Omega_2}[g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}]\}, \end{aligned} \tag{12}$$

where $0 \le \alpha_n, \beta_n \le 1$, $n \ge 0$.

Algorithm 1 can be viewed as a Gauss–Seidel method for solving system of extended general variational inequalities (SEGVI) (7).

For adequate and suitable conditions of spaces and operators, we can obtain several new and known iteration schemes for solving system of extended general variational inequalities (SEGVI) and related problems. It has been shown [22] that problem (1) has a solution under some suitable conditions.

Now, we investigate the convergence analysis of Algorithm 1. This is the core of our following result.

Theorem 1. *Let* $T_1, T_2, g_1, g_2, h_1, h_2 : H \to H$ *be relaxed* (α_{T_1}, k_{T_1}), (α_{T_2}, k_{T_2}), (α_{g_1}, k_{g_1}), (α_{g_2}, k_{g_2}), (α_{h_1}, k_{h_1}), (α_{h_2}, k_{h_2})*-cocoercive and* $l_{T_1}, l_{T_2}, l_{g_1}, l_{g_2}, l_{h_1}, l_{h_2}$*-Lipschitz continuous operators, respectively. If the following conditions hold:*

(i) $0 < \theta_1 = \sqrt{1 - 2\rho_1 k_{T_1} + (2\alpha_{T_1} + \rho_1)\rho_1 l_{T_1}^2} < 1$,
$0 < \theta_2 = \sqrt{1 - 2\rho_2 k_{T_2} + (2\alpha_{T_2} + \rho_2)\rho_2 l_{T_2}^2} < 1$,

(ii) $2k_{h_1} - (2\alpha_{h_1} + 1)l_{h_1}^2 < 1$, $2k_{h_2} - (2\alpha_{h_2} + 1)l_{h_2}^2 < 1$,
$2k_{g_1} - (2\alpha_{g_1} + 1)l_{g_1}^2 < 1$, $2k_{g_2} - (2\alpha_{g_2} + 1)l_{g_2}^2 < 1$,

(iii) $\alpha_n, \beta_n \in [0,1]$ *for all* $n \ge 0$, $1 - \nu = \alpha_n(\delta_2 + \theta_2) \ge 0$,
$1 - \varepsilon_1 = \alpha_n(1 - \mu_2) - \beta_n(\delta_1 + \theta_1) \ge 0$, $1 - \varepsilon_2 = \beta_n(1 - \mu_1) \ge 0$,
such that

$$\sum_{n=0}^{\infty} \alpha_n(\delta_2 + \theta_2) = \infty,$$

$$\sum_{n=0}^{\infty} \alpha_n(1-\mu_2) - \beta_n(\delta_1+\theta_1) = \infty,$$

$$\sum_{n=0}^{\infty} \beta_n(1-\mu_1) = \infty,$$

where

$$\delta_1 = \sqrt{1-2k_{g_1}+(2\alpha_{g_1}+1)l_{g_1}^2}, \quad \delta_2 = \sqrt{1-2k_{g_2}+(2\alpha_{g_2}+1)l_{g_2}^2},$$
$$\mu_1 = \sqrt{1-2k_{h_1}+(2\alpha_{h_1}+1)l_{h_1}^2}, \quad \mu_2 = \sqrt{1-2k_{h_2}+(2\alpha_{h_2}+1)l_{h_2}^2},$$

then sequences $\{x_n\}$ *and* $\{y_n\}$ *obtained from Algorithm 1 converge to* x *and* y*, respectively.*

Proof. Let $x, y \in H$ with $h_1(y) \in \Omega_1, h_2(y) \in \Omega_2$ be a solution of (7). Then, from (11) and (12), we have

$$\begin{aligned}
\|x_{n+1}-x\| &= \|(1-\alpha_n)x_n + \alpha_n\{x_n - h_2(x_n) + P_{\Omega_2}[g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}]\} \\
&\quad - (1-\alpha_n)x - \alpha_n\{x - h_2(x) + P_{\Omega_2}[g_2(y) - \rho_2 T_2 y]\}\| \\
&\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\|x_n - x - (h_2(x_n) - h_2(x))\| \\
&\quad + \alpha_n\|P_{\Omega_2}[g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}] - P_{\Omega_2}[g_2(y) - \rho_2 T_2 y]\| \\
&\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\|x_n - x - (h_2(x_n) - h_2(x))\| \\
&\quad + \alpha_n\|g_2(y_{n+1}) - \rho_2 T_2 y_{n+1} - g_2(y) + \rho_2 T_2 y\| \\
&\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\|x_n - x - (h_2(x_n) - h_2(x))\| \\
&\quad + \alpha_n\| - (y_{n+1} - y) + g_2(y_{n+1}) - g_2(y)\| \\
&\quad + \alpha_n\|y_{n+1} - y - \rho_2(T_2 y_{n+1} - T_2 y)\|.
\end{aligned} \tag{13}$$

Since operator T_2 is relaxed (α_{T_2}, k_{T_2})-cocoercive with constant $\alpha_{T_2} > 0, k_{T_2} > 0$ and l_{T_2}-Lipschitz continuous, then it follows that

$$\begin{aligned}
&\|y_{n+1} - y - \rho_2(T_2 y_{n+1} - T_2 y)\|^2 \\
&= \|y_{n+1} - y\|^2 - 2\rho_2\langle T_2 y_{n+1} - T_2 y, y_{n+1} - y\rangle + \rho_2^2\|T_2 y_{n+1} - T_2 y\|^2 \\
&\leq \|y_{n+1} - y\|^2 + 2\rho_2\alpha_{T_2}\|T_2 y_{n+1} - T_2 y\|^2 - 2\rho_2 k_{T_2}\|y_{n+1} - y\|^2 + \rho_2^2\|T_2 y_{n+1} - T_2 y\|^2 \\
&\leq \|y_{n+1} - y\|^2 + 2\rho_2\alpha_{T_2} l_{T_2}^2\|y_{n+1} - y\|^2 - 2\rho_2 k_{T_2}\|y_{n+1} - y\|^2 + \rho_2^2 l_{T_2}^2\|y_{n+1} - y\|^2 \\
&= (1 - 2\rho_2 k_{T_2} + (2\alpha_{T_2} + \rho_2)\rho_2\, l_{T_2}^2)\|y_{n+1} - y\|^2.
\end{aligned} \tag{14}$$

In a similar way, we have

$$\begin{aligned}
&\|x_n - x - (h_2(x_n) - h_2(x))\|^2 \\
&= \|x_n - x\|^2 - 2\langle x_n - x, h_2(x_n) - h_2(x)\rangle + \|h_2(x_n) - h_2(x)\|^2 \\
&\leq \|x_n - x\|^2 + 2\alpha_{h_2}\|h_2(x_n) - h_2(x)\|^2 - 2k_{h_2}\|x_n - x\|^2 + \|h_2(x_n) - h_2(x)\|^2 \\
&\leq \|x_n - x\|^2 + 2\alpha_{h_2} l_{h_2}^2\|x_n - x\|^2 - 2k_{h_2}\|x_n - x\|^2 + l_{h_2}^2\|x_n - x\|^2 \\
&= (1 - 2k_{h_2} + (2\alpha_{h_2} + 1)l_{h_2}^2)\|x_n - x\|^2
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\|y_{n+1} - y - (g_2(y_{n+1}) - g_2(y))\|^2 \\
&= \|y_{n+1} - y\|^2 - 2\langle y_{n+1} - y, g_2(y_{n+1}) - g_2(y)\rangle + \|g_2(y_{n+1}) - g_2(y)\|^2 \\
&\leq \|y_{n+1} - y\|^2 + 2\alpha_{g_2}\|g_2(y_{n+1}) - g_2(y)\|^2 - 2k_{g_2}\|y_{n+1} - y\|^2 + \|g_2(y_{n+1}) - g_2(y)\|^2 \\
&= (1 - 2k_{g_2} + (2\alpha_{g_2} + 1)l_{g_2}^2)\|y_{n+1} - y\|^2,
\end{aligned} \tag{16}$$

where we have used the property of operators h_2, g_2, respectively. Combining (13)–(16), we obtain

$$\begin{aligned}
&\|x_{n+1}-x\| \\
&\le (1-\alpha_n)\|x_n-x\| + \alpha_n\sqrt{1-2k_{h_2}+(2\alpha_{h_2}+1)l_{h_2}^2}\,\|x_n-x\| \\
&\quad + \alpha_n\sqrt{1-2k_{g_2}+(2\alpha_{g_2}+1)l_{g_2}^2}\,\|y_{n+1}-y\| \\
&\quad + \alpha_n\sqrt{1-2\rho_2 k_{T_2}+(2\alpha_{T_2}+\rho_2)\rho_2\, l_{T_2}^2}\,\|y_{n+1}-y\| \\
&= \left(1-\alpha_n\left(1-\sqrt{1-2k_{h_2}+(2\alpha_{h_2}+1)l_{h_2}^2}\,\right)\right)\|x_n-x\| \\
&\quad + \alpha_n\left(\sqrt{1-2k_{g_2}+(2\alpha_{g_2}+1)l_{g_2}^2}+\sqrt{1-2\rho_2 k_{T_2}+(2\alpha_{T_2}+\rho_2)\rho_2\, l_{T_2}^2}\right)\|y_{n+1}-y\|.
\end{aligned} \tag{17}$$

From (10) and (12), we have

$$\begin{aligned}
\|y_{n+1}-y\| &= \|(1-\beta_n)y_n+\beta_n\{y_n-h_1(y_n)+P_{\Omega_1}(g_1(x_n)-\rho_1 T_1 x_n)\} \\
&\quad -(1-\beta_n)y-\beta_n\{y-h_1(y)+P_{\Omega_1}(g_1(x)-\rho_1 T_1 x)\}\| \\
&\le (1-\beta_n)\|y_n-y\|+\beta_n\|y_n-y-(h_1(y_n)-h_1(y))\| \\
&\quad +\beta_n\|P_{\Omega_1}(g_1(x_n)-\rho_1 T_1 x_n)-P_{\Omega_1}(g_1(x)-\rho_1 T_1 x)\| \\
&\le (1-\beta_n)\|y_n-y\|+\beta_n\|y_n-y-(h_1(y_n)-h_1(y))\| \\
&\quad +\beta_n\|g_1(x_n)-\rho_1 T_1 x_n-g_1(x)+\rho_1 T_1 x\| \\
&\le (1-\beta_n)\|y_n-y\|+\beta_n\|y_n-y-(h_1(y_n)-h_1(y))\| \\
&\quad +\beta_n\|x_n-x-(g_1(x_n)-g_1(x))\| \\
&\quad +\beta_n\|x_n-x-\rho_1(T_1 x_n-T_1 x)\|.
\end{aligned} \tag{18}$$

In a similar way, from the property of operators h_1, g_1, we get

$$\|y_n-y-(h_1(y_n)-h_1(y))\|^2 \le (1-2k_{h_1}+(2\alpha_{h_1}+1)l_{h_1}^2)\|y_n-y\|^2, \tag{19}$$

$$\|x_n-x-(g_1(x_n)-g_1(x))\|^2 \le (1-2k_{g_1}+(2\alpha_{g_1}+1)l_{g_1}^2)\|x_n-x\|^2, \tag{20}$$

$$\|x_n-x-\rho_1(T_1 x_n-T_1 x)\|^2 \le (1-2\rho_1 k_{T_1}+(2\alpha_{T_1}+\rho_1)\rho_1\, l_{T_1}^2)\|x_n-x\|^2. \tag{21}$$

Combining (18)–(21), we have

$$\begin{aligned}
&\|y_{n+1}-y\| \\
&\le (1-\beta_n)\|y_n-y\|+\beta_n\sqrt{1-2k_{h_1}+(2\alpha_{h_1}+1)l_{h_1}^2}\,\|y_n-y\| \\
&\quad +\beta_n\sqrt{1-2k_{g_1}+(2\alpha_{g_1}+1)l_{g_1}^2}\,\|x_n-x\| \\
&\quad +\beta_n\sqrt{1-2\rho_1 k_{T_1}+(2\alpha_{T_1}+\rho_1)\rho_1\, l_{T_1}^2}\,\|x_n-x\| \\
&= \left(1-\beta_n\left(1-\sqrt{1-2k_{h_1}+(2\alpha_{h_1}+1)l_{h_1}^2}\,\right)\right)\|y_n-y\| \\
&\quad +\beta_n\left(\sqrt{1-2k_{g_1}+(2\alpha_{g_1}+1)l_{g_1}^2}+\sqrt{1-2\rho_1 k_{T_1}+(2\alpha_{T_1}+\rho_1)\rho_1\, l_{T_1}^2}\right)\|x_n-x\|.
\end{aligned} \tag{22}$$

From (17) and (22), put

$$\sqrt{1-2k_{h_1}+(2\alpha_{h_1}+1)l_{h_1}^2}=\mu_1,\ \ \sqrt{1-2k_{h_2}+(2\alpha_{h_2}+1)l_{h_2}^2}=\mu_2,$$
$$\sqrt{1-2k_{g_1}+(2\alpha_{g_1}+1)l_{g_1}^2}=\delta_1,\ \ \sqrt{1-2k_{g_2}+(2\alpha_{g_2}+1)l_{g_2}^2}=\delta_2,$$
$$\sqrt{1-2\rho_1 k_{T_1}+(2\alpha_{T_1}+\rho_1)\rho_1\, l_{T_1}^2}=\theta_1,\ \ \sqrt{1-2\rho_2 k_{T_2}+(2\alpha_{T_2}+\rho_2)\rho_2\, l_{T_2}^2}=\theta_2,$$

we obtain

$$\begin{aligned}\|x_{n+1}-x\|+\|y_{n+1}-y\| &\le (1-\alpha_n(1-\mu_2))\|x_n-x\|+\alpha_n(\delta_2+\theta_2)\|y_{n+1}-y\|\\ &\quad+(1-\beta_n(1-\mu_1))\|y_n-y\|+\beta_n(\delta_1+\theta_1)\|x_n-x\|\\ &\le (1-\alpha_n(1-\mu_2)+\beta_n(\delta_1+\theta_1))\|x_n-x\|\\ &\quad+\alpha_n(\delta_2+\theta_2)\|y_{n+1}-y\|+(1-\beta_n(1-\mu_1))\|y_n-y\|.\end{aligned}$$

Thus,

$$\begin{aligned}&\|x_{n+1}-x\|+(1-\alpha_n(\delta_2+\theta_2))\|y_{n+1}-y\|\\ &\le (1-\alpha_n(1-\mu_2)+\beta_n(\delta_1+\theta_1))\|x_n-x\|+(1-\beta_n(1-\mu_1))\|y_n-y\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x\|+\nu\|y_{n+1}-y\| &\le \max\{\varepsilon_1,\varepsilon_2\}\cdot(\|x_n-x\|+\|y_n-y\|)\\ &=\varepsilon(\|x_n-x\|+\|y_n-y\|),\end{aligned} \tag{23}$$

where

$$\begin{aligned}&\nu=1-\alpha_n(\delta_2+\theta_2),\ \ \varepsilon_1=1-\alpha_n(1-\mu_2)+\beta_n(\delta_1+\theta_1),\\ &\varepsilon_2=1-\beta_n(1-\mu_1),\ \ \varepsilon=\max\{\varepsilon_1,\varepsilon_2\}.\end{aligned}$$

From conditions, we obtain

$$\varepsilon<1.$$

By Lemma 2, it follows from (23) that

$$\lim_{n\to\infty}(\|x_{n+1}-x\|+\nu\|y_{n+1}-y\|)=0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x\|=0=\lim_{n\to\infty}\|y_{n+1}-y\|.$$

This completes the proof. □

Corollary 1. **([2], Theorem 4)** Let $T_1, T_2, g_1, g_2, h_1, h_2 : H \to H$ be strongly monotone with constants $k_{T_1}>0$, $k_{T_2}>0$, $k_{g_1}>0$, $k_{g_2}>0$, $k_{h_1}>0$, $k_{h_2}>0$ and l_{T_1}, l_{T_2}, l_{g_1}, l_{g_2}, l_{h_1}, l_{h_2}-Lipschitz continuous operators, respectively. If the following conditions hold:

(i) $0<\theta_1=\sqrt{1-2\rho_1 k_{T_1}+\rho_1^2\, l_{T_1}^2}<1,\ \ 0<\theta_2=\sqrt{1-2\rho_2 k_{T_2}+\rho_2^2\, l_{T_2}^2}<1,$

(ii) $2k_{h_1}-l_{h_1}^2<1,\ \ 2k_{h_2}-l_{h_2}^2<1,\ \ 2k_{g_1}-l_{g_1}^2<1,\ \ 2k_{g_2}-l_{g_2}^2<1,$

(iii) $\alpha_n, \beta_n \in [0,1]$ for all $n \geq 0$, $1-\nu = \alpha_n(\delta_2+\theta_2) \geq 0$,
$1-\varepsilon_1 = \alpha_n(1-\mu_2) - \beta_n(\delta_1+\theta_1) \geq 0$, $1-\varepsilon_2 = \beta_n(1-\mu_1) \geq 0$,
such that

$$\sum_{n=0}^{\infty} \alpha_n(\delta_2+\theta_2) = \infty,$$

$$\sum_{n=0}^{\infty} \alpha_n(1-\mu_2) - \beta_n(\delta_1+\theta_1) = \infty,$$

$$\sum_{n=0}^{\infty} \beta_n(1-\mu_1) = \infty,$$

where

$$\delta_1 = \sqrt{1-2k_{g_1}+l_{g_1}^2}, \ \ \delta_2 = \sqrt{1-2k_{g_2}+l_{g_2}^2},$$
$$\mu_1 = \sqrt{1-2k_{h_1}+l_{h_1}^2}, \ \ \mu_2 = \sqrt{1-2k_{h_2}+l_{h_2}^2},$$

then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 1 converge to x and y, respectively.

Proof. In Theorem 1, from Definition 1, we take $\alpha_{T_1} = \alpha_{T_2} = \alpha_{g_1} = \alpha_{g_2} = \alpha_{h_1} = \alpha_{h_2} = 0$, we get the result of Corollary 1. □

On the other hand, using Lemma 3, one can easily show that $x, y \in H$ with $h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ is a solution of (7) if and only if $x, y \in H$ with $h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ satisfies

$$h_1(y) = P_{\Omega_1}(z), \tag{24}$$
$$h_2(x) = P_{\Omega_2}(w), \tag{25}$$
$$z = g_1(x) - \rho_1 T_1 x, \tag{26}$$
$$w = g_2(y) - \rho_2 T_2 y. \tag{27}$$

This alternative problem can be used to propose and analyze the following iteration scheme for solving system (7).

Algorithm 2. *For given $x_0, y_0 \in H$ with $h_1(y_0) \in \Omega_1, h_2(x_0) \in \Omega_2$, find x_{n+1} and y_{n+1} by the iteration schemes*

$$y_{n+1} = (1-\beta_n)y_n + \beta_n\{y_n - h_1(y_n) + P_{\Omega_1}(z_n)\}, \tag{28}$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{x_n - h_2(x_n) + P_{\Omega_2}(w_n)\}, \tag{29}$$
$$z_n = g_1(x_n) - \rho_1 T_1 x_n, \tag{30}$$
$$w_n = g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}, \tag{31}$$

where $\alpha_n, \beta_n \in [0,1]$ for all $n \geq 0$.

Now, we consider the convergence analysis of Algorithm 2, using the method of Theorem 1. For the sake of completeness and to convey an idea, we include all the details.

Theorem 2. *Let operators $T_1, T_2, g_1, g_2, h_1, h_2 : H \to H$ be relaxed (α_{T_1}, k_{T_1}), (α_{T_2}, k_{T_2}), (α_{g_1}, k_{g_1}), (α_{g_2}, k_{g_2}), (α_{h_1}, k_{h_1}), (α_{h_2}, k_{h_2})-cocoercive and l_{T_1}, l_{T_2}, l_{g_1}, l_{g_2}, l_{h_1}, l_{h_2}-Lipschitz continuous, respectively. If the following conditions hold:*

(i) $0 < \theta_1 = \sqrt{1-2\rho_1 k_{T_1} + (2\alpha_{T_1}+\rho_1)\rho_1 l_{T_1}^2} < 1$,
$0 < \theta_2 = \sqrt{1-2\rho_2 k_{T_2} + (2\alpha_{T_2}+\rho_2)\rho_2 l_{T_2}^2} < 1$,

(ii) $2k_{h_1} - (2\alpha_{h_1} + 1)l_{h_1}^2 < 1, \quad 2k_{h_2} - (2\alpha_{h_2} + 1)l_{h_2}^2 < 1,$
$2k_{g_1} - (2\alpha_{g_1} + 1)l_{g_1}^2 < 1, \quad 2k_{g_2} - (2\alpha_{g_2} + 1)l_{g_2}^2 < 1,$

(iii) $\alpha_n, \beta_n \in [0,1]$ *for all* $n \geq 0, \quad 1 - \nu = \alpha_n(\delta_2 + \theta_2) \geq 0,$
$1 - \varepsilon_1 = \alpha_n(1 - \mu_2) - \beta_n(\delta_1 + \theta_1) \geq 0, \quad 1 - \varepsilon_2 = \beta_n(1 - \mu_1) \geq 0$
such that

$$\sum_{n=0}^{\infty} \alpha_n(\delta_2 + \theta_2) = \infty,$$

$$\sum_{n=0}^{\infty} \alpha_n(1 - \mu_2) - \beta_n(\delta_1 + \theta_1) = \infty,$$

$$\sum_{n=0}^{\infty} \beta_n(1 - \mu_1) = \infty,$$

where

$$\delta_1 = \sqrt{1 - 2k_{g_1} + (2\alpha_{g_1} + 1)l_{g_1}^2}, \quad \delta_2 = \sqrt{1 - 2k_{g_2} + (2\alpha_{g_2} + 1)l_{g_2}^2},$$
$$\mu_1 = \sqrt{1 - 2k_{h_1} + (2\alpha_{h_1} + 1)l_{h_1}^2}, \quad \mu_2 = \sqrt{1 - 2k_{h_2} + (2\alpha_{h_2} + 1)l_{h_2}^2},$$

then sequences $\{x_n\}$ *and* $\{y_n\}$ *which are defined by Algorithm 2 converge to* x *and* y*, respectively.*

Proof. Let $x, y \in H$ with $h_1(y) \in \Omega_1$, $h_2(x) \in \Omega_2$ be a solution of (7). Then, from (6), (14), (16), (27) and (31), we have

$$\begin{aligned}
&\|w_n - w\| \\
&= \|g_2(y_{n+1}) - \rho_2 T_2 y_{n+1} - (g_2(y) - \rho_2 T_2 y)\| \\
&\leq \|y_{n+1} - y - (g_2(y_{n+1}) - g_2(y))\| + \|y_{n+1} - y - \rho_2(T_2 y_{n+1} - T_2 y)\| \\
&\leq \left(\sqrt{1 - 2k_{g_2} + (2\alpha_{g_2} + 1)l_{g_2}^2} + \sqrt{1 - 2\rho_2 k_{T_2} + (2\alpha_{T_2} + \rho_2)\rho_2\, l_{T_2}^2}\right)\|y_{n+1} - y\|.
\end{aligned} \tag{32}$$

Thus, from (11), (15), (27), (29) and (32),

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n\|x_n - x - (h_2(x_n) - h_2(x))\| + \alpha_n\|P_{\Omega_2}(w_n) - P_{\Omega_2}(w)\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n\sqrt{1 - 2k_{h_2} + (2\alpha_{h_2} + 1)l_{h_2}^2}\,\|x_n - x\| \\
&\quad + \alpha_n\left(\sqrt{1 - 2k_{g_2} + (2\alpha_{g_2} + 1)l_{g_2}^2} + \sqrt{1 - 2\rho_2 k_{T_2} + (2\alpha_{T_2} + \rho_2)\rho_2\, l_{T_2}^2}\right)\|y_{n+1} - y\|.
\end{aligned} \tag{33}$$

From (20), (21), (26) and (30),

$$\begin{aligned}
&\|z_n - z\| \\
&= \|g_1(x_n) - g_1(x) - \rho_1(T_1 x_n - T_1 x)\| \\
&\leq \|x_n - x - (g_1(x_n) - g_1(x))\| + \|x_n - x - \rho_1(T_1 x_n - T_1 x)\| \\
&\leq \sqrt{1 - 2k_{g_1} + (2\alpha_{g_1} + 1)l_{g_1}^2}\,\|x_n - x\| + \sqrt{1 - 2\rho_1 k_{T_1} + (2\alpha_{T_1} + \rho_1)\rho_1 l_{T_1}^2}\,\|x_n - x\| \\
&= \left(\sqrt{1 - 2k_{g_1} + (2\alpha_{g_1} + 1)l_{g_1}^2} + \sqrt{1 - 2\rho_1 k_{T_1} + (2\alpha_{T_1} + \rho_1)\rho_1 l_{T_1}^2}\right)\|x_n - x\|.
\end{aligned} \tag{34}$$

Thus, from (10), (19), (28) and (34),

$$\begin{aligned} &\|y_{n+1} - y\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n\|y_n - y - (h_1(y_n) - h_1(y))\| + \beta_n\|P_{\Omega_1}(z_n) - P_{\Omega_1}(z)\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n\sqrt{1 - 2k_{h_1} + (2\alpha_{h_1}+1)l_{h_1}^2}\,\|y_n - y\| \\ &\quad + \beta_n\left(\sqrt{1 - 2k_{g_1} + (2\alpha_{g_1}+1)l_{g_1}^2} + \sqrt{1 - 2\rho_1 k_{T_1} + (2\alpha_{T_1}+\rho_1)\rho_1 l_{T_1}^2}\right)\|x_n - x\|. \end{aligned} \tag{35}$$

Now, we put

$$\begin{aligned} &\sqrt{1 - 2k_{h_1} + (2\alpha_{h_1}+1)l_{h_1}^2} = \mu_1, \ \ \sqrt{1 - 2k_{h_2} + (2\alpha_{h_2}+1)l_{h_2}^2} = \mu_2, \\ &\sqrt{1 - 2k_{g_1} + (2\alpha_{g_1}+1)l_{g_1}^2} = \delta_1, \ \ \sqrt{1 - 2k_{g_2} + (2\alpha_{g_2}+1)l_{g_2}^2} = \delta_2, \\ &\sqrt{1 - 2\rho_1 k_{T_1} + (2\alpha_{T_1}+\rho_1)\rho_1\, l_{T_1}^2} = \theta_1, \ \ \sqrt{1 - 2\rho_2 k_{T_2} + (2\alpha_{T_2}+\rho_2)\rho_2\, l_{T_2}^2} = \theta_2, \end{aligned}$$

then (33) and (35) have

$$\|x_{n+1} - x\| \leq (1 - \alpha_n(1-\mu_2))\|x_n - x\| + \alpha_n(\delta_2 + \theta_2)\|y_{n+1} - y\|, \tag{36}$$

$$\|y_{n+1} - y\| \leq (1 - \beta_n(1-\mu_1))\|y_n - y\| + \beta_n(\delta_1 + \theta_1)\|x_n - x\|. \tag{37}$$

Adding (36) and (37), we have

$$\begin{aligned} \|x_{n+1} - x\| + \|y_{n+1} - y\| \leq &\Big(1 - \alpha_n(1-\mu_2) + \beta_n(\delta_1+\theta_1)\Big)\|x_n - x\| \\ &+ \alpha_n(\delta_2+\theta_2)\|y_{n+1} - y\| + (1 - \beta_n(1-\mu_1))\|y_n - y\|. \end{aligned}$$

Thus,

$$\begin{aligned} &\|x_{n+1} - x\| + (1 - \alpha_n(\delta_2+\theta_2))\|y_{n+1} - y\| \\ &\leq \Big(1 - \alpha_n(1-\mu_2) + \beta_n(\delta_1+\theta_1)\Big)\|x_n - x\| + (1 - \beta_n(1-\mu_1))\|y_n - y\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x\| + \nu\|y_{n+1} - y\| &\leq \max\{\varepsilon_1, \varepsilon_2\} \cdot (\|x_n - x\| + \|y_n - y\|) \\ &= \varepsilon(\|x_n - x\| + \|y_n - y\|), \end{aligned} \tag{38}$$

where

$$\begin{aligned} &\nu = 1 - \alpha_n(\delta_2+\theta_2) \geq 0, \ \ \varepsilon_1 = 1 - (\alpha_n(1-\mu_2) - \beta_n(\delta_1+\theta_1)), \\ &\varepsilon_2 = 1 - \beta_n(1-\mu_1), \ \ \varepsilon = \max\{\varepsilon_1, \varepsilon_2\}. \end{aligned}$$

From conditions, we get

$$\varepsilon < 1.$$

Therefore, by Lemma 2, it follows from (38) that

$$\lim_{n\to\infty}(\|x_{n+1} - x\| + \nu\|y_{n+1} - y\|) = 0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x\| = 0 = \lim_{n\to\infty}\|y_{n+1} - y\|.$$

This completes the proof. □

Corollary 2. **([2], Theorem 6)** Let $T_1, T_2, g_1, g_2, h_1, h_2$: $H \to H$ be strongly monotone with constants $k_{T_1} > 0$, $k_{T_2} > 0$, $k_{g_1} > 0$, $k_{g_2} > 0$, $k_{h_1} > 0$, $k_{h_2} > 0$ and l_{T_1}, l_{T_2}, l_{g_1}, l_{g_2}, l_{h_1}, l_{h_2}-Lipschitz continuous operators, respectively. If the following conditions hold:

(i) $0 < \theta_1 = \sqrt{1 - 2\rho_1 k_{T_1} + \rho_1^2 l_{T_1}^2} < 1, \;\; 0 < \theta_2 = \sqrt{1 - 2\rho_2 k_{T_2} + \rho_2^2 l_{T_2}^2} < 1,$

(ii) $2k_{h_1} - l_{h_1}^2 < 1, \;\; 2k_{h_2} - l_{h_2}^2 < 1, \;\; 2k_{g_1} - l_{g_1}^2 < 1, \;\; 2k_{g_2} - l_{g_2}^2 < 1,$

(iii) $\alpha_n, \beta_n \in [0, 1]$ for all $n \geq 0$, $\;1 - \nu = \alpha_n(\delta_2 + \theta_2) \geq 0,$
$1 - \varepsilon_1 = \alpha_n(1 - \mu_2) - \beta_n(\delta_1 + \theta_1) \geq 0, \;\; 1 - \varepsilon_2 = \beta_n(1 - \mu_1) \geq 0,$
such that

$$\sum_{n=0}^{\infty} \alpha_n(\delta_2 + \theta_2) = \infty,$$

$$\sum_{n=0}^{\infty} \alpha_n(1 - \mu_2) - \beta_n(\delta_1 + \theta_1) = \infty,$$

$$\sum_{n=0}^{\infty} \beta_n(1 - \mu_1) = \infty,$$

where

$$\delta_1 = \sqrt{1 - 2k_{g_1} + l_{g_1}^2}, \;\; \delta_2 = \sqrt{1 - 2k_{g_2} + l_{g_2}^2},$$
$$\mu_1 = \sqrt{1 - 2k_{h_1} + l_{h_1}^2}, \;\; \mu_2 = \sqrt{1 - 2k_{h_2} + l_{h_2}^2},$$

then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 2 converge to x and y, respectively.

Proof. In Theorem 2, we take $\alpha_{T_1} = \alpha_{T_2} = \alpha_{g_1} = \alpha_{g_2} = \alpha_{h_1} = \alpha_{h_2} = 0$, and we get the result of Corollary 2. □

Open Problem Do Theorems 1 and 2 hold for a Banach space or other spaces?

4. Conclusions

Theorems 1 and 2 generalize and improve the results which are discussed in [2] and others. The system of extended general variational inequalities includes various classes of variational inequalities and optimization problems as special cases, and its results proved in this paper continue to hold for these problems. It is expected that this class will motivate and inspire further research in this area.

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