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Interpolative Reich–Rus–Ćirić Type Contractions on Partial Metric Spaces

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Abstract: By giving a counter-example, we point out a gap in the paper by Karapinar (Adv. Theory Nonlinear Anal. Its Appl. 2018, 2, 85–87) where the given fixed point may be not unique and we present the corrected version. We also state the celebrated fixed point theorem of Reich–Rus–Ćirić in the framework of complete partial metric spaces, by taking the interpolation theory into account. Some examples are provided where the main result in papers by Reich (Can. Math. Bull. 1971, 14, 121–124; Boll. Unione Mat. Ital. 1972, 4, 26–42 and Boll. Unione Mat. Ital. 1971, 4, 1–11.) is not applicable.

Keywords: partial metric; interpolative Reich–Rus–Ćirić type contraction; fixed point

MSC: 46T99; 47H10; 54H25

1. Introduction and Preliminaries

In 1922, Banach laid the foundations of metric fixed point theory by proposing his prominent fixed point result. To put a finer point on it, Banach observed that if a self-mapping T , defined on complete metric space (X, ρ) , fulfills the contraction inequality, i.e., there exists a constant $k \in [0, 1)$ such that

$$\rho(T\xi, T\eta) \leq k\rho(\xi, \eta) \text{ for all } \xi, \eta \in X, \quad (1)$$

then it possesses a unique fixed point in X . In 1968, Kannan [1] proposed a new fixed point result. He considered the following contraction type:

$$\rho(T\xi, T\eta) \leq \lambda [\rho(\xi, T\xi) + \rho(\eta, T\eta)] \text{ for all } \xi, \eta \in X,$$

where $\lambda \in [0, \frac{1}{2})$. Very recently, in [2], the acclaimed theorem of Kannan was revisited by taking the interpolation theory into account. For a metric space (X, ρ) , the self-mapping $T : X \rightarrow X$ is said to be an *interpolative Kannan type contraction*, if there are constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\rho(T\xi, T\eta) \leq \lambda [\rho(\xi, T\xi)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha}, \quad (2)$$

for all $\xi, \eta \in X$ with $\xi \neq T\xi$. The main result in [2] via an interpolative Kannan type contraction is

Theorem 1 ([2]). *In the framework of a complete metric space (X, ρ) , if a mapping $T : X \rightarrow X$ forms an interpolative Kannan type contraction, then it possesses a unique fixed point in X .*

Note that when the inequality (2) holds for all $\xi, \eta \in X$, and if T possesses a fixed point (say $v \in X$), then $T\xi = v$ for each $\xi \in X$, that is, T is a constant mapping, which is the trivial case, so the fixed point of T is unique. The appropriate condition on ξ and η in (2) should be $\xi, \eta \in X \setminus \text{Fix}(T)$, where $\text{Fix}(T) = \{u \in X, Tu = u\}$. In this case, the author [2] ensured the existence of a unique fixed point. There is a gap, that is, such fixed point is not necessarily unique. The following example illustrates our concern.

Example 1. Set $X = \{0, 1, 2, 3\}$ that is endowed with the Euclidean metric $\rho(\xi, \eta) = |\xi - \eta|$. Consider

$$T0 = 0, \quad T1 = 1, \quad T2 = T3 = 1.$$

Let $\xi, \eta \in X \setminus \text{Fix}(T)$. Then, $(\xi, \eta) \in \{(2, 3), (3, 2), (2, 2), (3, 3)\}$. Thus, (2) is satisfied for all $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$. It is evident that both 0 and 1 are fixed points for the self-mapping T .

As a correction of Theorem 1, we should state

Theorem 2. Let (X, ρ) be a complete metric space. A self-mapping $T : X \rightarrow X$ possesses a fixed point in X , if there exist constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\rho(T\xi, T\eta) \leq \lambda [\rho(\xi, T\xi)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha}$$

for all $\xi, \eta \in X \setminus \text{Fix}(T)$.

The following theorem was proved by Reich, Rus and Ćirić [3–7] independently to combine and improve both Banach and Kannan fixed point theorems.

Theorem 3. In the framework of a complete metric space (X, ρ) , if $T : X \rightarrow X$ forms a Reich–Rus–Ćirić contraction mapping, i.e.,

$$\rho(T\xi, T\eta) \leq \lambda [\rho(\xi, \eta) + \rho(\xi, T\xi) + \rho(\eta, T\eta)], \tag{3}$$

for all $\xi, \eta \in X$, where $\lambda \in [0, \frac{1}{3})$, then T possesses a unique fixed point.

Notice that several variations of Reich contractions (3) can be stated. We may state the following:

$$\rho(T\xi, T\eta) \leq a\rho(\xi, \eta) + b\rho(\xi, T\xi) + c\rho(\eta, T\eta),$$

where $a, b, c \in (0, \infty)$ such that $0 \leq a + b + c < 1$.

In this paper, we shall investigate the validity of the interpolation approach for Reich contractions in the context of partial metric spaces that was introduced by Matthews [8].

Definition 1. Let X be a non-empty set. A function $p : X \times X \rightarrow [0, \infty)$ is said to be a partial metric, if the following conditions are fulfilled for each $\xi, \eta, \zeta \in X$,

$$\begin{aligned} (P1) \quad & \xi = \eta \Leftrightarrow p(\xi, \xi) = p(\eta, \eta) = p(\xi, \eta); \\ (P2) \quad & p(\xi, \xi) \leq p(\xi, \eta); \\ (P3) \quad & p(\xi, \eta) = p(\eta, \xi); \\ (P4) \quad & p(\xi, \eta) \leq p(\xi, \zeta) + p(\zeta, \eta) - p(\zeta, \zeta). \end{aligned} \tag{4}$$

In this case, (X, p) is said to be a partial metric space.

The function $\rho_p : X \times X \rightarrow [0, \infty)$ defined as

$$\rho_p(\xi, \eta) = 2p(\xi, \eta) - p(\xi, \xi) - p(\eta, \eta) \tag{5}$$

is a standard metric on X . It is natural to define the basic topological concepts, in particular, convergence of a sequence, fundamental (Cauchy) sequence criteria, continuity of the mappings, and completeness of the topological space in the framework of partial metric spaces; see, e.g., [8–18].

Definition 2. *In the framework of a partial metric space (X, p) , we say that*

- (i) *A sequence $\{\xi_n\}$ converges to the limit ξ if $p(\xi, \xi) = \lim_{n \rightarrow \infty} p(\xi, \xi_n)$;*
- (ii) *A sequence $\{\xi_n\}$ is fundamental or Cauchy if $\lim_{n, m \rightarrow \infty} p(\xi_n, \xi_m)$ exists and is finite;*
- (iii) *A partial metric space (X, p) is complete if each fundamental sequence $\{\xi_n\}$ converges to a point $\xi \in X$ such that $p(\xi, \xi) = \lim_{n, m \rightarrow \infty} p(\xi_n, \xi_m)$;*
- (iv) *A mapping $F : X \rightarrow X$ is continuous at a point $\xi_0 \in X$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(\xi_0, \delta)) \subseteq B_p(F\xi_0, \epsilon)$.*

For what follows, we shall recall the following lemma that can be derived easily (see [8]).

Lemma 1. *Let p be a partial metric on a non-empty set X and ρ_p be the corresponding standard metric space on the same set X .*

- (a) *A sequence $\{\xi_n\}$ is fundamental in the framework of a partial metric (X, p) if and only if it is a fundamental sequence in the setting of the corresponding standard metric space (X, ρ_p) .*
- (b) *A partial metric space (X, p) is complete if and only if the corresponding standard metric space (X, ρ_p) is complete. Moreover,*

$$\lim_{n \rightarrow \infty} \rho_p(\xi, \xi_n) = 0 \Leftrightarrow p(\xi, \xi) = \lim_{n \rightarrow \infty} p(\xi, \xi_n) = \lim_{n, m \rightarrow \infty} p(\xi_n, \xi_m). \tag{6}$$

- (c) *If $\xi_n \rightarrow \xi$ as $n \rightarrow \infty$ in a partial metric space (X, p) with $p(\xi, \xi) = 0$, then we have*

$$\lim_{n \rightarrow \infty} p(\xi_n, \eta) = p(\xi, \eta) \text{ for every } \eta \in X.$$

In this paper, we initiate the notion of *interpolative Reich–Rus–Ćirić type contractions* on partial metric spaces. We also present two examples illustrating our approach.

2. Main Results

We start this section by introducing the notion of *interpolative Reich–Rus–Ćirić type contractions*.

Definition 3. *In the framework of a partial metric space (X, p) , a mapping $T : X \rightarrow X$ is called an *interpolative Reich–Rus–Ćirić type contraction*, if there are constants $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that*

$$p(T\xi, T\eta) \leq \lambda [p(\xi, \eta)]^\beta \cdot [p(\xi, T\xi)]^\alpha \cdot [p(\eta, T\eta)]^{1-\alpha-\beta} \tag{7}$$

for all $\xi, \eta \in X \setminus \text{Fix}(T)$,

Theorem 4. *In the framework of a partial metric space (X, p) , if $T : X \rightarrow X$ is an *interpolative Reich–Rus–Ćirić type contraction*, then T has a fixed point in X .*

Proof. We take an arbitrary point $\xi_0 \in (X, p)$ and build an iterative sequence $\{\xi_n\}$ by $\xi_n = T^n(\xi_0)$ for each positive integer n . If there exists n_0 such that $\xi_{n_0} = \xi_{n_0+1}$, then ξ_{n_0} is a fixed point of T . The proof

is completed. Henceforward, assume that $\xi_n \neq \xi_{n+1}$ for each $n \geq 0$. By substituting the values $\xi = \xi_n$ and $\eta = \xi_{n-1}$ in (7), we find that

$$\begin{aligned} p(\xi_{n+1}, \xi_n) &= p(T\xi_n, T\xi_{n-1}) \leq \lambda [p(\xi_n, \xi_{n-1})]^\beta [p(\xi_n, T\xi_n)]^\alpha \cdot [p(\xi_{n-1}, T\xi_{n-1})]^{1-\alpha-\beta} \\ &= \lambda [p(\xi_n, \xi_{n-1})]^\beta \cdot [p(\xi_n, \xi_{n+1})]^\alpha \cdot [p(\xi_{n-1}, \xi_n)]^{1-\alpha-\beta} \tag{8} \\ &= \lambda [p(\xi_{n-1}, \xi_n)]^{1-\alpha} \cdot [p(\xi_n, \xi_{n+1})]^\alpha. \end{aligned}$$

By a calculation, we derive

$$[p(\xi_n, \xi_{n+1})]^{1-\alpha} \leq \lambda [p(\xi_{n-1}, \xi_n)]^{1-\alpha} \tag{9}$$

from the inequality (8). We conclude that $\{p(\xi_{n-1}, \xi_n)\}$ is a non-increasing sequence with non-negative terms. Thus, there is a nonnegative constant ℓ such that $\lim_{n \rightarrow \infty} p(\xi_{n-1}, \xi_n) = \ell$. Note that $\ell \geq 0$. Indeed, from (9), we deduce that

$$p(\xi_n, \xi_{n+1}) \leq \lambda p(\xi_{n-1}, \xi_n) \leq \lambda^n p(\xi_0, \xi_1). \tag{10}$$

Regarding $\lambda < 1$, and by taking $n \rightarrow \infty$ in the inequality (10), we deduce that $\ell = 0$.

For what follows, we shall prove that $\{\xi_n\}$ is a fundamental (Cauchy) sequence by employing standard tools. More precisely, starting with the triangle inequality, we shall get the following estimation:

$$\begin{aligned} p(\xi_n, \xi_{n+r}) &\leq p(\xi_n, \xi_{n+1}) + \dots + p(\xi_{n+r-1}, \xi_{n+r}) \\ &\leq \lambda^n p(\xi_0, \xi_1) + \dots + \lambda^{n+r-1} p(\xi_0, \xi_1) \tag{11} \\ &\leq \frac{\lambda^n}{1-\lambda} p(\xi_0, \xi_1). \end{aligned}$$

Letting $n \rightarrow \infty$ in the inequality (11), we ascertain that $\{\xi_n\}$ is a fundamental sequence.

Hence, $\lim_{n,m \rightarrow \infty} p(\xi_n, \xi_m) = 0$, that is, $\{\xi_n\}$ is a fundamental sequence in (X, p) . By Lemma 1, $\{\xi_n\}$ is also Cauchy in (X, ρ_p) . More particularly, since (X, p) is complete, (X, ρ_p) is also complete. Hence, there exists $\xi \in X$ such that

$$p(\xi, \xi) = \lim_{n \rightarrow \infty} p(\xi, \xi_n) = \lim_{n,m \rightarrow \infty} p(\xi_n, \xi_m) = 0, \tag{12}$$

which implies that

$$\lim_{n \rightarrow \infty} \rho_p(\xi, \xi_n) = 0. \tag{13}$$

As a next step, we make evident that the limit ξ of the iterative sequence $\{\xi_n\}$ is a fixed point of the given mapping T . Assume that $\xi \neq T\xi$, so $p(\xi, T\xi) > 0$. Recall that $x_n \neq Tx_n$ for each $n \geq 0$. By letting $\xi = \xi_n$ and $\eta = \xi$ in (7), we determine that

$$\begin{aligned} p(\xi_{n+1}, T\xi) &= p(T\xi_n, T\xi) \\ &\leq \lambda [p(\xi_n, \xi)]^\beta \cdot [p(\xi_n, T\xi_n)]^\alpha \cdot [p(\xi, T\xi)]^{1-\alpha-\beta} \tag{14} \\ &= \lambda [p(\xi_n, \xi)]^\beta \cdot [p(\xi_n, \xi_{n+1})]^\alpha \cdot [p(\xi, T\xi)]^{1-\alpha-\beta}. \end{aligned}$$

Letting $n \rightarrow \infty$ in the inequality (14), we find out $p(\xi, T\xi) = 0$, so $\xi = T\xi$, which is a contradiction. Thus, $T\xi = \xi$. \square

The following examples illustrate Theorem 4.

Example 2. Let $X = \{1, 3, 4, 7\}$ be a set endowed with the classical partial metric $\rho(\xi, \eta) = \max\{\xi, \eta\}$, that is,

| | | | | |
|-------------------|---|---|---|---|
| $\rho(\xi, \eta)$ | 1 | 3 | 4 | 7 |
| 1 | 1 | 3 | 4 | 7 |
| 3 | 3 | 3 | 4 | 7 |
| 4 | 4 | 4 | 4 | 7 |
| 7 | 7 | 7 | 7 | 7 |

We define a self-mapping T on X by $T : \begin{pmatrix} 1 & 3 & 4 & 7 \\ 1 & 3 & 1 & 3 \end{pmatrix}$. It is clear that T is not a Reich–Rus–Ćirić contraction. Indeed, there is no $\lambda \in [0, \frac{1}{3})$ such that the following inequality is fulfilled:

$$\begin{aligned} \rho(T1, T3) = \rho(1, 3) = 3 &\leq \lambda(\rho(1, 3) + \rho(T1, 1) + \rho(3, T3)) \\ &= \lambda(\rho(1, 3) + \rho(1, 1) + \rho(3, 3)) \\ &= 7\lambda. \end{aligned}$$

On the other hand, choose $\alpha = \frac{1}{2}$, $\beta = \frac{2}{5}$ and $\lambda = \frac{7}{10}$. Let $\xi, \eta \in X \setminus \text{Fix}(T)$; then, $(\xi, \eta) \in \{(4, 7), (7, 4), (4, 4), (7, 7)\}$. Without loss of generality, we have

Case 1: $\xi = \eta = 4$. Here,

$$\rho(T\xi, T\eta) = 1 \leq 4\lambda = \lambda [\rho(\xi, \eta)]^\beta \cdot [\rho(\xi, T\xi)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha-\beta}.$$

Case 2: $\xi = \eta = 7$. we have

$$\rho(T\xi, T\eta) = 3 \leq 7\lambda = \lambda [\rho(\xi, \eta)]^\beta \cdot [\rho(\xi, T\xi)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha-\beta}.$$

Case 3: $\xi = 4$ and $\eta = 7$. Here,

$$\begin{aligned} \rho(T\xi, T\eta) = 3 &\leq \lambda 7^{1-\alpha} 4^\alpha \\ &= \lambda [\rho(\xi, \eta)]^\beta \cdot [\rho(\xi, T\xi)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha-\beta}. \end{aligned}$$

Thus, the self-mapping T is an interpolative Reich–Rus–Ćirić type contraction and 1, 3 are the desired fixed points. Note that, in the setting of interpolative Reich–Rus–Ćirić type contractions, the constant lies between 0 and 1, although in the classical version it is restricted by 1/3.

Example 3. Following Example 1, let $X = \{0, 1, 2, 3\}$. Consider

$$T0 = 0, \quad T1 = 1, \quad T2 = T3 = 1.$$

Consider $p(x, y) = \max\{x, y\}$. Clearly, (7) holds for all $\xi, \eta \in X \setminus \text{Fix}(T)$ [by taking $\lambda = \frac{\sqrt{2}}{2}$, $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{3}$]. Note that T has two fixed points, which are 0 and 1.

On the other hand, taking $x = 0$ and $y = 1$, we have, for any $\lambda \in [0, \frac{1}{3})$,

$$p(Tx, Ty) = 1 > 2\lambda = \lambda[p(x, y) + p(x, Tx) + p(y, Ty)],$$

that is, Corollary 4 in [9] is not applicable.

The following is an immediate consequence of our main result.

Corollary 1. In the framework of a standard metric space (X, ρ) , if $T : X \rightarrow X$ is an interpolative Reich–Rus–Ćirić type contraction, that is,

$$\rho(T\xi, T\eta) \leq \lambda [\rho(\xi, \eta)]^\beta \cdot [\rho(\xi, T\xi)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha-\beta} \quad (15)$$

for all $\xi, \eta \in X \setminus \text{Fix}(T)$, then T possesses a fixed point in X .

Proceeding as Theorem 4, we shall extend Theorem 2 to partial metric spaces.

Theorem 5. Let (X, p) be a complete partial metric space and $T : X \rightarrow X$ be such that

$$p(T\xi, T\eta) \leq \lambda \cdot [p(\xi, T\xi)]^\alpha \cdot [p(\eta, T\eta)]^{1-\alpha} \quad (16)$$

for all $\xi, \eta \in X \setminus \text{Fix}(T)$, where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$. Then, T possesses a fixed point in X .

3. Conclusions

The main contribution of the paper to ensure the existence of fixed points for interpolative Reich–Rus–Ćirić type contraction mappings on partial metric spaces. We also removed the gap in [2].

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