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# Edge Irregular Reflexive Labeling for Disjoint Union of Generalized Petersen Graph

**Juan L. G. Guirao [1,*], Sarfraz Ahmad [2], Muhammad Kamran Siddiqui [3] and Muhammad Ibrahim [4]**

1 Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Hospital de Marina, 30203 Cartagena, Región de Murcia, Spain
2 Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan; sarfrazahmad@cuilahore.edu.pk
3 Department of Mathematics, COMSATS University Islamabad, Sahiwal Campus, Sahiwal 57000, Pakistan; kamransiddiqui75@gmail.com
4 Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Multan 60800, Pakistan; mibtufail@gmail.com
* Correspondence: juan.garcia@upct.es

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**Abstract:** A graph labeling is the task of integers, generally spoken to by whole numbers, to the edges or vertices, or both of a graph. Formally, given a graph $G = (V, E)$ a vertex labeling is a capacity from $V$ to an arrangement of integers. A graph with such a capacity characterized is known as a vertex-labeled graph. Similarly, an edge labeling is an element of $E$ to an arrangement of labels. For this situation, the graph is called an edge-labeled graph. We examine an edge irregular reflexive $k$-labeling for the disjoint association of the cycle related graphs and decide the correct estimation of the reflexive edge strength for the disjoint association of $s$ isomorphic duplicates of the cycle related graphs to be specific Generalized Peterson graphs.

**Keywords:** edge irregular reflexive labeling; reflexive edge strength; generalized peterson graphs

**MSC:** 05C12, 05C90

## 1. Introduction.

All graphs considered in this paper are basic, limited and undirected. Chartrand et al. [1] proposed the accompanying issue. Appoint a positive whole number mark from the set $\{1, 2, ...k\}$ to the edges of a basic associated graph of request no less than three in such a path, to the point that the graph winds up unpredictable, i.e., the weight (mark entirety) at every vertex is particular. What is the base estimation of the biggest mark $k$ over all such sporadic assignments? This parameter of the graph $G$ is notable as the irregularity strength of the graph $G$. A phenomenal review on the anomaly quality is given by Lahel in [2]. For ongoing outcomes, see the papers by Amar and Togni in [3], Dimitz et al. in [4], Gyarfas in [5] and Nierhoff in [6].

Propelled by these papers, an edge irregular $k$-labeling as a vertex naming $\Gamma : V(G) \to \{1, 2, ..., k\}$ was characterized, with the end goal that for each two unique edges $gq$ and $g^{'}q^{'}$ there is $w_\Gamma(gq) \neq w_\Gamma(g^{'}q^{'})$, where the heaviness of an edge $gq \in E(G)$ is $w_\Gamma(gq) = \Gamma(g) + \Gamma(q)$. The base $k$ for which the graph $G$ has an edge irregular $k$-labeling is called the edge irregularity strength of the graph $G$, indicated by $es(G)$. In [7], the limits of the parameters $es(G)$ are evaluated. Furthermore, the correct estimation of the edge irregularity strength for a few groups of graphs is resolved; in particular for stars, twofold stars and also the cartesian product of two paths.

Baca et al. [8], characterized the total labeling $\Gamma : V(G) \cup E(G) \rightarrow \{1, 2, ..., k\}$ to be an edge irregular total $k$-labeling of the graph $G$ if for each two distinctive edge $gq$ and $g^{'}q^{'}$ of $G$ one has $w_{\Gamma}(gq) = \Gamma(g) + \Gamma(gq) + \Gamma(q) \neq w_{\Gamma}(g^{'}q^{'}) = \Gamma(g^{'}) + \Gamma(g^{'}q^{'}) + \Gamma(q^{'})$. The total edge irregularity strength, $tes(G)$, is characterized as the base $k$ for which $G$ has an edge edge irregular total $k$-labeling. Evaluations of these parameters are acquired, which gives the exact estimations of the total irregularity strength for paths, cycles, stars, and haggles diagrams. Additionally, results on the aggregate inconsistency quality can be found in [9,10].

The fundamental issue for the sporadic marking emerges from a thought of graphs with distinct degree. In a simple graph, it is not conceivable to develop a graph in which each vertex has a novel degree; be that as it may, this is conceivable in multigraphs (graphs in which we permit different edges between the contiguous vertices). The inquiry at that point moved toward becoming: "what is the smallest number of parallel edges between two vertices required to guarantee that the graph shows vertex irregularity?" This issue is proportional to the marking issue as depicted toward the start of this area.

Ryan et al. [11] proclaimed that the vertex labels ought to speak to circles at the vertex. The result was two-overlap; first, every vertex name was required to be a considerable number, since each circle added two to the vertex degree; and second, not at all like in all out unpredictable marking, the mark 0 was allowed to speak to a loopless vertex. Edges continued to be marked by whole numbers from one to $k$.

Thus, they defined labellings $\Gamma_e : E(G) \rightarrow \{1, 2, ..., k_e\}$ and $\Gamma_v : V(G) \rightarrow \{0, 2, ..., 2k_v\}$, and then, labeling $\Gamma$ is a total $k$-labeling of $G$ defined such that $\Gamma(x) = \Gamma_v(r)$ if $r \in V(G)$ and $\Gamma(r) = \Gamma_e(r)$ if $r \in E(G)$, where $k = max\{k_e, 2k_v\}$.

The total $k$-labeling $\Gamma$ is called an edge irregular reflexive $k$-labeling of the graph $G$ if for every two different edges $gq$ and $g^{'}q^{'}$ of $G$, one has $wt(gq) = \Gamma_v(g) + \Gamma_e(gq) + \Gamma_v(q) \neq wt(g^{'}q^{'}) = \Gamma_v(g^{'}) + \Gamma_e(g^{'}q^{'}) + \Gamma_v(q^{'})$. The smallest value of $k$ for which such labeling exists is called the reflexive edge strength of the graph $G$ and is denoted by $res(G)$. For recent results see [12,13].

The after effect of this variety was not generally shown in the naming quality, however, it produced some essential results:

$$tes(K_5) = 5 \text{ whereas } res(K_5) = 4$$

The impact of this change was quick in the accompanying conjecture where we could evacuate the trouble, for some exemptions see [14].

**Conjecture 1.** *Any graph $G$ with maximum degree $\Delta(G)$ other than $K_5$ satisfies*

$$\mathrm{tes}(G) = \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta + 1}{2} \right\rceil\right\}.$$

Conjecture 1 has been verified for complete graphs and complete bipartite graphs [15,16], for the grid [17], for hexagonal grid graphs [18], for toroidal grid [19], for generalized prism [20], for categorical product of two cycles [21], for strong product of cycles and paths [22], for zigzag graphs [23], for star in [24], for the categorical product of cycle and path in [25], for convex polytopes in [26] and for the strong product of two paths in [27].

In terms of $res(G)$, Baca et al. [28] purpose the following conjecture and prove Theorem 1.

**Conjecture 2.** *Any graph $G$ with maximum degree $\Delta(G)$ satisfies*

$$\mathrm{res}(G) = \max\left\{\left\lceil \frac{|E(G)|}{3} + r \right\rceil, \left\lceil \frac{\Delta + 2}{2} \right\rceil\right\},$$

*where $r = 1$ for $|E(G)| \equiv 2, 3(mod\ \ 6)$, and zero otherwise.*

Conjecture 2 has been verified for the disjoint union of Gear graphs and Prism graphs in [11] and for Cycle graph in [28].

**Theorem 1.** *If G be a cycle graph [28], then* $res(C_n) = \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } n \not\equiv 2,3(mod \ \ 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } n \equiv 2,3(mod \ \ 6). \end{cases}$

## 2. Constructing an Edge Irregular Reflexive Labeling

Let us recall the following lemma.

**Lemma 1.** *Let G be any graph [28], then* $res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } n \not\equiv 2,3(mod \ \ 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } n \equiv 2,3(mod \ \ 6). \end{cases}$

The lower destined for $res(G)$ emerges from the way that the negligible edge weight under an edge edge irregular reflexive labeling in one, and the base of the maximal edge weight, that is $|E(G)|$ can be accomplished just as the aggregate of three numbers, somewhere around two of which are even.

In this paper, we investigate the $res(G)$ for disjoint union of $s$ isomorphic copies of Generalized Peterson graphs.

## 3. Applications of Graph Labeling

The field of graph theory assumes a fundamental job in different fields. In graph hypothesis, the principle issue is graph labeling. Graph labeling is the task of number's form 1 to $n$ for vertex, edges and both of the graphs separately. One of the vital territories in graph theory is graph labeling which is utilized in numerous applications like coding hypothesis, radar, cosmology, circuit structure, rocket direction, correspondence arrange tending to, x-beam crystallography, information base administration. Here we might want to improve the edge irregular reflexive $k$-labeling with applications in the field of software engineering. This paper gives a diagram of the labeling of graphs in heterogeneous fields to some degree, however, for the most part it centers around vital real territories of software engineering like information mining, picture preparing, cryptography, programming testing, data security, correspondence systems and so forth. These are different subjects in designing investigations and they are all the more effectively utilized in different areas like government parts and corporate segments. More preciously, the edge unpredictable reflexive $k$-labeling drives us to deal with the irregular situation in networking. For more details of graph labeling see [29].

## 4. Generalized Petersen Graph

The Generalized Petersen graph $P(n, m)$ has been studied extensively in recent years. Generalized Petersen graphs were first defined by Watkins [30]. Mominul Haque [31] determined the irregular total labelings of Generalized Petersen graphs. Jendrol and Žoldák [32] determined the irregularity strength of Generalized Petersen graphs. Chunling et al. [33] determined the total edge irregularity strength of Generalized Petersen graphs. Ahmad et al. [34] determined the total irregularity strength of Generalized Petersen graph. Naeem et al. [35] provide the total irregularity strength of disjoint union of isomorphic copies of the Generalized Petersen graph. Gera et al. [36] completely describe the spectrum of the Generalized Petersen graph $P(n, m)$. Yegnanarayanan [37] computed the spectrum, the Estrada index, Laplacian Estrada index, signless Laplacian Estrada index, normalized Laplacian Estrada index, and energy of Generalized Petersen graph $P(n, m)$.

In this paper, we investigate the reflexive edge irregularity strength of complete disjoint union of $s$ isomorphic copies of the Generalized Peterson graphs. First we define the vertex set and edge set of disjoint union of $s$ isomorphic copies of Generalized Peterson graph $P(n,m)$ in the following way.

$$V(sP(n,m)) = \{x_i^j, y_i^j : 1 \leq i \leq n; 1 \leq j \leq s\}$$

$$E(sP(n,m)) = \{x_i^j x_{i+m}^j, x_i^j y_i^j, y_i^j y_{i+1}^j : 1 \leq i \leq n; 1 \leq j \leq s\}$$

where the subscripts $i$ and $i+m$ are taken under modulo $n$.

**Theorem 2.** *For $s \geq 1$, $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. we have*

$$res(sP(n,m)) = \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$$

**Proof.** Since $(sP(n,m))$ has $3ns$ edges. Therefore From Lemma 1, we get

$$(sP(n,m)) \geq \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$$

Next, we will show that $res(sP(n,m)) \leq \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$

For this we define a $f$-labeling on $(sP(n,m))$ as follow:

$$k = \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$$

For $s = 1, 1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. and $i = 1, 2, ..., n$ we have the following labeling of vertices and edges along with their weights.

$$f(x_i^1) = 0, f(x_i^1 x_{i+1}^1) = i,$$

$$f(y_i^1) = \begin{cases} n-1, & \text{for } i = 1 \\ k, & \text{for } i = 2, 3, ..., n \end{cases}$$

$$f(x_i^1 y_i^1) = \begin{cases} 1, & \text{for } i = 1 \\ i-1, & \text{for } i = 2, 3, ..., n \end{cases}$$

$$f(y_i^1 y_{i+1}^1) = \begin{cases} 1, & \text{for } i = 1 \\ i-1, & \text{for } i = 2, 3, ..., n-1 \\ 2, & \text{for } i = n \end{cases}$$

$$wt(x_i^1 x_{i+1}^1) = i, wt(x_i^1 y_i^1) = n+i$$

$$wt(y_i^1 y_{i+1}^1) = \begin{cases} 2n+1, & \text{for } i = 1 \\ 2n+1+i, & \text{for } i = 2, 3, ..., n-1 \\ 2n+2, & \text{for } i = n \end{cases}$$

For $s \geq 2$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. We have the following labeling of vertices and edges along with their weights:

$$f(x_i^j) = \begin{cases} ns - n + 1, & \text{if } 1 \leq i < n, (s \ is \ even) \\ ns - n, & \text{if } 1 \leq i < n, (s \ is \ odd) \end{cases}$$

$$f(y_i^j) = \begin{cases} ns, & \text{if } 1 \leq i < n, (s \ is \ even) \\ ns + 1, & \text{if } 1 \leq i < n, (s \ is \ odd) \end{cases}$$

$$f(x_i^j x_{i+m}^j) = \begin{cases} ns - n - 2 + i, & \text{if } 1 \leq i < n, (s \ is \ even) \\ ns - n + i, & \text{if } 1 \leq i < n, (s \ is \ odd) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} ns - n - 1 + i, & \text{if } 1 \leq i < n, (s \ is \ even) \\ ns - n + 1 + i, & \text{if } 1 \leq i < n, (s \ is \ odd) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} ns - n + i, & \text{if } 1 \leq i < n, (s \ is \ even) \\ ns - n - 2 + i, & \text{if } 1 \leq i < n, (s \ is \ odd) \end{cases}$$

For $1 \leq i < n, \quad s \geq 2$ we have

$$wt(x_i^j x_{i+m}^j) = 3ns - 3n + i, wt(x_i^j y_i^j) = 3ns - 2n + i, wt(y_i^j y_{i+1}^j) = 3ns - n + i,$$

It is a matter of routine checking that there are no two edges of the same weight.

So, $f$ is an edge irregular reflexive labeling of $(sP(n_j, m))$ for $1 \leq m \leq \lfloor \frac{n-1}{2}. \rfloor$ and for $n \geq 3$. Which completes the proof. $\square$

**Theorem 3.** *Let $(sP(n, \frac{n}{2})), s \geq 1$ be isomorphic copies of the Generalized Petersen graphs with $n$ even, and $n = 4, 6, 8$. Then*

$$res(s(P(n, \frac{n}{2}))) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2, 3 (mod \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2, 3 (mod \ 6) \end{cases}$$

**Proof.** For $m = \frac{n}{2}$, $(sP(n, \frac{n}{2}))$ has $\frac{5ns}{2}$ edges. From Lemma 1, we get

$$res(s(P(n, \frac{n}{2}))) \geq \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2, 3 (mod \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2, 3 (mod \ 6) \end{cases}$$

Next, we will show that

$$res(s(P(n, \frac{n}{2}))) \leq \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2, 3 (mod \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2, 3 (mod \ 6) \end{cases}$$

For $n = 4, 6, 8$ and $j = 1$ we have the following labeling and weights of vertices and edges as follows:

$$f(x_i^1) = 0, f(x_i^1 x_{i+\frac{n}{2}}^1) = i,$$
$$wt(x_i^1 x_{i+\frac{n}{2}}^1) = i, wt(x_i^1 y_i^1) = \tfrac{n}{2} + i$$

For $n = 4$ and $j = 1$

$$f(y_i^1) = \begin{cases} 2, & \text{for } i = 1, 2 \\ 4, & \text{for } i = 3, 4 \end{cases}$$

$$f(x_i^1y_i^1)=\begin{cases} i, & \text{for } i=1,2\\ 4-i, & \text{for } i=3,4\end{cases}$$

$$f(y_i^1y_{i+1}^1)=\begin{cases} n-1, & \text{for } i=1,2\\ n-2, & \text{for } i=3,4\end{cases}$$

$$wt(x_i^1x_{i+\frac{n}{2}}^1)=i, wt(x_i^1y_i^1)=2+i$$

$$wt(y_i^1y_{i+1}^1)=\begin{cases} 2n-1, & \text{for } i=1\\ 2n-3+i, & \text{for } i=2,3\\ 2n, & \text{for } i=4\end{cases}$$

For $n=6$ and $j=1$

$$f(y_i^1)=\begin{cases} 2, & \text{for } i=1\\ 4, & \text{for } i=2,3\\ 6, & \text{for } i=4,5,6\end{cases}$$

$$f(x_i^1y_i^1)=\begin{cases} 2, & \text{for } i=1\\ 1, & \text{for } i=2\\ i-\frac{n}{2}, & \text{for } i=3,4,5,6\end{cases}$$

$$f(y_i^1y_{i+1}^1)=\begin{cases} \frac{n}{2}+2-i, & \text{for } i=1,3\\ i-\frac{n}{2}+1, & \text{for } i=2,4,5\\ n-1, & \text{for } i=6\end{cases}$$

$$wt(y_i^1y_{i+1}^1)=\begin{cases} n+3+i, & \text{for } i=1,2\\ n+4+i, & \text{for } i=3,4,5\\ \frac{3n}{2}+4, & \text{for } i=6\end{cases}$$

For $n=8$ and $j=1$

$$f(y_i^1)=\begin{cases} 4, & \text{for } i=1,2\\ 6, & \text{for } i=3,4\\ 8, & \text{for } i=5,6,7,8\end{cases}$$

$$f(x_i^1y_i^1)=\begin{cases} 1, & \text{for } i=1,3\\ 2, & \text{for } i=2,4\\ i-\frac{n}{2}, & \text{for } i=5,6,7,8\end{cases}$$

$$f(y_i^1y_{i+1}^1)=\begin{cases} \frac{n}{2}+2-i, & \text{for } 1\le i\le \frac{n}{2}\\ i-\frac{n}{2}+1, & \text{for } \frac{n}{2}+1\le i\le n\end{cases}$$

$$wt(y_i^1y_{i+1}^1)=\begin{cases} n+3+i, & \text{for } i=1,2\\ n+4+i, & \text{for } i=3,4,5\\ \frac{3n}{2}+5, & \text{for } i=6\end{cases}$$

For $j \geq 2$ and $n = 4$ we define a $f$-labeling on vertices and edges of $(s(P(4,2)))$ as follows:

$$f(x_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \leq i < 4, \quad s \equiv 1,2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \leq i < 4, \quad s \equiv 0(mod\ 3) \end{cases}$$

$$f(y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \leq i < 4, \quad s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \leq i < 4, \quad s \equiv 0,1(mod\ 3) \end{cases}$$

$$f(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \leq i < 2, \quad s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \leq i < 2, \quad s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil + i, & \text{if } 1 \leq i < 2, \quad j \equiv 1(mod\ 3) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 4 + i, & \text{if } 1 \leq i < 4, \quad s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \leq i < 4, \quad s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \leq i < 4, \quad s \equiv 1(mod\ 3) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 4 + i, & \text{if } 1 \leq i < 4, \quad s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 1 + i, & \text{if } 1 \leq i < 4, \quad s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \leq i < 4, \quad s \equiv 1(mod\ 3) \end{cases}$$

$$wt(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 2 + i, & \text{if } 1 \leq i < 2, \quad s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil + i, & \text{if } 1 \leq i < 2, \quad s \equiv 1(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil - 1 + i, & \text{if } 1 \leq i < 2, \quad s \equiv 0(mod\ 3) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + ns, & \text{if } 1 \leq i < 4, \quad s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil + i + ns, & \text{if } 1 \leq i < 4, \quad s \equiv 1(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil - 1 + i + ns, & \text{if } 1 \leq i < 4, \quad s \equiv 2(mod\ 3) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 4 + i + 2ns, & \text{if } 1 \leq i < 4, \quad s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + 2ns, & \text{if } 1 \leq i < 4, \quad s \equiv 1,2(mod\ 3) \end{cases}$$

For $j \geq 2$ and $n = 6$ we define a $f$-labeling on vertices and edges of $(s(P(6,3)))$ as follow:

$$f(x_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \leq i < 6, \quad s \equiv 0,2,4(mod\ 6) \\ \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \leq i < 6, \quad s \equiv 1,3,5(mod\ 6) \end{cases}$$

$$f(y_i^j) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \leq i < 6, \quad s \equiv 0,2,4(mod\ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{if } 1 \leq i < 6, \quad s \equiv 1,3,5(mod\ 6) \end{cases}$$

$$f(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \leq i < 3, \quad (s \ \ is \ \ even) \\ \lceil \frac{5n(s-1)}{6} \rceil + i, & \text{if } 1 \leq i < 3, \quad (s \ \ is \ \ odd) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \leq i < 6, \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 1 + i, & \text{if } 1 \leq i < 6, \quad s \equiv 0(mod\ 2) \\ \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \leq i < 6, \quad s \equiv 1(mod\ 2) \end{cases}$$

$$wt(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil + i, & \text{if } 1 \le i < 3 \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 3 + i + \frac{5ns}{6}, & \text{if } 1 \le i < 6, \quad s \equiv 2(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + \frac{5ns}{6}, & \text{if } 1 \le i < 6, \quad s \equiv 0,1,3,4,5(mod \;\; 6) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 1 + i + \frac{5ns}{3}, & \text{if } 1 \le i < 6 \end{cases}$$

For $j \ge 2$ and $n = 8$ we define a $f$-labeling on vertices and edges of $(s(P(8,4)))$ as follows:

$$f(x_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \le i < 8, \quad s \equiv 0,2(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 1,3,4,5(mod \;\; 6) \end{cases}$$

$$f(y_i^j) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 2,3,4,5(mod \;\; 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{if } 1 \le i < 8, \quad s \equiv 0,1(mod \;\; 6) \end{cases}$$

$$f(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 4, \quad (s \;\; is \;\; even) \\ \lceil \frac{5n(s-1)}{6} \rceil + i, & \text{if } 1 \le i < 4, \quad (s \;\; is \;\; odd) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 5 + i, & \text{if } 1 \le i < 8, \quad s \equiv 2(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{6} \rceil - 4 + i, & \text{if } 1 \le i < 8, \quad s \equiv 0,1,3,4,5(mod \;\; 6) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \le i < 8, \quad s \equiv 2,5(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 8, \quad s \equiv 0,3(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 8, \quad s \equiv 1,4(mod \;\; 6) \end{cases}$$

$$wt(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 1 + i, & \text{if } 1 \le i < 4, \quad s \equiv 2(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i, & \text{if } 1 \le i < 4, \quad s \equiv 3(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil + i, & \text{if } 1 \le i < 4, \quad s \equiv 0,1,4(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 3 + i, & \text{if } 1 \le i < 4, \quad s \equiv 5(mod \;\; 6) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 4 + i + \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 2,3,4,5(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 0(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 3 + i + \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 1(mod \;\; 6) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 3 + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 2,5(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 1,3(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 4 + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 4(mod \;\; 6) \\ \lceil \frac{5n(s-1)}{2} \rceil + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 0(mod \;\; 6) \end{cases}$$

It is easy to check that there are no two edges of the same weight.

So, $f$ is an edge irregular reflexive labeling of $(sP(n, \frac{n}{2}))$ for $m = \frac{n}{2}$. and for $n = 4.6, 8$ Which completes the proof. □

## 5. Conclusions

In this paper, we give detailed information about the Generalized Petersen graphs and their importance in different areas of science. We also give details about the application of graph labeling in computer science and data processing. More preciously, we have determined the edge irregular reflexive labeling for disjoint union of $s$ isomorphic copies of generalizes Petersen graphs $P(n,m)$ for $s \geq 1$, $n \geq 3$ and $1 \leq m < \frac{n}{2}$. and $(P(n,m))$ for $n = 4,6,8$ with $m = \frac{n}{2}$. We tried to find the edge irregular reflexive labeling for disjoint union of $s$ isomorphic copies of Generalized Petersen graphs $(P(n,m))$ for $n \geq 10$ with $m = \frac{n}{2}$ and $n$ even but so far without success. So we conclude the paper with the following open problem.

## 6. Open Problem

Find the edge irregular reflexive labeling for disjoint union of $s$ isomorphic copies of Generalized Petersen graphs $(P(n,\frac{n}{2}))$ for $n$ even and $n \geq 10$.

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