

Article

# Babenko's Approach to Abel's Integral Equations

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**Abstract:** The goal of this paper is to investigate the following Abel's integral equation of the second kind:  $y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = f(t)$ , ( $t > 0$ ) and its variants by fractional calculus. Applying Babenko's approach and fractional integrals, we provide a general method for solving Abel's integral equation and others with a demonstration of different types of examples by showing convergence of series. In particular, we extend this equation to a distributional space for any arbitrary  $\alpha \in R$  by fractional operations of generalized functions for the first time and obtain several new and interesting results that cannot be realized in the classical sense or by the Laplace transform.

**Keywords:** distribution; fractional calculus; convolution; series convergence; Laplace transform; Gamma function; Mittag-Leffler function

**JEL Classification:** 46F10; 26A33; 45E10; 45G05

## 1. Introduction

Abel's equations are related to a wide range of physical problems, such as heat transfer [1], nonlinear diffusion [2], the propagation of nonlinear waves [3], and applications in the theory of neutron transport and traffic theory. There have been many approaches including numerical analysis thus far to studying Abel's integral equations as well as their variants with many applications [4–13]. In 1930, Tamarkin [14] discussed integrable solutions of Abel's integral equation under certain conditions by several integral operators. Sumner [15] studied Abel's integral equation from the point of view of the convolutional transform. Minerbo and Levy [16] investigated a numerical solution of Abel's integral equation using orthogonal polynomials. In 1985, Hatcher [17] worked on a nonlinear Hilbert problem of power type, solved in closed form by representing a sectionally holomorphic function by means of an integral with power kernel, and transformed the problem to one of solving a generalized Abel's integral equation. Singh et al. [18] obtained a stable numerical solution of Abel's integral equation using an almost Bernstein operational matrix. Recently, Li and Zhao [19] used the inverse of Mikusinski's operator of fractional order, based on Mikusinski's convolution, to construct the solution of the integral equation of Abel's type. Jahanshahi et al. [20] solved Abel's integral equation numerically on the basis of approximations of fractional integrals and Caputo derivatives. Saleh et al. [21] investigated the numerical solution of Abel's integral equation by Chebyshev polynomials. Kumar et al. [22] proposed a new and simple algorithm for Abel's integral equation, namely, the homotopy perturbation transform method (HPTM), based on the Laplace transform algorithm, and made the calculation for approximate solutions much easier. The main advance of this proposal is its capability of obtaining rapid convergent series for singular integral equations of Abel type. Recently, Li et al. [23] studied the following Abel's integral equation:

$$g(x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^x \frac{f(\zeta)}{(x - \zeta)^\alpha} d\zeta, \quad \text{where } \alpha \in R,$$

and its variants in the distributional (Schwartz) sense based on fractional calculus of distributions and derived new results that are not achievable in the classical sense.

The current work is grouped as follows. In Section 2, we briefly introduce the necessary concepts and definitions of fractional calculus of distributions in  $\mathcal{D}'(R^+)$ , which is described in Section 4. In Section 3, we solve Abel’s integral equation of the second kind for  $\alpha > 0$  and its variants by Babenko’s approach, as well as fractional integrals with different types of illustrative examples. Often we obtain an infinite series as the solution of Abel’s integral equation and then show its convergence. In Section 4, we extend Abel’s integral equation into the distributional space  $\mathcal{D}'(R^+)$  for all  $\alpha \in R$  by a new technique of computing fractional operations of distributions. We produce some novel results that cannot be derived in the ordinary sense.

## 2. Fractional Calculus of Distributions in $\mathcal{D}'(R^+)$

In order to investigate Abel’s integral equation in the generalized sense, we introduce the following basic concepts in detail. We let  $\mathcal{D}(R)$  be the Schwartz space [24] of infinitely differentiable functions with compact support in  $R$  and  $\mathcal{D}'(R)$  be the space of distributions defined on  $\mathcal{D}(R)$ . Further, we define a sequence  $\phi_1(x), \phi_2(x), \dots, \phi_n(x), \dots$ , which converges to zero in  $\mathcal{D}(R)$  if all these functions vanish outside a certain fixed bounded interval and converges uniformly to zero (in the usual sense) together with their derivatives of any order. The functional  $\delta(x)$  is defined as

$$(\delta, \phi) = \phi(0),$$

where  $\phi \in \mathcal{D}(R)$ . Clearly,  $\delta$  is a linear and continuous functional on  $\mathcal{D}(R)$ , and hence  $\delta \in \mathcal{D}'(R)$ . We let  $\mathcal{D}'(R^+)$  be the subspace of  $\mathcal{D}'(R)$  with support contained in  $R^+$ .

We define

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then

$$(\theta(x), \phi(x)) = \int_0^\infty \phi(x) dx \quad \text{for } \phi \in \mathcal{D}(R),$$

which implies  $\theta(x) \in \mathcal{D}'(R)$ .

We let  $f \in \mathcal{D}'(R)$ . The distributional derivative of  $f$ , denoted by  $f'$  or  $df/dx$ , is defined as

$$(f', \phi) = -(f, \phi')$$

for  $\phi \in \mathcal{D}(R)$ .

Clearly,  $f' \in \mathcal{D}'(R)$  and every distribution has a derivative. As an example, we show that  $\theta'(x) = \delta(x)$ , although  $\theta(x)$  is not defined at  $x = 0$ . Indeed,

$$(\theta'(x), \phi(x)) = -(\theta(x), \phi'(x)) = -\int_0^\infty \phi'(x) dx = \phi(0) = (\delta(x), \phi(x)),$$

which claims

$$\theta'(x) = \delta(x).$$

It can be shown that the ordinary rules of differentiation apply also to distributions. For instance, the derivative of a sum is the sum of the derivatives, and a constant can be commuted with the derivative operator.

It follows from [24–26] that  $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathcal{D}'(R^+)$  is an entire function of  $\lambda$  on the complex plane, and

$$\left. \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \right|_{\lambda=-n} = \delta^{(n)}(x) \quad \text{for } n = 0, 1, 2, \dots \tag{1}$$

Clearly, the Laplace transform of  $\Phi_\lambda$  is given by

$$\mathcal{L}\{\Phi_\lambda(x)\} = \int_0^\infty e^{-sx} \Phi_\lambda(x) dx = \frac{1}{s^\lambda}, \quad \text{Re } \lambda > 0, \quad \text{Res} > 0,$$

which plays an important role in solving integral equations [27,28].

For the functional  $\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}$ , the derivative formula is simpler than that for  $x_+^\lambda$ . In fact,

$$\frac{d}{dx} \Phi_\lambda = \frac{d}{dx} \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{(\lambda-1)x_+^{\lambda-2}}{\Gamma(\lambda)} = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)} = \Phi_{\lambda-1}. \tag{2}$$

The convolution of certain pairs of distributions is usually defined as follows (see Gel'fand and Shilov [24], for example):

**Definition 1.** Let  $f$  and  $g$  be distributions in  $\mathcal{D}'(\mathbb{R})$  satisfying either of the following conditions:

- (a) Either  $f$  or  $g$  has bounded support (set of all essential points), or
- (b) the supports of  $f$  and  $g$  are bounded on the same side.

Then the convolution  $f * g$  is defined by the equation

$$((f * g)(x), \phi(x)) = (g(x), (f(y), \phi(x + y)))$$

for  $\phi \in \mathcal{D}(\mathbb{R})$ .

The classical definition of the convolution is as follows:

**Definition 2.** If  $f$  and  $g$  are locally integrable functions, then the convolution  $f * g$  is defined by

$$(f * g)(x) = \int_{-\infty}^\infty f(t)g(x - t)dt = \int_{-\infty}^\infty f(x - t)g(t)dt$$

for all  $x$  for which the integrals exist.

We note that if  $f$  and  $g$  are locally integrable functions satisfying either of the conditions (a) or (b) in Definition 1, then Definition 1 is in agreement with Definition 2. It also follows that if the convolution  $f * g$  exists by Definition 1 or 2, then the following equations hold:

$$f * g = g * f \tag{3}$$

$$(f * g)' = f * g' = f' * g \tag{4}$$

where all the derivatives above are in the distributional sense.

We let  $\lambda$  and  $\mu$  be arbitrary complex numbers. Then it is easy to show that

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu} \tag{5}$$

by Equation (2), without any help of analytic continuation mentioned in all current books.

We let  $\lambda$  be an arbitrary complex number and  $g(x)$  be the distribution concentrated on  $x \geq 0$ . We define the primitive of order  $\lambda$  of  $g$  as a convolution in the distributional sense:

$$g_\lambda(x) = g(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = g(x) * \Phi_\lambda. \tag{6}$$

We note that the convolution on the right-hand side is well defined, as supports of  $g$  and  $\Phi_\lambda$  are bounded on the same side.

Thus Equation (6) with various  $\lambda$  will not only give the fractional derivatives but also the fractional integrals of  $g(x) \in \mathcal{D}'(R^+)$  when  $\lambda \notin Z$ , and it reduces to integer-order derivatives or integrals when  $\lambda \in Z$ . We define the convolution

$$g_{-\lambda} = g(x) * \Phi_{-\lambda}$$

as the fractional derivative of the distribution  $g(x)$  with order  $\lambda$ , writing it as

$$g_{-\lambda} = \frac{d^\lambda}{dx^\lambda} g$$

for  $\text{Re}\lambda \geq 0$ . Similarly,  $\frac{d^\lambda}{dx^\lambda} g$  is interpreted as the fractional integral if  $\text{Re}\lambda < 0$ .

In 1996, Matignon [26] studied fractional derivatives in the sense of distributions for fractional differential equations with applications to control processing and defined the fractional derivative of order  $\lambda$  of a continuous causal (zero for  $t < 0$ ) function  $g$  as

$$\frac{d^\lambda}{dx^\lambda} g = g(x) * \Phi_{-\lambda},$$

which is a special case of Equation (6), as  $g$  belongs to  $\mathcal{D}'(R^+)$ . A very similar definition for the fractional derivatives of causal functions (not necessarily continuous) is given by Mainardi in [27].

As an example of finding a fractional derivative of a distribution, we let  $g(x) \in \mathcal{D}'(R^+)$  be given by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is irrational and positive,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the ordinary derivative of  $g(x)$  does not exist. However, the distributional derivative of  $g(x)$  does exist, and  $g'(x) = \delta(x)$  on the basis of the following:

$$(g'(x), \phi(x)) = -(g(x), \phi'(x)) = -\int_0^\infty \phi'(x) dx = \phi(0) = (\delta(x), \phi(x)),$$

as the measure of rational numbers is zero. Therefore,

$$\frac{d^{1.5}}{dx^{1.5}} g(x) = \frac{d^{0.5}}{dx^{0.5}} \delta(x) = \frac{x_+^{-1.5}}{\Gamma(-0.5)} = -\frac{1}{2\sqrt{\pi}} x_+^{-1.5}.$$

We note that the sequential fractional derivative holds in a distribution [26].

For a given function, its classical Riemann–Liouville derivative and/or Caputo derivative [29–31] may not exist in general [32–34]. Even if they do, the Riemann–Liouville derivative and the Caputo derivative are not necessarily the same. However, if  $g(x)$  is a distribution in  $\mathcal{D}'(R^+)$ , then the case is different. Let  $m - 1 < \text{Re}\lambda < m \in Z^+$ . From Equation (4), we derive that

$$\begin{aligned} g_{-\lambda}(x) &= g(x) * \frac{x_+^{-\lambda-1}}{\Gamma(-\lambda)} = g(x) * \frac{d^m}{dx^m} \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \\ &= \frac{d^m}{dx^m} \left( g(x) * \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} \right) = \frac{x_+^{m-\lambda-1}}{\Gamma(m-\lambda)} * g^{(m)}(x), \end{aligned}$$

which indicates there is no difference between the Riemann–Liouville derivative and the Caputo derivative of the distribution  $g(x)$  (both exist clearly). On the basis of this fact, we only call the fractional derivative of the distribution for brevity.

We mention that Podlubny [28] investigated fractional calculus of generalized functions by the distributional convolution and derived many identities of fractional integrals and derivatives related to  $\delta(x - a)$ ,  $\theta(x - a)$  and  $\Phi_\lambda$ , where  $a$  is a constant in  $R$ .

We note that the fractional integral (or the Riemann–Liouville fractional integral)  $D^{-\alpha}$  ( $= D_{0,t}^{-\alpha}$ ) of order  $\alpha \in R^+$  of function  $y(t)$  is defined by

$$D^{-\alpha}y(t) = D_{0,t}^{-\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1}y(\tau)d\tau \quad (t > 0)$$

if the integral exists.

### 3. Babenko’s Approach with Demonstration

We consider Abel’s integral equation of the second kind:

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = f(t), \quad t > 0, \tag{7}$$

where we start with  $\alpha > 0$  and where  $\lambda$  is a constant. This equation was initially introduced and investigated by Hille and Tamarkin [35] in 1930, who considered Volterra’s equation (a more general integral equation):

$$y(t) = f(t) + \lambda \int_0^t K(t, \tau)y(\tau)d\tau$$

by the Liouville–Neumann series. In 1997, Gorenflo and Mainardi studied Abel’s integral equations of the first and second kind in their survey paper [36], with emphasis on the method of the Laplace transforms, which is a common treatment of such fractional integral equations. They also outlined the method used by Yu. I. Babenko in his book [37] for solving various types of fractional integral and differential equations. The method itself is close to the Laplace transform method, but it can be used in more cases than the Laplace transform method, such as solving integral equations with variable coefficients. Clearly, it is always necessary to prove convergence of the series obtained as solutions, although it is not a simple task in the general case [28].

In this section, we apply Babenko’s method to solve many Abel’s integral equations of the second kind, as well as their variants with variable coefficients, and we show convergence of the infinite series by utilizing the rapid growth of the Gamma function. We note that if an infinite series is uniformly convergent in every bounded interval of the variable  $t$ , then term-wise integrations and differentiations are allowed.

We can write Equation (7) in the form

$$(1 + \lambda D^{-\alpha})y(t) = f(t)$$

by the Riemann–Liouville fractional integral operator. This implies that

$$y(t) = (1 + \lambda D^{-\alpha})^{-1}f(t) = \sum_{n=0}^{\infty} (-1)^n \lambda^n D^{-\alpha n} f(t) \tag{8}$$

by Babenko’s method [28].

Because, for many functions  $f(t)$ , all the fractional integrals on the right-hand side of Equation (8) can be evaluated, Equation (8) gives the formal solution in the form of a series if it converges.

A two-parameter function of the Mittag–Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where  $\alpha, \beta > 0$ .

It follows from the Mittag-Leffler function that

$$\begin{aligned}
 E_{1,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \\
 E_{1,2}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z} \\
 E_{2,1}(z^2) &= \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z).
 \end{aligned}$$

**Demonstration of Examples**

**Example 1.** Let  $\lambda$  and  $a$  be constants and  $\alpha > 0$ . Then Abel’s integral equation:

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = at^\beta \tag{9}$$

has the solution

$$y(t) = a\Gamma(\beta + 1)t^\beta E_{\alpha,\beta+1}(-\lambda t^\alpha),$$

where  $\beta > -1$ .

**Proof.** Clearly, we have

$$D^{-\alpha n} at^\beta = aD^{-\alpha n} t^\beta = \frac{a\Gamma(\beta + 1)}{\Gamma(\alpha n + \beta + 1)} t^{\beta+\alpha n}.$$

Hence

$$\begin{aligned}
 y(t) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n D^{-\alpha n} at^\beta = \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{a\Gamma(\beta + 1)}{\Gamma(\alpha n + \beta + 1)} t^{\beta+\alpha n} \\
 &= a\Gamma(\beta + 1)t^\beta \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{t^{\alpha n}}{\Gamma(\alpha n + \beta + 1)} \\
 &= a\Gamma(\beta + 1)t^\beta E_{\alpha,\beta+1}(-\lambda t^\alpha),
 \end{aligned}$$

which is convergent for all  $t \in R^+$ . □

This example implies that the following Abel’s integral equation:

$$y(t) = 2\sqrt{t} - \int_0^t \frac{y(\tau)}{\sqrt{t - \tau}} d\tau \tag{10}$$

has the solution

$$y(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (\pi t)^{n/2}}{\Gamma(n/2 + 1)}.$$

Indeed, Equation (10) can be converted into

$$y(t) + \frac{\sqrt{\pi}}{\Gamma(1/2)} \int_0^t \frac{y(\tau)}{\sqrt{t - \tau}} d\tau = 2\sqrt{t}.$$

By Equation (9) we come to

$$\begin{aligned}
 y(t) &= 2\Gamma(1/2 + 1)t^{1/2}E_{1/2,3/2}(-\sqrt{\pi}t^{1/2}) = \sum_{n=0}^{\infty} \frac{(-1)^n(\pi t)^{(n+1)/2}}{\Gamma(n/2 + 1/2 + 1)} \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\pi t)^{n/2}}{\Gamma(n/2 + 1)}
 \end{aligned}$$

using

$$\Gamma(1/2 + 1) = \frac{\sqrt{\pi}}{2}.$$

**Example 2.** Abel’s integral equation:

$$y(t) + \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = t + \frac{4}{3}t^{3/2}$$

has the solution  $y(t) = t$ .

**Proof.** First we note that

$$\begin{aligned}
 \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau &= \frac{\sqrt{\pi}}{\Gamma(1/2)} \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau, \\
 D^{-n/2}t &= \frac{t^{(n+2)/2}}{\Gamma(n/2 + 2)}, \quad \text{and} \\
 D^{-n/2}\frac{4t^{3/2}}{3} &= \frac{4}{3} \frac{\Gamma(3/2 + 1)}{\Gamma(n/2 + 3/2 + 1)} t^{(n+3)/2} = \frac{\sqrt{\pi}t^{(n+3)/2}}{\Gamma((n + 3)/2 + 1)}.
 \end{aligned}$$

We infer from Equation (8) that

$$\begin{aligned}
 y(t) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n D^{-\alpha n} (t + \frac{4}{3}t^{3/2}) = \lim_{m \rightarrow \infty} \sum_{n=0}^m (-1)^n \pi^{n/2} D^{-n/2} (t + \frac{4}{3}t^{3/2}) \\
 &= \lim_{m \rightarrow \infty} \left\{ (t + \frac{4}{3}t^{3/2}) - \sqrt{\pi} \left( \frac{t^{3/2}}{\Gamma(1/2 + 2)} + \frac{\sqrt{\pi}}{\Gamma(2 + 1)} t^2 \right) + \right. \\
 &\quad \left. \pi \left( \frac{t^2}{2} + \frac{4}{3} \frac{t^{3/2+1}}{3/2 + 1} \right) + \dots + (-1)^m \pi^{m/2} D^{-m/2} (t + \frac{4}{3}t^{3/2}) \right\} \\
 &= t + \lim_{m \rightarrow \infty} (-1)^m \pi^{m/2} D^{-m/2} \frac{4}{3} t^{3/2}
 \end{aligned}$$

by cancellations. Furthermore,

$$\lim_{m \rightarrow \infty} (-1)^m \pi^{m/2} D^{-m/2} \frac{4}{3} t^{3/2} = 0$$

for all  $t > 0$ . Indeed,

$$\begin{aligned}
 (-1)^m \pi^{m/2} D^{-m/2} \frac{4}{3} t^{3/2} &= \frac{4}{3} (-1)^m \pi^{m/2} \frac{\Gamma(3/2 + 1)}{\Gamma(m/2 + 3/2 + 1)} t^{3/2+m/2} \\
 &= \sqrt{\pi} t^{3/2} \frac{(-1)^m (\pi t)^{m/2}}{\Gamma(m/2 + 3/2 + 1)},
 \end{aligned}$$

and the series

$$\sum_{m=0}^{\infty} \frac{(\pi t)^{m/2}}{\Gamma(m/2 + 3/2 + 1)} = E_{1/2,3/2+1}(\sqrt{\pi}t)$$

is convergent. Hence  $y(t) = t$  is the solution.  $\square$

**Remark 1.** Kumar et al. [22] considered Example 2 by applying the aforesaid HPTM and found an approximate solution that converges to the solution  $y(t) = t$  for  $t$  only between 0 and 1, with more calculations involving Laplace transforms.

**Example 3.** Abel’s integral equation:

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = \phi(t)$$

has the solution

$$y(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \lambda^n \frac{\phi^{(k)}(0)}{\Gamma(\alpha n + k + 1)} t^{k+\alpha n},$$

where

$$\phi(t) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} t^k$$

for  $t > 0$ .

**Proof.** By Equation (8), we obtain

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n D^{-\alpha n} \phi(t) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \lambda^n \frac{\phi^{(k)}(0)}{k!} D^{-\alpha n} t^k \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \lambda^n \frac{\phi^{(k)}(0)}{\Gamma(\alpha n + k + 1)} t^{k+\alpha n}. \end{aligned}$$

It remains to show that the above series is convergent. We note that

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n + \alpha)}{\Gamma(n) n^\alpha} = 1$$

implies

$$\Gamma(\alpha n + k + 1) \sim \Gamma(k + 1) n^{\alpha n}$$

when  $k$  is large [38]. Therefore, the convergence of

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \lambda^n \frac{\phi^{(k)}(0)}{\Gamma(\alpha n + k + 1)} t^{k+\alpha n}$$

is equivalent to that of

$$\sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} t^k \sum_{n=0}^{\infty} (-1)^n \left( \frac{\lambda t^\alpha}{n^\alpha} \right)^n,$$

which is convergent because it is the product of two convergent series for all  $t > 0$ .

This clearly implies that the following Abel’s integral equation:

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = e^t$$



has the solution

$$y(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \lambda^n \frac{t^{k+\alpha n}}{\Gamma(\alpha n + k + 1)},$$

and Abel’s integral equation:

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = \frac{t}{t^2 + b^2}$$

has the solution

$$y(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n \lambda^n \frac{(-1)^k (2k + 1)!}{b^{2k+2} \Gamma(\alpha n + 2k + 2)} t^{\alpha n + 2k + 1}$$

for  $0 < t < b$ . Indeed,

$$\begin{aligned} \phi(t) &= \frac{t}{t^2 + b^2} = \frac{t}{b^2 \left(1 + \left(\frac{t}{b}\right)^2\right)} \\ &= \frac{t}{b^2} \sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{b}\right)^{2k} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{b^{2k+2}}, \end{aligned}$$

which infers that

$$\phi^{(2k+1)}(0) = \frac{(-1)^k (2k + 1)!}{b^{2k+2}} \quad \text{and} \quad \phi^{(2k)}(0) = 0.$$

Now we consider some variants of Abel’s integral equation.  $\square$

**Example 4.** Let  $\alpha > 0$ . Then the integral equation

$$y(t) + \frac{\lambda t}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = t$$

has the solution

$$y(t) = tE_{\alpha,2}(-\lambda t^{\alpha+1}).$$

**Proof.** We can write the original equation in the form

$$(1 + \lambda t D^{-\alpha})y(t) = t.$$

Therefore

$$\begin{aligned} y(t) &= (1 + \lambda t D^{-\alpha})^{-1} t = \sum_{n=0}^{\infty} (-1)^n \lambda^n t^n D^{-\alpha n} t \\ &= \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{t^n}{\Gamma(\alpha n + 2)} t^{\alpha n + 1} \\ &= t \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{t^{\alpha n + n}}{\Gamma(\alpha n + 2)} = tE_{\alpha,2}(-\lambda t^{\alpha+1}). \end{aligned}$$

Similarly, the integral equation

$$y(t) + \frac{\lambda t^\beta}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = t^\gamma \quad \text{for } t > 0$$

has the solution

$$y(t) = \Gamma(\gamma + 1) t^\gamma E_{\alpha,\gamma+1}(-\lambda t^{\alpha+\beta}), \tag{11}$$

where  $\gamma > -1$  and  $\alpha + \beta > 0$ .

Furthermore, the integral equation for  $\beta > -1$ :

$$y(t) - t^\beta \int_0^t y(\tau) d\tau = 1$$

has the solution  $y(t) = e^{t^{\beta+1}}$  by noting that  $E_{1,1}(t) = e^t$ .

Clearly, the integral equation

$$y(t) - \int_0^t (t - \tau)y(\tau) d\tau = t$$

has the solution  $y(t) = tE_{2,2}(t^2) = \sinh(t)$  by noting that  $E_{2,2}(t^2) = \sinh(t)/t$ , and the integral equation

$$y(t) - \int_0^t (t - \tau)y(\tau) d\tau = 1$$

has the solution  $y(t) = \cosh(t)$  as  $E_{2,1}(t^2) = \cosh(t)$ .

Using the following formula [28]:

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right\},$$

we derive that the following integral equation for  $m = 1, 2, \dots, \beta > -1$ , and  $\lambda \neq 0$ :

$$y(t) + \lambda t^\beta \int_0^t y(\tau) d\tau = t^m$$

has the solution

$$y(t) = \frac{(-1)^m m!}{\lambda^m t^{\beta m}} \left\{ e^{-\lambda t^{\beta+1}} - \sum_{k=0}^{m-1} \frac{(-\lambda)^k t^{(\beta+1)k}}{k!} \right\}.$$

Indeed, from Equation (11), we infer that the solution is

$$\begin{aligned} y(t) &= m! t^m E_{1,m+1}(-\lambda t^{\beta+1}) \\ &= \frac{(-1)^m m!}{\lambda^m t^{\beta m}} \left\{ e^{-\lambda t^{\beta+1}} - \sum_{k=0}^{m-1} \frac{(-\lambda)^k t^{(\beta+1)k}}{k!} \right\}, \end{aligned}$$

as  $\gamma = m$  and  $\alpha = 1$ .

Finally, we claim that the following integral equation for  $\beta > -1/2$ :

$$y(t) + \frac{\lambda t^\beta}{\sqrt{\pi}} \int_0^t \frac{y(\tau)}{\sqrt{t - \tau}} d\tau = 1$$

has the solution  $y(t) = E_{1/2,1}(-\lambda t^{\beta+1/2})$ , where

$$E_{1/2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k/2 + 1)} = e^{z^2} \operatorname{erfc}(-z)$$

and  $\operatorname{erfc}(z)$  is the error function complement defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt.$$

To end this section, we point out that some exact solutions of equations considered in the examples can be easily obtained using the Laplace transform mentioned earlier; for example, applying the Laplace transform to Equation (9) from Example 1, we obtain

$$y^*(s) = \mathcal{L}\{y(t)\} = a\Gamma(\beta + 1) \frac{s^{\alpha-(\beta+1)}}{s^\alpha + \lambda}$$

by the formula

$$\int_0^\infty t^\beta e^{-st} dt = \frac{\Gamma(\beta + 1)}{s^{\beta+1}}, \quad \text{Res} > 0.$$

This implies the following by the inverse Laplace transform and Equation (1.80) in [28]:

$$y(t) = a\Gamma(\beta + 1)t^\beta E_{\alpha,\beta+1}(-\lambda t^\alpha).$$

Similarly, for the equation from Example 2, we immediately derive

$$y^*(s) = \mathcal{L}\{y(t)\} = \frac{1}{s^2}$$

from

$$\mathcal{L}\{t + \frac{4}{3}t^{3/2}\} = \frac{1}{s^2} + \frac{\sqrt{\pi}}{s^{2.5}},$$

which indicates that  $y(t) = t$ . However, it seems much harder for the Laplace transform to deal with the variants of Abel’s integral equation with variable coefficients, such as

$$y(t) + \lambda t^\beta \int_0^t y(\tau) d\tau = t^m,$$

which is solved above by Babenko’s approach. □

#### 4. Abel’s Integral Equation in Distributions

We let  $f(t) \in \mathcal{D}'(R^+)$  be given. We now study Abel’s integral equation of the second kind:

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} y(\tau) d\tau = f(t) \tag{12}$$

in the distributional space  $\mathcal{D}'(R^+)$ , where  $\alpha \in R$  and  $\lambda$  is a constant.

Clearly, Equation (12) is equivalent to the convolutional equation:

$$(\delta + \lambda\Phi_\alpha) * y(t) = f(t)$$

in  $\mathcal{D}'(R^+)$ , although it is undefined in the classical sense for  $\alpha \leq 0$ . We note that the distributional convolution  $\Phi_\alpha * y(t)$  is well defined for arbitrary  $\alpha \in R$  if  $y(t) \in \mathcal{D}'(R^+)$  by Definition 1 (b).

Applying Babenko’s method, we can write out  $y(t)$  as

$$y(t) = (\delta + \lambda\Phi_\alpha)^{-1} * f(x),$$

where  $(\delta + \lambda\Phi_\alpha)^{-1}$  is the distributional inverse operator of  $\delta + \lambda\Phi_\alpha$  in terms of convolution. Using the binomial expression of  $(\delta + \lambda\Phi_\alpha)^{-1}$ , we can formally obtain

$$y(t) = \sum_{n=0}^\infty (-1)^n \lambda^n \Phi_\alpha^n * f(t) = \sum_{n=0}^\infty (-1)^n \lambda^n \Phi_{n\alpha} * f(t) \tag{13}$$

by using the following formula [25]:

$$\Phi_\alpha^n = \underbrace{\Phi_\alpha * \Phi_\alpha * \dots * \Phi_\alpha}_{n \text{ times}} = \Phi_{n\alpha},$$

and  $\delta$  is a unit distribution in terms of convolution.

Because, for many distributions  $f(t)$ , all the fractional integrals (if  $\alpha > 0$ ) or derivatives (if  $\alpha < 0$ ) on the right-hand side of Equation (13) can be evaluated, Equation (13) gives the formal solution in the form of a series if it converges. We must point out that Equation (12) becomes the differential equation:

$$y + \lambda y * \delta^{(m)} = y + \lambda y^{(m)} = f$$

if  $\alpha = -m$  (undefined in the classical sense) because of Equation (1), which can be converted into a system of linear differential equations for consideration [24].

**Example 5.** Abel’s integral equation:

$$y(t) + \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = t_+^{-1.5} \tag{14}$$

has its solution in the space  $\mathcal{D}'(R^+)$ :

$$y(t) = t_+^{-1.5} + 2\pi\delta(t) - 2\pi t_+^{-1/2} + 2\pi^2 E_{1/2,1}(-\sqrt{\pi t_+}).$$

**Proof.** Equation (14) can be written as

$$y(t) + \frac{\sqrt{\pi}}{\Gamma(1/2)} \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = t_+^{-1.5}.$$

From Equation (13),

$$\begin{aligned} y(t) &= \sum_{n=0}^\infty (-1)^n \pi^{n/2} \Phi_{1/2}^n * t_+^{-1.5} = \sum_{n=0}^\infty (-1)^n \pi^{n/2} \Phi_{n/2} * \frac{t_+^{-1.5}}{\Gamma(-0.5)} \Gamma(-0.5) \\ &= \sum_{n=0}^\infty (-1)^n \pi^{n/2} \Gamma(-0.5) \Phi_{n/2} * \Phi_{-0.5} = \sum_{n=0}^\infty (-1)^n \pi^{n/2} \Gamma(-0.5) \Phi_{n/2-0.5} \\ &= \sum_{n=0}^\infty (-1)^n \pi^{n/2} \Gamma(-0.5) \frac{t_+^{n/2-1.5}}{\Gamma(n/2-0.5)} \\ &= t_+^{-1.5} + 2\pi\delta(t) - 2\pi t_+^{-1/2} + 2 \sum_{n=3}^\infty (-1)^{n+1} \pi^{(n+1)/2} \frac{t_+^{n/2-1.5}}{\Gamma(n/2-0.5)} \\ &= t_+^{-1.5} + 2\pi\delta(t) - 2\pi t_+^{-1/2} + 2\pi^2 E_{1/2,1}(-\sqrt{\pi t_+}) \end{aligned}$$

using  $\Gamma(-0.5) = -2\sqrt{\pi}$ .  $\square$

**Remark 2.** Equation (14) cannot be discussed in the classical sense, because the fractional integral of  $t_+^{-1.5}$  does not exist in the normal sense. Clearly, the distribution  $t_+^{-1.5} + 2\pi\delta(t)$  is a singular generalized function in  $\mathcal{D}'(R^+)$ , while  $-2\pi t_+^{-1/2} + 2\pi^2 E_{1/2,1}(-\sqrt{\pi t_+})$  is regular (locally integrable).

**Example 6.** For  $m = 0, 1, 2, \dots$ , the integral equation

$$y(t) + t^m \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = \delta(t)$$

has the solution

$$y(t) = \delta(t) - t_+^{m-1/2} + \pi t_+^{2m} E_{1/2,1}(-\sqrt{\pi} t_+^{m+1/2}).$$

**Proof.** Clearly,

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (-1)^n \pi^{n/2} t_+^{mn} \Phi_{n/2} * \delta(t) = \sum_{n=0}^{\infty} (-1)^n \pi^{n/2} t_+^{mn} \Phi_{n/2} \\ &= \delta(t) - \pi^{1/2} t_+^m \frac{t_+^{-1/2}}{\Gamma(1/2)} + \sum_{n=2}^{\infty} (-1)^n \pi^{n/2} t_+^{mn} \Phi_{n/2} \\ &= \delta(t) - t_+^{m-1/2} + \sum_{n=2}^{\infty} (-1)^n \pi^{n/2} \frac{t_+^{mn-1+n/2}}{\Gamma(n/2)} \\ &= \delta(t) - t_+^{m-1/2} + \pi t_+^{2m} \sum_{k=0}^{\infty} (-1)^k \pi^{k/2} \frac{t_+^{mk+k/2}}{\Gamma(k/2+1)} \\ &= \delta(t) - t_+^{m-1/2} + \pi t_+^{2m} E_{1/2,1}(-\sqrt{\pi} t_+^{m+1/2}), \end{aligned}$$

which is a distribution in  $\mathcal{D}'(R^+)$ .

We note that if  $f$  is a distribution in  $\mathcal{D}'(R)$  and  $g$  is an infinitely differentiable function, then the product  $fg$  is defined by

$$(fg, \phi) = (f, g\phi)$$

for all functions  $\phi \in \mathcal{D}(R)$ . Therefore, the product

$$t^m \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau$$

is well defined, as  $t^m$  is an infinitely differentiable function.

In particular, Abel's integral equation:

$$y(t) + \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = \delta(t)$$

has the solution

$$y(t) = \delta(t) - t_+^{-1/2} + \pi E_{1/2,1}(-\sqrt{\pi} t_+),$$

and the integral equation

$$y(t) + t \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = \delta(t)$$

has the solution

$$y(t) = \delta(t) - t_+^{1/2} + \pi t_+^2 E_{1/2,1}(-\sqrt{\pi} t_+^{1+1/2}).$$

We mention that Abel's integral equations or integral equations with more general weakly singular kernels play important roles in solving partial differential equations, in particular, parabolic equations in which naturally the independent variable has the meaning of time. Gorenflo and Mainardi [36] described the occurrence of Abel's integral equations of the first and second kind in the problem of the heating (or cooling) of a semi-infinite rod by influx (or efflux) of heat across the boundary into (or from) its interior. They used Abel's integral equations to solve the following equation of heat flow:

$$u_t - u_{xx} = 0, \quad u = u(x, t)$$

in the semi-infinite intervals  $0 < x < \infty$  and  $0 < t < \infty$  of space and time, respectively. In this dimensionless equation,  $u = u(x, t)$  refers to temperature. Assuming a vanishing initial temperature, that is,  $u(x, 0) = 0$  for  $0 < x < \infty$ , and a given influx across the boundary  $x = 0$  from  $x < 0$  to  $x > 0$ ,

$$-u_x(0, t) = p(t).$$

The interested readers are referred to [36] for the detailed methods and further references.  $\square$

## 5. Conclusions

Applying Babenko's method and fractional calculus, we have studied Abel's integral equation of the second kind as well as its variants with variable coefficients, and we have solved many types of different integral equations by showing the convergence of infinite series using the rapid growth of the Gamma function. In particular, we have discussed Abel's equation in the distributional sense for any arbitrary  $\alpha \in \mathbb{R}$  by fractional operations of generalized functions for the first time, and we have derived several new and interesting results that cannot be realized in the classical sense or by the Laplace transform. For example, Abel's integral equation:

$$y(t) + \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} d\tau = t_+^{-\sqrt{2}} + \delta^{(m)}(t)$$

cannot be considered in the classical sense because of the term  $t_+^{-\sqrt{2}} + \delta^{(m)}(t)$ , but it is well defined and solvable in the distributional sense.

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