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Existence of Solution, Filippov's Theorem and Compactness of the Set of Solutions for a Third-Order Differential Inclusion with Three- Point Boundary Conditions

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Abstract: In this paper, we study a third-order differential inclusion with three-point boundary conditions. We prove the existence of a solution under convexity conditions on the multi-valued right-hand side; the proof is based on a nonlinear alternative of Leray-Schauder type. We also study the compactness of the set of solutions and establish some Filippov's- type results for this problem.

Keywords: differential inclusion; boundary value problem; fixed point theorem; selection theory; Filippov's Theorem

1. Introduction

Various aspects of the theory of third-order differential inclusions with boundary conditions attract the attention of many researchers (e.g., [1–10]).

In the present paper we study third-order differential inclusions of the form

$$-u'''(t) \in F(t, u(t)), \; t \in (0,1), \tag{1}$$

with boundary conditions of the form:

$$u'(0) = u'(1) = \alpha u(\eta), \; u(0) = \beta u(\eta), \tag{2}$$

where α, β, and η are constants in $\mathbb{R}$, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ a multi-valued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

This paper is a continuation of the work in [11], where the authors discussed the existence of solutions of the problem (1)–(2) when the multi-valued map F is nonconvex and lower semi-continuous.

The aim of our present paper is to provide some existence results for the problem (1)–(2) under assumptions of convexity and upper semi-continuity of the right-hand side. To this end, we use a nonlinear alternative of Leray-Shauder type, some hypothesis of Carathéodory type, and some facts of the selection theory. More exactly, we discuss the existence of solutions for the problem (1)–(2) when F is convex and upper semi-continuous and satisfies a Carathéodory condition. We also prove that the set of solutions is compact, and we end our results by presenting a Filippov's-type result concerning the existence of solutions to the considered problem. An illustrative example of a boundary value problem satisfying the mentioned conditions is also given.

The paper is divided into three sections. In the second section, we give some necessary background material. In Section 3, we prove our main results.

2. Preliminaries

In this section we introduce some notations, definitions, and preliminary facts which will be used in the remainder of the paper. Let $C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$, equipped with the norm

$$\|u\| = \sup\{|u(t)|, \text{ for all } t \in [0,1]\}.$$

We also denote the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable by $L^1([0,1],\mathbb{R})$, normed by

$$\|u\|_{L^1} = \int_0^1 |u(t)|\, dt.$$

By $AC^i([0,1],\mathbb{R})$ we denote the space of $i-$times differentiable functions $u : [0,1] \to \mathbb{R}$, whose i^{th} derivative, $u^{(i)}$is absolutely continuous.

Let (X,d) be a metric space induced from a normed space $(X,\|.\|)$. Denote $\mathcal{P}_0(X) = \{A \in \mathcal{P}(X) : A \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is closed}\}$, $\mathcal{P}_b(X) = \{A \in \mathcal{P}_0(X) : A \text{ is bounded}\}$, $\mathcal{P}_{comp}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is compact}\}$, and $\mathcal{P}_{cv}(X) = \{A \in \mathcal{P}_0(X) : A \text{ is convex}\}$.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$ and $d(b,A) = \inf_{a\in A} d(a,b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [12]).

Let E be a separable Banach space, Y a nonempty closed subset of E and $G : Y \to \mathcal{P}_{cl}(E)$ a multi-valued map. G is said to be upper semi-continuous $(u.s.c)$ at the point $y_0 \in Y$ if for every open $W \subseteq Y$ such that $G(y_0) \subset W$ there exists a neighborhood $V(y_0)$ of y_0 such that $G(V(y_0)) \subset W$. We say that G has a fixed point if there is $x \in Y$ such that $x \in G(x)$. G is also said to be completely continuous if $G(\Omega)$ is relatively compact for every $\Omega \in \mathcal{P}_b(Y)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is upper semi-continuous $(u.s.c)$ if and only if G has a closed graph; that is, $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply that $y_* \in G(x_*)$.

For more details on the multi-valued maps, see the books of Aubin and Cellina [13], Aubin and Frankowska [14], Deimling [15], Gorniewicz [16], and Hu and Papageorgiou [17].

We recall here some definitions and Lemmas needed below.

Definition 1. *A multi-valued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(1) $t \to F(t,u)$ *is measurable for each* $u \in \mathbb{R}$,
(2) $u \to F(t,u)$ *is upper semi-continuous for almost all* $t \in (0,1)$, *and further a Carathéodory function* F *is called* L^1-*Carathéodory if*
(3) *for each* $r > 0$, *there exists* $\Phi_r \in L^1((0,1),\mathbb{R}^+)$, *such that*

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leqslant \Phi_r(t)$$

for all $\|u\| \leq r$ *and for a.e.* $t \in (0,1)$.

For each $u \in C((0,1),\mathbb{R})$, define the set of selections of F by

$$S_{F,u} = \left\{v \in L^1((0,1),\mathbb{R}) : v(t) \in F(t,u(t)) \text{ for a.e. } t \in (0,1)\right\}.$$

Lemma 1 ([18]). *Let E be a Banach space, let $F : [0,T] \times E \to \mathcal{P}_{comp,cv}(E)$ be a L^1-Carathéodory multi-valued map, and let Θ be a linear continuous mapping from $L^1([0,1], E)$ to $C([0,1], E)$. Then, the operator*

$$\Theta \circ S_F : C([0,1], E) \to \mathcal{P}_{comp,cv}(C([0,1], E))$$

$$u \to (\Theta \circ S_F)(u) = \Theta(S_{F,u})$$

is a closed graph operator in $C([0,1], E) \times C([0,1], E)$.

Lemma 2 (See [16], Theorem 19.7). *Let X be a separable metric space and G a multi-valued map with nonempty closed values. Then, G has a measurable selection.*

Lemma 3 ([19]). *Assume $1 - \beta - \alpha\eta \neq 0$, then for $y \in C([0,1], \mathbb{R})$ the problem*

$$u'''(t) + y(t) = 0,\ t \in (0,1), \tag{3}$$

$$u'(0) = u'(1) = \alpha u(\eta),\ u(0) = \beta u(\eta), \tag{4}$$

where α, β, and η are constants with $\alpha \in \left[0, \frac{1}{\eta}\right)$, $0 < \eta < 1$, $\beta \neq 1 - \alpha\eta$, has a unique solution

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s)\,y(s)\,ds - \\ &\quad - \frac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\int_0^\eta (\eta - s)^2 y(s)\,ds. \end{aligned}$$

3. Main Results

3.1. Existence of Solutions

Before giving some results on the existence of solutions for the problem (1) and (2), let us introduce the following hypotheses which are assumed hereafter:

(H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{comp,cv}(\mathbb{R})$ is Carathéodory,
(H_2) there exists a function $p \in C([0,1], \mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|w| : w \in F(t,u)\} \leq p(t), \text{ for each } (t,u) \in [0,1] \times \mathbb{R}.$$

Theorem 1. *Assume that $(H_1), (H_2)$ hold, then the boundary value problem (1) and (2) has at least one solution on $[0,1]$.*

Proof. Define the operator $T : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ by

$$\begin{aligned} T(u) = &\Big\{h \in C([0,1], \mathbb{R}) : h(t) = -\tfrac{1}{2}\textstyle\int_0^t (t-s)^2 f(u)\,ds + \\ &+ \tfrac{1}{2}\left[t^2 + \eta^2 \tfrac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\textstyle\int_0^1 (1-s) f(u)\,ds - \\ &- \tfrac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\textstyle\int_0^\eta (\eta - s)^2 f(u)\,ds\Big\}, \end{aligned}$$

for $f \in \mathcal{S}_{F,u}$, we will show that T satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps.

Step 1: Let us begin by proving that T is convex for each $u \in C([0,1], \mathbb{R})$.

Let $h_1, h_2 \in Tu$. Then, there exist $w_1, w_2 \in \mathcal{S}_{F,u}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w_i(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w_i(s)\,ds - \\ &\quad -\frac{\alpha t + \beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 w_i(s)\,ds, \quad i = 1,2. \end{aligned}$$

Let $0 \le \mu \le 1$. So, for each $t \in [0,1]$, we have

$$\begin{aligned} \mu h_1(t) + (1-\mu)h_2(t) &= \frac{1}{2}\int_0^t (t-s)^2 (\mu w_1(s) + (1-\mu)w_2(s))\,ds + \\ &\quad + \frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right] \times \\ &\quad \times \int_0^1 (1-s)(\mu w_1(s) + (1-\mu)w_2(s))\,ds - \\ &\quad - \frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2(\mu w_1(s) + (1-\mu)w_2(s))\,ds. \end{aligned}$$

Since $\mathcal{S}_{F,u}$ is convex, it follows that $\mu h_1 + (1-\mu)h_2 \in Tu$.

Step 2: In this step, we prove that T maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

For a positive number r, let $B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \le r\}$ be a bounded ball in $C([0,1],\mathbb{R})$. So, for each $h \in Tu$, $u \in B_r$, there exists $w \in \mathcal{S}_{F,u}$ such that

$$\begin{aligned} h(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w(s)\,ds - \\ &\quad -\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 w(s)\,ds, \end{aligned}$$

$$\begin{aligned} |h(t)| &\le \left(1 + \eta^2\frac{\alpha+|\beta|}{2|1-\alpha\eta-\beta|}\right)\int_0^1 p(s)\,ds + \frac{\alpha+|\beta|}{2|1-\alpha\eta-\beta|}\int_0^\eta p(s)\,ds, \\ &= R. \end{aligned}$$

So,

$$\|h\|_\infty \leqslant R.$$

Step 3: Here we verify that T maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.

Let $t_1, t_2 \in [0,1]$, with $t_1 < t_2$ and B_r be a bounded set of $C([0,1],\mathbb{R})$. So, for each $h \in Tu$, we obtain

$$\begin{aligned} |h(t_2) - h(t_1)| &\le \frac{1}{2}\int_{t_1}^{t_2} (t_2-s)^2 |w(s)|\,ds + \\ &\quad + \frac{1}{2}\left[\left(t_2^2 - t_1^2\right) + \eta^2\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\right]\int_0^1 (1-s)\,|w(s)|\,ds + \\ &\quad + \frac{\alpha(t_2-t_1)}{2|1-\alpha\eta-\beta|}\int_0^\eta (\eta-s)^2 |w(s)|\,ds + \\ &\quad + \frac{1}{2}\int_0^{t_1}\left((t_1-s)^2 - (t_2-s)^2\right)|w(s)|\,ds, \end{aligned}$$

$$
\begin{aligned}
\leq\; & \frac{1}{2}\int_{t_1}^{t_2}(t_2-s)^2\,p(s)\,ds+\\
&+\frac{1}{2}\left[\left(t_2^2-t_1^2\right)+\eta^2\frac{\alpha(t_2-t_1)}{|1-\alpha\eta-\beta|}\right]\int_0^1(1-s)\,p(s)\,ds+\\
&+\frac{\alpha(t_2-t_1)}{2|1-\alpha\eta-\beta|}\int_0^\eta(\eta-s)^2\,p(s)\,ds+\\
&+\frac{1}{2}\int_0^{t_1}\left((t_1-s)^2-(t_2-s)^2\right)p(s)\,ds.
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently from $u\in B_r$ as $t_2-t_1\to 0$. As T satisfies the above three assumptions, it follows by Ascoli–Arzela's theorem that $T:C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

Step 4: In this step we prove that T has a closed graph.

Let $u_n\to u_*$, $h_n\in T(u_n)$ and $h_n\to h_*$. Then, we need to show that $h_*\in Tu_*$.
Associated with $h_n\in T(u_n)$, there exists $w_n\in\mathcal{S}_{F,u_n}$ such that for each $t\in[0,1]$,

$$
\begin{aligned}
h_n(t) \;=\; & -\frac{1}{2}\int_0^t(t-s)^2\,w_n(s)\,ds+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)\,w_n(s)\,ds-\\
&-\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta(\eta-s)^2\,w_n(s)\,ds.
\end{aligned}
$$

So, we have to show that there exists $w_*\in\mathcal{S}_{F,u_*}$ such that for each $t\in[0,1]$,

$$
\begin{aligned}
h_*(t) \;=\; & -\frac{1}{2}\int_0^t(t-s)^2\,w_*(s)\,ds+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)\,w_*(s)\,ds-\\
&-\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta(\eta-s)^2\,w_*(s)\,ds.
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta:L^1([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ given by

$$
\begin{aligned}
w \;\to\; & \Theta w(t)=-\frac{1}{2}\int_0^t(t-s)^2\,w(s)\,ds+\\
&+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)\,w(s)\,ds-\\
&-\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta(\eta-s)^2\,w(s)\,ds.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|h_n(t)-h_*(t)\| \;=\; & \left\|-\frac{1}{2}\int_0^t(t-s)^2\,(w_n(s)-w_*(s))\,ds+\right.\\
&+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1(1-s)\,(w_n(s)-w_*(s))\,ds-\\
&\left.-\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta(\eta-s)^2\,(w_n(s)-w_*(s))\,ds\right\|,
\end{aligned}
$$

then $\|h_n(t)-h_*(t)\|\to 0$ as $n\to\infty$.
So, it follows from Lemma 1 that $\Theta\circ S_F$ is a closed graph operator.

Further, we have $h_n(t) \in \Theta(S_{F,u_n})$. Since $u_n \to u_*$, therefore,

$$\begin{aligned} h_*(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w_*(s)\,ds + \frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w_*(s)\,ds - \\ &\quad -\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 w_*(s)\,ds, \end{aligned}$$

for some $w_* \in S_{F,u_*}$.

Step 5: We end our proof by discussing an a priori bounds on solutions.

Let u be a solution of (1) and (2). So, there exists $w \in L^1([0,1],\mathbb{R})$ with $w \in S_{F,u}$ such that for $t \in [0,1]$, we have

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds + \frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,w(s)\,ds - \\ &\quad -\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 w(s)\,ds. \end{aligned}$$

In view of (H_2), for each $t \in [0,1]$, we obtain

$$|u(t)| \le \left[1+\eta^2\frac{\alpha+|\beta|}{|1-\alpha\eta-\beta|}\right]\int_0^1 p(s)\,ds = L.$$

Let us set

$$U = \{u \in C([0,1],\mathbb{R}) : \|u\| < L+1\}.$$

Note that the operator $T : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda T x$ for some $\lambda \in (0,1)$.

Consequently, by the nonlinear alternative of Leray-Schauder type (see [20]), we deduce that T has a fixed point $u \in \overline{U}$ which is a solution of the problem (1) and (2). This completes the proof. □

Example 1. *Consider the boundary value problem given by*

$$-u'''(t) \in \left[-u^2\log(t+2)+1, t\frac{e^u}{1+e^u}+2\right],\ t \in (0,1), \tag{5}$$

$$u'(0) = u'(1) = u\left(\tfrac{1}{5}\right),\ \ u(0) = -u\left(\tfrac{1}{5}\right), \tag{6}$$

where $\alpha = 1, \beta = -1, \eta = \frac{1}{5}$. *and* $F(t,u(t)) = \left[-u^2\log(t+2)+1, t\frac{e^u}{1+e^u}+2\right]$.

For $f \in F$, we have

$$|f| \leqslant \max\left(-u^2\log(t+2)+1, t\frac{e^u}{1+e^u}+2\right) \leqslant 3,\ u \in \mathbb{R}.$$

Applying Theorem 1, we get that F is a Carathéodory multi-valued map and there exists a function $p(t) \in C([0,1],\mathbb{R}^+)$ $\|F(t,u)\|_{\mathcal{P}} \le p(t)$ for each $(t,u) \in [0,1]\times\mathbb{R}$, where $p(t) = 3$. By a simple calculus we get $R = \frac{17}{5}, L = \frac{47}{15}$. Then, the boundary value problem (5) and (6) has at least one solution on $[0,1]$.

3.2. Compactness of the Set of Solutions

Theorem 2. *Under Assumptions* (H_1), (H_2), *the set of solutions to Problem* (1) *and* (2) *is not empty, and it is compact.*

Proof. Let $S = \{u \in C([0,1], \mathbb{R}) : u$ solutions of the problem (1) and (2)$\}$. From Theorem 1, $S \neq \emptyset$. Now, we prove that S is compact.

Let $(u_n)_{n \in \mathbb{N}} \in S$, then there exist $v_n \in S_{F.u_n}$ such that

$$\begin{aligned} u_n(t) &= -\frac{1}{2}\int_0^t (t-s)^2 v_n(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s)\, v_n(s)\,ds - \\ &\quad -\frac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\int_0^\eta (\eta - s)^2 v_n(s)\,ds. \end{aligned}$$

From (H_2), we can prove that there exists an $M > 0$ such that

$$\|u_n\|_\infty \leq M, \text{ for every } n \geq 1.$$

As in Theorem 1, we can show by using (H_2) that the set $\{u_n,\ n \geq 1\}$ is equicontinuous in $C([0,1], R)$; hence, by Arzela-Ascoli's theorem we can conclude that there exists a subsequence $\{u_{n_k}\}$ such that u_{n_k} converges to some u in $C([0,1], R)$. We shall now prove that there exists $v(.) \in F(., y(.))$ such that

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 v(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s)\, v(s)\,ds - \\ &\quad -\frac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\int_0^\eta (\eta - s)^2 v(s)\,ds. \end{aligned}$$

Additionally, as $F(t,.)$ is upper semi-continuous, then for every $\epsilon > 0$, there exists $n_0(\epsilon)$, such that for every $n \geq n_0$ we have

$$v_n(t) \in F(t, u_n(t)) \subset F(t, u(t)) + \epsilon B(0,1), \text{ a.e. } t \in [0,1].$$

Since $F(.,.)$ has compact values, there exists a subsequence v_{n_m} such that

$$v_{n_m}(.) \to v(.) \text{ as } m \to \infty$$

and

$$v(t) \in F(t, u(t)), \text{ a.e. } t \in [0,1], \text{ and } \forall m \in \mathbb{N}.$$

It is clear that

$$v_{n_m}(t) \leq p(t), \text{ a.e. } t \in [0,1].$$

By Lebesgue's dominated convergence theorem, we conclude that

$$v \in L^1[0,1], \mathbb{R}) \Longrightarrow v \in S_{F,u}.$$

Thus,

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 v(s)\,ds + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s)\, v(s)\,ds - \\ &\quad -\frac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\int_0^\eta (\eta - s)^2 v(s)\,ds. \end{aligned}$$

Then, $S \in \mathcal{P}_{comp}(C([0,1], \mathbb{R}))$. $\square$

3.3. Filippov's Theorem

Now, we present a Filippov's result for the problem (1) and (2). Let $u \in AC^2([0,1], \mathbb{R})$ be a solution of the following problem:

$$u'''(t) + g(t) = 0, \; t \in (0,1), \tag{7}$$

$$u'(0) = u'(1) = \alpha u(\eta), \; u(0) = \beta u(\eta). \tag{8}$$

We will consider the following two assumptions:

$(\mathcal{C}_1)$ The function $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is such that

(a) for all $v \in \mathbb{R}$, the map $t \to F(t, v)$ is measurable,
(b) the map $\gamma_* : t \to d(g(t), F(t, u(t))$ is integrable.

$(\mathcal{C}_2)$ There exists a function $p(t) \in C([0,1], \mathbb{R}^+)$ such that, for a.e. $t \in [0,1]$

$$H_d(F(t, w_1), F(t, w_2)) \leq p(t)|w_1(t) - w_2(t)|, \text{for all } a.e \in [0,1].$$

Theorem 3. *Assume that the conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ hold. If*

$$\left(1 + \left|\eta^2 \frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|\right) \|p\|_{L^1} < 1,$$

then the problem (1) *and* (2) *has at least one solution v satisfying, for a.e. $t \in [0,1]$, the estimates*

$$|v(t) - u(t)| \leq \phi(t),$$

where

$$\phi(t) \leq \left(1 + L + \frac{L-1}{\eta^2}\right)\left(K\|p\|_{L^1} + \|\gamma_*\|_{L^1}\right),$$

$$K = \frac{L}{1 - L\|p\|_{L^1}}\|\gamma_*\|, \text{ and } L = 1 + \eta^2\left|\frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|.$$

Proof. Let $g_0 = -u'''$ and $v_0(t) = u(t)$ for a.e. $t \in [0,1]$, i.e. Then, by Lemma 3

$$\begin{aligned} v_0(t) \;=\; & -\tfrac{1}{2}\int_0^t (t-s)^2 g_0(s)\,ds + \\ & +\tfrac{1}{2}\left[t^2 + \eta^2 \tfrac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\, g_0(s)\,ds - \\ & -\tfrac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 g_0(s)\,ds. \end{aligned} \tag{9}$$

Let $U_1 : [0,1] \to \mathcal{P}(\mathbb{R})$ be given by $U_1(t) = F(t, v_0(t)) \cap B(g(t), \gamma_*)$. The multi-valued map $U_1(t)$ is measurable (see Proposition III.4 in [12]), so there exists a function $t \to g_1(t)$ which is a measurable selection for U_1.

Let

$$\begin{aligned} v_1(t) \;=\; & -\tfrac{1}{2}\int_0^t (t-s)^2 g_1(s)\,ds + \\ & +\tfrac{1}{2}\left[t^2 + \eta^2 \tfrac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\, g_1(s)\,ds - \\ & -\tfrac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 g_1(s)\,ds. \end{aligned} \tag{10}$$

Then, we have

$$\begin{aligned} |v_1(t) - v_0(t)| \;\leq\; & \frac{1}{2}\int_0^t (t-s)^2 \gamma_*(s)\,ds + \frac{1}{2}\left|t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right| \int_0^1 (1-s)\,\gamma_*(s)\,ds + \\ & + \frac{1}{2}\left|\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right| \int_0^\eta (\eta - s)^2 \gamma_*(s)\,ds, \\ \leq\; & \left(1 + \eta^2\left|\frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|\right) \int_0^1 \gamma_*(s)\,ds. \end{aligned}$$

So, we obtain

$$\|v_1(t) - v_0(t)\|_\infty \leq \left(1 + \eta^2 \left|\frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|\right) \|\gamma_*\|_{L^1}. \tag{11}$$

In the same way, the multi-valued map $U_2(t) = F(t, v_1(t)) \cap B(g_1(t), p(t)|v_1(t) - v_0(t)|)$ is measurable with nonempty closed values (see [12,16,19]). By Lemma 2 (Kuratowski–Ryll–Nardzewski selection theorem), there exists a function g_2 which is a mesurable selection of U_2.

Let the function

$$\begin{aligned} v_2(t) = & -\tfrac{1}{2}\int_0^t (t-s)^2 g_2(s)\,ds + \\ & +\tfrac{1}{2}\left[t^2 + \eta^2 \tfrac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s) g_2(s)\,ds - \\ & -\tfrac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\int_0^\eta (\eta - s)^2 g_2(s)\,ds. \end{aligned} \tag{12}$$

Then

$$\begin{aligned} |v_2(t) - v_1(t)| \leq & \frac{1}{2}\int_0^t (t-s)^2 p(t)|v_1(t) - v_0(t)|\,ds + \\ & +\frac{1}{2}\left|t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right|\int_0^1 (1-s) p(t)|v_1(t) - v_0(t)|\,ds + \\ & +\frac{1}{2}\left|\frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right|\int_0^\eta (\eta - s)^2 p(t)|v_1(t) - v_0(t)|\,ds. \end{aligned}$$

Hence

$$\begin{aligned} |v_2(t) - v_1(t)| \leq & \frac{1}{2}\int_0^t p(t)\|v_1(t) - v_0(t)\|_\infty\,ds + \\ & +\frac{1}{2}\left|1 + \eta^2 \frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|\int_0^1 p(t)\|v_1(t) - v_0(t)\|_\infty\,ds + \\ & +\frac{1}{2}\eta^2\left|\frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|\int_0^\eta p(t)\|v_1(t) - v_0(t)\|_\infty\,ds, \end{aligned}$$

$$|v_2(t) - v_1(t)| \leq \left(1 + \eta^2\left|\frac{\alpha + \beta}{1 - \alpha\eta - \beta}\right|\right)^2 \|p\|_{L^1}\|\gamma_*\|_{L^1}. \tag{13}$$

As above, the multi-valued map $U_3(t) = F(t, v_2(t)) \cap B(g_2(t), p(t)|v_2(t) - v_1(t)|)$ is measurable, so there exists a measurable selection g_3 of U_3. Consider the function

$$\begin{aligned} v_3(t) = & -\tfrac{1}{2}\int_0^t (t-s)^2 g_3(s)\,ds + \\ & +\tfrac{1}{2}\left[t^2 + \eta^2 \tfrac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right]\int_0^1 (1-s) g_3(s)\,ds - \\ & -\tfrac{\alpha t + \beta}{2(1 - \alpha\eta - \beta)}\int_0^\eta (\eta - s)^2 g_3(s)\,ds. \end{aligned} \tag{14}$$

Then

$$\begin{aligned} |v_3(t) - v_2(t)| \leq & \frac{1}{2}\int_0^t (t-s)^2 p(t)|v_2(t) - v_1(t)|\,ds + \\ & +\frac{1}{2}\left|t^2 + \eta^2 \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\right|\int_0^1 (1-s) p(t)|v_2(t) - v_1(t)|\,ds + \\ & +\left|\frac{1}{2}\frac{\alpha t + \beta}{(1 - \alpha\eta - \beta)}\right|\int_0^\eta (\eta - s)^2 p(t)|v_2(t) - v_1(t)|\,ds. \end{aligned}$$

Hence

$$\begin{aligned}|v_3(t)-v_2(t)| &\le \frac{1}{2}\int_0^t p(t)\,\|v_2(t)-v_1(t)\|_\infty\,ds+\\ &+\frac{1}{2}\left|1+\eta^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\int_0^1 p(t)\,\|v_2(t)-v_1(t)\|_\infty\,ds+\\ &+\left|\eta^2\frac{\alpha+\beta}{2(1-\alpha\eta-\beta)}\right|\int_0^\eta p(t)\,\|v_2(t)-v_1(t)\|_\infty\,ds,\end{aligned}$$

$$|v_3(t)-v_2(t)| \le \left(1+\eta^2\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right)^3\|p\|_{L^1}^2\,\|\gamma_*\|_{L^1}\,. \tag{15}$$

Repeating the process for $n=0,1,2,3,....$, we arrive at the following bound:

$$|v_n(t)-v_{n-1}(t)| \le \left(1+\eta^2\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right)^n\|p\|_{L^1}^{n-1}\,\|\gamma_*\|_{L^1}\,. \tag{16}$$

Suppose that (15) holds for some n, now it is left to check (16) for $n+1$. The multi-valued map $U_{n+1}(t)=F(t,v_n(t))\cap B(g_n(t),p(t)\,|v_n(t)-v_{n-1}(t)|)$ is measurable (see Proposition III.4 in [8]); then, there exists a function $t\to g_{n+1}(t)$, which is a measurable selection for U_{n+1}.

We consider

$$\begin{aligned}v_{n+1}(t) &= -\frac{1}{2}\int_0^t (t-s)^2 g_{n+1}(s)\,ds+\\ &+\frac{1}{2}\left[t^2+\eta^2\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right]\int_0^1 (1-s)\,g_{n+1}(s)\,ds-\\ &-\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 g_{n+1}(s)\,ds.\end{aligned}$$

Then

$$\begin{aligned}|v_{n+1}(t)-v_n(t)| &\le \frac{1}{2}\int_0^t p(t)\,\|v_n(t)-v_{n-1}(t)\|_\infty\,ds+\\ &+\frac{1}{2}\left|1+\eta^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\int_0^1 p(t)\,\|v_n(t)-v_{n-1}(t)\|_\infty\,ds+\\ &+\frac{\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\int_0^\eta p(t)\,\|v_n(t)-v_{n-1}(t)\|_\infty\,ds,\\ &\le \left(1+\eta^2\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right)^n\|p\|_{L^1}^{n-1}\,\|v_1(t)-v_0(t)\|_\infty\,.\end{aligned}$$

Since $\left(1+\eta^2\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right)\|p\|_{L^1}<1$, we deduce that $\{v_n\}$ is a Cauchy sequence in $C([0,1],\mathbb{R})$, converging uniformly to a function $v\in C([0,1],\mathbb{R})$. From the definition of U_n, $n\in\mathbb{N}$,

$$|g_{n+1}(t)-g_n(t)| \le p(t)\,|v_n(t)-v_{n-1}(t)| \text{ for } n\in\mathbb{N}, a.e, t\in[0,1]\,.$$

Hence, for almost every $t\in[0,1]$, the sequence $\{g_n(t):n\in\mathbb{N}\}$ is Cauchy in $\mathbb{R}$, then $\{g_n(t):n\in\mathbb{N}\}$ converges almost everywhere to a measurable function $\{g(.)\}$ in $\mathbb{R}$.

Moreover, since $g_0 = -u'''$ and by using the least inequality, we get

$$
\begin{aligned}
|g_n(t) - g_0(t)| &\leq |g_n(t) - g_{n-1}(t)| + |g_{n-1}(t) - g_{n-2}(t)| + \ldots + |g_2(t) - g_1(t)| + \\
&\quad + |g_1(t) - g_0(t)|, \\
&\leq \textstyle\sum_{k=1}^{n-1} p(t) |v_k(t) - v_{k-1}(t)| + |g_1(t) - g_0(t)|, \\
&\leq p(t) \textstyle\sum_{k=1}^{\infty} \left(1 + \eta^2 \left|\dfrac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right)^k \|p\|_{L^1}^{k-1} \|\gamma_*\|_{L^1} + \gamma_*(t), \\
&\leq Kp(t) + \gamma_*(t),
\end{aligned}
$$

where

$$
K = \frac{L}{1 - L\|p\|_{L^1}} \|\gamma_*\|, \text{ and } L = 1 + \eta^2 \left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|.
$$

Then, for all $n \in \mathbb{N}$,

$$
|g_n(t)| \leq Kp(t) + \gamma_*(t). \tag{17}
$$

By (17), we deduce that g_n converges to g in $L^1([0.1], \mathbb{R})$. Consequently,

$$
\begin{aligned}
v(t) &= -\tfrac{1}{2} \textstyle\int_0^t (t-s)^2 g(s)\, ds + \\
&\quad + \tfrac{1}{2}\left[t^2 + \eta^2 \tfrac{\alpha t+\beta}{1-\alpha\eta-\beta}\right] \textstyle\int_0^1 (1-s) g(s)\, ds - \\
&\quad - \tfrac{\alpha t+\beta}{2(1-\alpha\eta-\beta)} \textstyle\int_0^\eta (\eta-s)^2 g(s)\, ds
\end{aligned} \tag{18}
$$

is a solution for the problem (1) and (2) with conditions $v'(0) = v'(1) = \alpha v(\eta)$, $v(0) = \beta v(\eta)$. Then, $v \in S_F$.

Finally, we prove that the solution $v(t)$ verifies the estimate:

$$
|u(t) - v(t)| \leq \phi(t) \text{ for all } t \in [0,1].
$$

$$
\begin{aligned}
|u(t) - v(t)| &= \left| -\frac{1}{2}\int_0^t (t-s)^2 (g_0(s) - g(s))\, ds + \right. \\
&\quad + \frac{1}{2}\left[t^2 + \eta^2 \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right] \int_0^1 (1-s)(g_0(s) - g(s))\, ds - \\
&\quad \left. - \frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)} \int_0^\eta (\eta-s)^2 (g_0(s) - g(s))\, ds \right|, \\
&\leq \frac{1}{2}\int_0^t (t-s)^2 |g_0(s) - g(s)|\, ds + \\
&\quad + \frac{1}{2}\left|t^2 + \eta^2 \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\right| \int_0^1 (1-s) |g_0(s) - g(s)|\, ds + \\
&\quad + \left|\frac{\alpha t+\beta}{2(1-\alpha\eta-\beta)}\right| \int_0^\eta (\eta-s)^2 |g_0(s) - g(s)|\, ds, \\
&\leq \left(1 + \frac{\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right| + \frac{1}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right) \int_0^1 |g_0(s) - g_n(s)|\, ds + \\
&\quad + \left(1 + \frac{\eta^2}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right| + \frac{1}{2}\left|\frac{\alpha+\beta}{1-\alpha\eta-\beta}\right|\right) \int_0^1 |g(s) - g_n(s)|\, ds,
\end{aligned}
$$

As $n \to \infty$, we conclude that

$$\begin{aligned}|u(t)-v(t)| &\leq \left(1+L+\frac{L-1}{\eta^2}\right)\int_0^1 (Kp(t)+\gamma_*(t))\,ds,\\ &\leq \left(1+L+\frac{L-1}{\eta^2}\right)(K\|p\|_{L^1}+\|\gamma_*\|_{L^1}).\end{aligned}$$

□

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References

1. Affane, D.; Azzam-Laouir, D. A control problem governed by a second order differential inclusion. *Appl. Anal.* **2009**, *88*, 1677–1690.
2. Agarwal, R.P.; Grace, S.R.; O'Regan, D. Oscillation theorems for second order differential inclusions. *Int. J. Dyn. Syst. Differ. Equ.* **2007**, *1*, 85–88.
3. Andres, J.; Malaguti, L.; Pavlackova, M. Strictly localized bounding functions for vector second-order boundary value problems. *Nonlinear Anal.* **2009**, *71*, 6019–6028.
4. Avgerinos, E.P.; Papageorgiou, N.S.; Yannakakis, N. Periodic solutions for second order differential inclusions with nonconvex and unbounded multifunction. *Acta Math. Hung.* **1999**, *83*, 303–314.
5. Benchohra, M.; Ntouyas, S.K. Controllability of second-order differential inclusions in Banach spaces with nonlocal conditions. *J. Optim. Theory Appl.* **2000**, *107*, 559–571.
6. Domachowski, S.; Gulgowski, J. A global bifurcation theorem for convex-valued differential inclusions. *Z. Anal. Anwend.* **2004**, *23*, 275–292.
7. Erbe, L.; Krawcewicz, W. Nonlinear Boundary value problems for differential inclusions $y'' \in F(t, y, y')$. *Ann. Polon. Math* **1991**, *3*, 195–296.
8. Krawcewicz, E.W. Existence of solutions to boundary value problems for impulsive second order differential inclusions. *Rocky Mt. J. Math.* **1992**, *22*, 519–539.
9. Grace, S.R.; Agarwal, R.P.; O'Regan, D. A selection of oscillation criteria for second-order differential inclusions. *Appl. Math. Lett.* **2009**, *22*, 153–158.
10. Kyritsi, S.; Matzakos, N.; Papageorgiou, N.S. Periodic problems for strongly nonlinear second-order differential inclusions. *J. Differ. Equ.* **2002**, *183*, 279–302.
11. Rezaiguia, A.; Kelaiaia, S. Existenece Results For Third-Order Differential Incusions With Three-Point Boundary Value Problems. *Acta Math. Univ. Comen.* **2016**, *85*, 311–318.
12. Castaing, C.; Valadier, M. Convex Analysis and Measurable Multifunctions. In *Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany, 1977.
13. Aubin, J.P.; Cellina, A. *Differential Inclusions*; Springer: Berlin/Heidelberg, Germany, 1984.
14. Aubin, J.P.; Frankowska, H. *Set-Valued Analysis*; Birkhauser: Boston, MA, USA, 1990.
15. Deimling, K. *Multivalued Differential Equations*; Walter De Gruyter: Berlin, Germany, 1992.
16. Gorniewicz, L. Topological Fixed Point Theory of Multivalued Map pings. In *Mathematics and Its Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999.
17. Hu, S.; Papageorgiou, N. *Handbook of Multivalued Analysis, Volume I: Theory*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1997.
18. Lasota, A.; Opial, Z. An application of the Kakutani—Ky Fan theorem in the theory of ordinary differential equations. *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **1965**, *13*, 781–786.
19. Rezaiguia, A.; Kelaiaia, S. Existence of a Positive Solusion for a Third-order Tree Point Boundary Value Problem. *Mat. Vesn.* **2016**, *68*, 12–25.
20. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer Monographs inMathematics: New York, NY, USA, 2003.