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Curvature Invariants for Statistical Submanifolds of Hessian Manifolds of Constant Hessian Curvature

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Abstract: We consider statistical submanifolds of Hessian manifolds of constant Hessian curvature. For such submanifolds we establish a Euler inequality and a Chen-Ricci inequality with respect to a sectional curvature of the ambient Hessian manifold.

Keywords: statistical manifolds; Hessian manifolds; Hessian sectional curvature; scalar curvature; Ricci curvature

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1. Introduction

It is well-known that curvature invariants play the most fundamental role in Riemannian geometry. Curvature invariants provide the intrinsic characteristics of Riemannian manifolds which affect the behavior in general of the Riemannian manifold. They are the main Riemannian invariants and the most natural ones. Curvature invariants also play key roles in physics. For instance, the magnitude of a force required to move an object at constant speed, according to Newton's laws, is a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein's general theory of relativity, by the curvatures of spacetime. All sorts of shapes, from soap bubbles to red cells are determined by various curvatures.

Classically, among the curvature invariants, the most studied were sectional, scalar and Ricci curvatures.

Chen [1] established a generalized Euler inequality for submanifolds in real space forms. Also a sharp relationship between the Ricci curvature and the squared mean curvature for any Riemannian submanifold of a real space form was proved in [2], which is known as the Chen-Ricci inequality.

Statistical manifolds introduced, in 1985, by Amari have been studied in terms of information geometry. Since the geometry of such manifolds includes the notion of dual connections, also called conjugate connections in affine geometry, it is closely related to affine differential geometry. Further, a statistical structure is a generalization of a Hessian one.

In [3], Aydin and the present authors obtained geometrical inequalities for the scalar curvature and the Ricci curvature associated to the dual connections for submanifolds in statistical manifolds of constant curvature. We want to point-out that, generally, the dual connections are not metric; then one cannot define a sectional curvature with respect to them by the standard definitions. However there exists a sectional curvature on a statistical manifold defined by B. Opozda (see [4]).

We mention that in [5] we established a Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature by using another sectional curvature.

As we know, submanifolds in Hessian manifolds have not been considered until now.

In the present paper we deal with statistical submanifolds in Hessian manifolds of constant Hessian curvature c. It is known [6] that such a manifold is a statistical manifold of null constant curvature and also a Riemannian space form of constant sectional curvature $-c/4$ (with respect to the sectional curvature defined by the Levi-Civita connection).

2. Statistical Manifolds and Their Submanifolds

A *statistical manifold* is an m-dimensional Riemannian manifold $(\tilde{M}^m, g)$ endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\tilde{g}\left(X,Y\right) = \tilde{g}\left(\tilde{\nabla}_Z X, Y\right) + \tilde{g}\left(X, \tilde{\nabla}^*_Z Y\right), \tag{1}$$

for any $X, Y, Z \in \Gamma(T\tilde{M}^m)$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called *dual connections* (see [7–9]), and it is easily shown that $(\tilde{\nabla}^*)^* = \tilde{\nabla}$. The pair $(\tilde{\nabla}, g)$ is said to be a *statistical structure*. If $(\tilde{\nabla}, g)$ is a statistical structure on $\tilde{M}^m$, then $(\tilde{\nabla}^*, g)$ is a statistical structure too [8].

On the other hand, any torsion-free affine connection $\tilde{\nabla}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0, \tag{2}$$

where $\tilde{\nabla}^0$ is the Levi-Civita connection on $\tilde{M}^m$.

Denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

A statistical structure $(\tilde{\nabla}, g)$ is said to be *of constant curvature* $\varepsilon \in \mathbb{R}$ if

$$\tilde{R}(X,Y)Z = \varepsilon\{g(Y,Z)X - g(X,Z)Y\}. \tag{3}$$

A statistical structure $(\tilde{\nabla}, g)$ of constant curvature 0 is called a *Hessian structure.*

The curvature tensor fields $\tilde{R}$ and $\tilde{R}^*$ of the dual connections satisfy

$$g(\tilde{R}^*(X,Y)Z, W) = -g(Z, \tilde{R}(X,Y)W). \tag{4}$$

From (4) it follows immediately that if $(\tilde{\nabla}, g)$ is a statistical structure of constant curvature ε, then $(\tilde{\nabla}^*, g)$ is also a statistical structure of constant curvature ε. In particular, if $(\tilde{\nabla}, g)$ is Hessian, $(\tilde{\nabla}^*, g)$ is also Hessian [6].

On a Hessian manifold $(\tilde{M}^m, \tilde{\nabla})$, let $\gamma = \tilde{\nabla} - \tilde{\nabla}^0$. The tensor field $\tilde{Q}$ of type (1, 3) defined by $\tilde{Q}(X,Y) = [\gamma_X, \gamma_Y]$, $X, Y \in \Gamma(T\tilde{M}^m)$ is said to be the *Hessian curvature tensor* for $\tilde{\nabla}$ (see [4,6]). It satisfies

$$\tilde{R}(X,Y) + \tilde{R}^*(X,Y) = 2\tilde{R}^0(X,Y) + 2\tilde{Q}(X,Y).$$

By using the Hessian curvature tensor $\tilde{Q}$, a Hessian sectional curvature can be defined on a Hessian manifold.

Let $p \in \tilde{M}^m$ and π a plane in $T_p\tilde{M}^m$. Take an orthonormal basis $\{X, Y\}$ of π and set

$$\tilde{K}(\pi) = g(\tilde{Q}(X,Y)Y, X).$$

The number $\tilde{K}(\pi)$ is independent of the choice of an orthonormal basis and is called the *Hessian sectional curvature*.

A Hessian manifold has constant Hessian sectional curvature c if and only if (see [6])

$$\tilde{Q}(X,Y,Z,W) = \frac{c}{2}[g(X,Y)g(Z,W) + g(X,W)g(Y,Z)],$$

for all vector fields on $\tilde{M}^m$.

If $(\tilde{M}^m, g)$ is a statistical manifold and M^n an n-dimensional submanifold of $\tilde{M}^m$, then (M^n, g) is also a statistical manifold with the induced connection by $\tilde{\nabla}$ and induced metric g. In the case that

$(\tilde{M}^m, g)$ is a semi-Riemannian manifold, the induced metric g has to be non-degenerate. For details, see [10].

In the geometry of Riemannian submanifolds (see [11]), the fundamental equations are the Gauss and Weingarten formulas and the equations of Gauss, Codazzi and Ricci.

Let denote the set of the sections of the normal bundle to M^n by $\Gamma(T^{\perp}M^n)$.

In our case, for any $X, Y \in \Gamma(TM^n)$, according to [10], the corresponding Gauss formulas are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{5}$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \tag{6}$$

where $h, h^* : \Gamma(TM^n) \times \Gamma(TM^n) \to \Gamma(T^{\perp}M^n)$ are symmetric and bilinear, called the *imbedding curvature tensor* of M^n in $\tilde{M}^m$ for $\tilde{\nabla}$ and the *imbedding curvature tensor* of M^n in $\tilde{M}^m$ for $\tilde{\nabla}^*$, respectively.

In [10], it is also proved that (∇, g) and (∇^*, g) are dual statistical structures on M^n.

Since h and h^* are bilinear, there exist linear transformations A_ξ and A^*_ξ on TM^n defined by

$$g(A_\xi X, Y) = g(h(X, Y), \xi), \tag{7}$$

$$g(A^*_\xi X, Y) = g(h^*(X, Y), \xi), \tag{8}$$

for any $\xi \in \Gamma(T^{\perp}M^n)$ and $X, Y \in \Gamma(TM^n)$. Further, see [10], the corresponding Weingarten formulas are

$$\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^{\perp}_X \xi, \tag{9}$$

$$\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi, \tag{10}$$

for any $\xi \in \Gamma(T^{\perp}M^n)$ and $X \in \Gamma(TM^n)$. The connections $\nabla^{\perp}_X$ and $\nabla^{*\perp}_X$ given by (9) and (10) are Riemannian dual connections with respect to induced metric on $\Gamma(T^{\perp}M^n)$.

Let $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_m\}$ be orthonormal tangent and normal frames, respectively, on M^n. Then the mean curvature vector fields are defined by

$$H = \frac{1}{n}\sum_{i=1}^n h(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^m \left(\sum_{i=1}^n h^\alpha_{ii}\right) e_\alpha,\ h^\alpha_{ij} = g(h(e_i, e_j), e_\alpha), \tag{11}$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^n h^*(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^m \left(\sum_{i=1}^n h^{*\alpha}_{ii}\right) e_\alpha,\ h^{*\alpha}_{ij} = g(h^*(e_i, e_j), e_\alpha), \tag{12}$$

for $1 \le i, j \le n$ and $n + 1 \le \alpha \le m$.

The corresponding Gauss, Codazzi and Ricci equations are given by the following result.

Proposition 1. *[10] Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ be dual connections on a statistical manifold $\tilde{M}^m$ and ∇ the induced connection by $\tilde{\nabla}$ on a statistical submanifold M^n. Let $\tilde{R}$ and R be the Riemannian curvature tensors for $\tilde{\nabla}$ and ∇, respectively. Then*

$$g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(h(X, Z), h^*(Y, W)) - g(h^*(X, W), h(Y, Z)), \tag{13}$$

$$(\tilde{R}(X, Y)Z)^{\perp} = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) -$$
$$-\{\nabla^{\perp}_Y h(X, Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z)\},$$

$$g(R^{\perp}(X, Y)\xi, \eta) = g(\tilde{R}(X, Y)\xi, \eta) + g([A^*_\xi, A_\eta]X, Y), \tag{14}$$

where $R^{\perp}$ is the Riemannian curvature tensor of $\nabla^{\perp}$ on $T^{\perp}M^n$, $\xi, \eta \in \Gamma(T^{\perp}M^n)$ and $[A^*_{\xi}, A_{\eta}] = A^*_{\xi}A_{\eta} - A_{\eta}A^*_{\xi}$.

For the equations of Gauss, Codazzi and Ricci with respect to the connection $\tilde{\nabla}^*$ on M^n, we have

Proposition 2. *[10] Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ be dual connections on a statistical manifold $\tilde{M}^m$ and ∇^* the induced connection by $\tilde{\nabla}^*$ on a statistical submanifold M^n. Let $\tilde{R}^*$ and R^* be the Riemannian curvature tensors for $\tilde{\nabla}^*$ and ∇^*, respectively. Then*

$$g(\tilde{R}^*(X,Y)Z,W) = g(R^*(X,Y)Z,W) + g(h^*(X,Z),h(Y,W)) - g(h(X,W),h^*(Y,Z)), \tag{15}$$

$$\begin{aligned}(\tilde{R}^*(X,Y)Z)^{\perp} = \nabla^{*\perp}_X h^*(Y,Z) - h^*(\nabla^*_X Y, Z) - h^*(Y, \nabla^*_X Z) - \\ -\{\nabla^{*\perp}_Y h^*(X,Z) - h^*(\nabla^*_Y X, Z) - h^*(X, \nabla^*_Y Z)\},\end{aligned}$$

$$g(R^{*\perp}(X,Y)\xi,\eta) = g(\tilde{R}^*(X,Y)\xi,\eta) + g([A_{\xi}, A^*_{\eta}]X,Y), \tag{16}$$

where $R^{*\perp}$ is the Riemannian curvature tensor of $\nabla^{\perp *}$ on $T^{\perp}M^n$, $\xi, \eta \in \Gamma(T^{\perp}M^n)$ and $\left[A_{\xi}, A^*_{\eta}\right] = A_{\xi}A^*_{\eta} - A^*_{\eta}A_{\xi}$.

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [3].

3. Euler Inequality and Chen-Ricci Inequality

First we obtain a Euler inequality for submanifolds in a Hessian manifold of constant Hessian curvature.

Let $\tilde{M}^m(c)$ be a Hessian manifold of constant Hessian curvature c. Then it is flat with respect to the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$. Moreover $\tilde{M}^m(c)$ is a Riemannian space form of constant sectional curvature $-c/4$ (with respect to the Levi-Civita connection $\tilde{\nabla}^0$).

Let M^n be an n-dimensional statistical submanifold of $\tilde{M}^m(c)$ and $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_m\}$ be orthonormal tangent and normal frames, respectively, on M^n.

We denote by τ^0 the scalar curvature of the Levi-Civita connection ∇^0 on M^n. Gauss equation implies

$$2\tau^0 = n^2\|H^0\|^2 - \|h^0\|^2 - n(n-1)\tfrac{c}{4}, \tag{17}$$

where H^0 and h^0 are the mean curvature vector and the second fundamental form, respectively, with respect to the Levi-Civita connection.

Let τ be the scalar curvature of M^n (with respect to the Hessian curvature tensor Q). Then, from (13) and (15), we have:

$$2\tau = \frac{1}{2}\sum_{i,j=1}^{n}[g(R(e_i,e_j)e_j,e_i) + g(R^*(e_i,e_j)e_j,e_i) - 2g(R^0(e_i,e_j)e_j,e_i)] =$$

$$= n^2 g(H,H^*) - \sum_{i,j=1}^{n} g(h(e_i,e_j),h^*(e_i,e_j)) - 2\tau^0 =$$

$$= n^2 g(H,H^*) - \sum_{r=n+1}^{m}\sum_{i,j=1}^{n} h^r_{ij}h^{*r}_{ij} - 2\tau^0 =$$

$$= n^2 g(H,H^*) - \frac{1}{4}\sum_{r=n+1}^{m}\sum_{i,j=1}^{n}[(h^r_{ij} + h^{*r}_{ij})^2 - (h^r_{ij} - h^{*r}_{ij})^2] - 2\tau^0 =$$

$$= n^2 g(H, H^*) - \|h^0\|^2 + \frac{1}{4} \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} (h_{ij}^r - h_{ij}^{*r})^2 - 2\tau^0.$$

By (17), it follows that

$$2\tau \geq n^2 g(H, H^*) - n^2 \|H^0\|^2 + n(n-1)\frac{c}{4} =$$

$$= n^2 g(H, H^*) - \frac{n^2}{4} g(H + H^*, H + H^*) + n(n-1)\frac{c}{4} =$$

$$= \frac{n^2}{2} g(H, H^*) - \frac{n^2}{4} g(H, H) - \frac{n^2}{4} g(H^*, H^*) + n(n-1)\frac{c}{4} =$$

$$= -\frac{n^2}{4} \|H - H^*\|^2 + n(n-1)\frac{c}{4}.$$

Summing up, we proved the following.

Theorem 1. *Let M^n be a statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature c. Then the scalar curvature satisfies:*

$$2\tau \geq -\frac{n^2}{4} \|H - H^*\|^2 + n(n-1)\frac{c}{4}.$$

Moreover, the equality holds at any pont $p \in M^n$ if and only if $h = h^*$. In this case, the scalar curvature is constant, $2\tau = n(n-1)\frac{c}{4}$.

We want to point-out that τ is non-positive on standard examples of Hessian manifolds.

Next we establish a Chen-Ricci inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature.

Recall that

$$2\tau = \frac{1}{2} \sum_{i,j=1}^{n} [g(R(e_i, e_j)e_j, e_i) + g(R^*(e_i, e_j)e_j, e_i) - 2g(R^0(e_i, e_j)e_j, e_i)] =$$

$$= n^2 g(H, H^*) - \sum_{i,j=1}^{n} g(h(e_i, e_j), h^*(e_i, e_j)) - 2\tau^0.$$

Then

$$2\tau = \frac{n^2}{2} [g(H + H^*, H + H^*) - g(H, H) - g(H^*, H^*)] -$$

$$-\frac{1}{2} [g(h(e_i, e_j) + h^*(e_i, e_j), h(e_i, e_j) + h^*(e_i, e_j)) -$$

$$-g(h(e_i, e_j), h(e_i, e_j)) - g(h^*(e_i, e_j), h^*(e_i, e_j))] - 2\tau^0 =$$

$$= 2n^2 g(H^0, H^0) - 2\left\|h^0\right\|^2 - 2\tau^0 -$$

$$-\frac{n^2}{2} g(H, H) - \frac{n^2}{2} g(H^*, H^*) + \frac{1}{2} \|h\|^2 + \frac{1}{2} \|h^*\|^2,$$

where H^0 and h^0 are the mean curvature vector and the second fundamental form, respectively, with respect to the Levi-Civita connection.

By using (17), we get

$$2\tau = n(n-1)\tfrac{c}{2} + 2\tau^0 - \tfrac{n^2}{2} g(H, H) - \tfrac{n^2}{2} g(H^*, H^*) + \tfrac{1}{2} \|h\|^2 + \tfrac{1}{2} \|h^*\|^2. \tag{18}$$

On the other hand, we may write:

$$\|h\|^2 = \sum_{\alpha=n+1}^{m} [(h_{11}^{\alpha})^2 + (h_{22}^{\alpha} + \ldots + h_{nn}^{\alpha})^2 + 2 \sum_{1\le i<j\le n} (h_{ij}^{\alpha})^2 - 2 \sum_{2\le i\neq j\le n} h_{ii}^{\alpha} h_{jj}^{\alpha}] =$$

$$= \frac{1}{2} \sum_{\alpha=n+1}^{m} [(h_{11}^{\alpha} + h_{22}^{\alpha} + \ldots + h_{nn}^{\alpha})^2 + (h_{11}^{\alpha} - h_{22}^{\alpha} - \ldots - h_{nn}^{\alpha})^2] -$$

$$- \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] + 2 \sum_{\alpha=n+1}^{m} \sum_{j=1}^{n} (h_{1j}^{\alpha})^2 \geq$$

$$\geq \frac{n^2}{2} \|H\|^2 - \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2].$$

In the same manner, one obtains

$$\|h^*\|^2 \geq \frac{n^2}{2} \|H^*\|^2 - \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2].$$

Substituting the above inequalities in (18), it follows that

$$2\tau \geq n(n-1)\frac{c}{2} + 2\tau^0 - \frac{n^2}{4} \|H\|^2 - \frac{n^2}{4} \|H^*\|^2 -$$

$$- \frac{1}{2} \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] - \frac{1}{2} \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2] =$$

$$= n(n-1)\frac{c}{2} + 2\tau^0 - \frac{n^2}{4} \|H\|^2 - \frac{n^2}{4} \|H^*\|^2 -$$

$$- \frac{1}{2} \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [(h_{ii}^{\alpha} + h_{ii}^{*\alpha})(h_{jj}^{\alpha} + h_{jj}^{*\alpha}) - (h_{ij}^{\alpha} + h_{ij}^{*\alpha})^2] +$$

$$+ \frac{1}{2} \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} (h_{ii}^{\alpha} h_{jj}^{*\alpha} - h_{ij}^{\alpha} h_{ij}^{*\alpha}) + \frac{1}{2} \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} (h_{ii}^{*\alpha} h_{jj}^{\alpha} - h_{ij}^{\alpha} h_{ij}^{*\alpha}) =$$

$$= n(n-1)\frac{c}{2} + 2\tau^0 - \frac{n^2}{4} \|H\|^2 - \frac{n^2}{4} \|H^*\|^2 -$$

$$-2 \sum_{\alpha=n+1}^{m} \sum_{2\le i\neq j\le n} [h_{ii}^{0\alpha} h_{jj}^{0\alpha} - (h_{ij}^{0\alpha})^2] + \frac{1}{2} \sum_{2\le i\neq j\le n} [g(R(e_i, e_j)e_j, e_i) + g(R^*(e_i, e_j)e_j, e_i)].$$

Gauss equation for the Levi-Civita connection and the definition of the Hessian sectional curvature imply

$$2\tau \geq n(n-1)\frac{c}{2} + 2\tau^0 - \frac{n^2}{4} \|H\|^2 - \frac{n^2}{4} \|H^*\|^2 -$$

$$-(n-1)(n-2)\frac{c}{2} - 2 \sum_{2\le i\neq j\le n} K^0(e_i \wedge e_j) + \sum_{2\le i\neq j\le n} [K(e_i \wedge e_j) + K^0(e_i \wedge e_j)].$$

But the Ricci curvature R^0 with respect to the Levi-Civita connection is given by

$$2Ric^0(X) = 2\tau^0 - \sum_{2\le i\neq j\le n} K^0(e_i \wedge e_j),$$

and, similarly,

$$2Ric(X) = 2\tau - \sum_{2 \le i \ne j \le n} K(e_i \wedge e_j).$$

Consequently

$$Ric(X) \ge (n-1)\frac{c}{2} - \frac{n^2}{8}\|H\|^2 - \frac{n^2}{8}\|H^*\|^2 + Ric^0(X).$$

The vector field $X = e_1$ satisfies the equality case if and only if

$$\begin{cases} h_{11}^{\alpha} = h_{22}^{\alpha} + ... + h_{nn}^{\alpha},\ h_{1j}^{\alpha} = 0,\ \forall j \in \{2, ..., n\}, \forall \alpha \in \{n+1, ..., m\}, \\ h_{11}^{*\alpha} = h_{22}^{*\alpha} + ... + h_{nn}^{*\alpha},\ h_{1j}^{*\alpha} = 0,\ \forall j \in \{2, ..., n\}, \forall \alpha \in \{n+1, ..., m\}, \end{cases}$$

or, equivalently,

$$\begin{cases} 2h(X,X) = nH(p),\ h(X,Y) = 0,\ \forall Y \in T_pM^n \text{ orthogonal to } X, \\ 2h^*(X,X) = nH^*(p),\ h^*(X,Y) = 0,\ \forall Y \in T_pM^n \text{ orthogonal to } X. \end{cases}$$

Therefore, we proved the following Chen-Ricci inequality.

Theorem 2. *Let M^n be a statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature c. Then the Ricci curvature of a unit vector $X \in T_pM^n$ satisfies:*

$$Ric(X) \ge (n-1)\frac{c}{2} - \frac{n^2}{8}\|H\|^2 - \frac{n^2}{8}\|H^*\|^2 + Ric^0(X).$$

Moreover, the equality case holds if and only if

$$\begin{cases} 2h(X,X) = nH(p),\ h(X,Y) = 0,\ \forall Y \in T_pM^n \textit{ orthogonal to } X, \\ 2h^*(X,X) = nH^*(p),\ h^*(X,Y) = 0,\ \forall Y \in T_pM^n \textit{ orthogonal to } X. \end{cases}$$

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