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Primes and the Lambert W function

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Abstract: The Lambert W function, implicitly defined by $W(x)\,e^{W(x)} = x$, is a relatively "new" special function that has recently been the subject of an extended upsurge in interest and applications. In this note, I point out that the Lambert W function can also be used to gain a new perspective on the distribution of the prime numbers.

Keywords: primes; Lambert W function; prime counting function; the n'th prime

MSC: 11A41 (Primes); 11N05 (Distribution of primes)

1. Introduction

The Lambert W function, implicitly defined by the relation $W(x)\,e^{W(x)} = x$, has a long and quite convoluted 250-year history, but only recently has it become common to view this particular function as one of the standard "special functions" of mathematics [1]. Applications range over an extremely broad range [1,2], from combinatorics (for instance in the enumeration of rooted trees) [1], to delay differential equations [1], to falling objects subject to linear drag [3], to the evaluation of the numerical constant in Wien's displacement law [4,5], to quantum statistics [6], to constructing the "tortoise" coordinate for Schwarzschild black holes [7], *etcetera*. In this brief note I will indicate some apparently new applications of the Lambert W function to the distribution of primes, specifically to the prime counting function $\pi(x)$ and estimating the n'th prime p_n.

2. The Prime Counting Function $\pi(x)$

Theorem 1. *The prime counting function* $\pi(x)$ *satisfies*

$$\pi(x) < \frac{x}{W(x)} = e^{W(x)}\,; \qquad (\forall x \geq 0). \tag{1}$$

Proof. First observe that $x \geq p_{\pi(x)}$. Second recall the standard result that $p_n > n \ln n$ for $n \geq 1$. (See Rosser [8], or any standard reference book on prime numbers, for example [9,10].) Then we have $x \geq p_{\pi(x)} > \pi(x) \ln \pi(x)$, so implying $x > \pi(x) \ln \pi(x)$. Invert, noting that the right hand side (RHS) is monotone increasing, to see that $\pi(x) < x/W(x)$, certainly for $\pi(x) \geq 1$ (corresponding to $x \geq 2$). Then explicitly check validity of the inequality on the domain $x \in [0, 2)$. Finally, use the definition of the Lambert W function to note $x/W(x) = e^{W(x)}$. □

Corollary 1.

$$\ln \pi(x) < W(x)\,; \qquad (\forall x \geq 0). \tag{2}$$

Theorem 2. *The prime number theorem,* $\pi(x) \sim x/\ln x$, *is equivalent to the statement*

$$\pi(x) \sim \frac{x}{W(x)} = e^{W(x)}\,; \qquad (x \to \infty). \tag{3}$$

Proof. Trivial. Note that asymptotically $W(x) \sim \ln x$. (The only potential subtlety is that we are using the principal branch of the W function, denoted $W_0(x)$ whenever there is any risk of confusion [1].) □

Corollary 2.

$$\ln \pi(x) \sim W(x)\,; \qquad (x \to \infty). \tag{4}$$

So we have derived *both* a strict upper bound on $\pi(x)$ *and* an asymptotic equality. This is, so far, essentially a re-packaging of well-known results in terms of the Lambert W function.

When it comes to developing an analogous lower bound on $\pi(x)$, the situation is considerably more subtle. First consider the well-known upper bound on p_n [9–11]:

$$p_n < n(\ln n + \ln\ln n) = n \ln(n \ln n); \qquad (n \geq 6). \tag{5}$$

Note that $x < p_{\pi(x)+1}$. Now consider the elementary inequality

$$\ln x \leq \frac{x}{e}\,, \tag{6}$$

with equality only at $x = e$, and observe that (here and below we shall generically use ϵ to represent an arbitrarily small positive number) this implies

$$\ln x = \frac{\ln(x^\epsilon)}{\epsilon} \leq \frac{x^\epsilon}{\epsilon\, e}\,; \qquad (\forall \epsilon > 0), \tag{7}$$

now with equality only at $x = e^{1/\epsilon}$. This inequality explicitly captures the well-known fact that the logarithm grows less rapidly than any positive power. Applied to the n'th prime this now implies

$$p_n < n \ln(n \ln n) \leq n \ln\left(\frac{n^{1+\epsilon}}{\epsilon\, e}\right) = n\{(1+\epsilon)\ln n - 1 - \ln \epsilon\}; \qquad (n \geq 6;\;\; \forall \epsilon > 0). \tag{8}$$

The inequality $p_n < n\{(1+\epsilon)\ln n - 1 - \ln\epsilon\}$ is much weaker than Equation (5), but much more tractable. Using logic identical to that of Theorem 1, it is easy to convert this into the inequality

$$x \leq p_{\pi(x)+1} < [\pi(x)+1] \ln\left\{\frac{[\pi(x)+1]^{1+\epsilon}}{\epsilon\, e}\right\}; \qquad (\pi(x) \geq 5;\;\; \forall \epsilon > 0). \tag{9}$$

This is now easily inverted to obtain:

Theorem 3. *The prime counting function* $\pi(x)$ *satisfies* $(x \geq 11;\;\; \forall \epsilon > 0)$:

$$\pi(x) > \frac{\dfrac{x}{1+\epsilon}}{W\left(\dfrac{x}{1+\epsilon}\,(\epsilon\, e)^{-1/(1+\epsilon)}\right)} - 1 \;=\; (\epsilon\, e)^{1/(1+\epsilon)} \exp W\left(\frac{x}{1+\epsilon}\,(\epsilon\, e)^{-1/(1+\epsilon)}\right) - 1. \tag{10}$$

For any fixed $\epsilon > 0$, this it is easy to check that inequality holds at least for $\pi(x) \geq 5$, corresponding to $x \geq 11$. But, depending on the specific value ϵ, the domain of validity may actually be larger. That is, $\forall \epsilon > 0$ the inequality holds for $\pi(x) \geq n_0(\epsilon)$ with $n_0(\epsilon) \leq 5$, corresponding to $x \geq x_0(\epsilon)$ with $x_0(\epsilon) < 11$. For instance, numerically solving $\frac{1}{1+\epsilon}\,(\epsilon\, e)^{-1/(1+\epsilon)} = 1$ gives $\epsilon_* = 0.2711715619...$, whence $\frac{1}{1+\epsilon_*} = (\epsilon_*\, e)^{1/(1+\epsilon_*)} = 0.7866758744...$. (As always, there is a trade-off between the tightness of the bound and domain of validity of the bound; sometimes it is worthwhile to consider a simpler less-stringent bound that is valid over a wider domain; sometimes it is worthwhile to consider an uglier but tighter bound that might be valid over a smaller domain).

Corollary 3. *After explicitly checking the domain of validity we have:*

$$\pi(x) > \frac{x}{(1+\epsilon_*)\, W(x)} - 1 = \frac{\exp W(x)}{1+\epsilon_*} - 1; \qquad (x \geq 3). \tag{11}$$

where numerically $\frac{1}{1+\epsilon_*} = 0.7866758744....$

The particular corollary above has perhaps the best trade-off in terms of elegance and wide domain of validity. Other related corollaries, again chosen for their (comparative) elegance and wide domain of applicability, may be determined analytically:

Corollary 4. *Setting* $\epsilon = 1$*, and explicitly checking the domain of validity, we have:*

$$\pi(x) > \frac{x/2}{W\left(\frac{x}{2e^{1/2}}\right)} - 1 = e^{1/2} \exp W\left(\frac{x}{2e^{1/2}}\right) - 1; \qquad (x \geq 6). \tag{12}$$

Corollary 5. *Choosing the specific case* $\epsilon = e^{-1}$*, and explicitly checking the domain of validity, we have:*

$$\pi(x) > \frac{\frac{x}{1+e^{-1}}}{W\left(\frac{x}{1+e^{-1}}\right)} - 1 = \exp W\left(\frac{x}{1+e^{-1}}\right) - 1; \qquad (x \geq 5). \tag{13}$$

Corollary 6. *Choosing the specific case* $\epsilon = e^{-3}$*, the domain of validity is the entire positive half line* $(x \geq 0)$*:*

$$\pi(x) > \frac{\frac{x}{1+e^{-3}}}{W\left(\frac{x}{1+e^{-3}}\, e^{2/(1+e^{-3})}\right)} - 1 = e^{-2/(1+e^{-3})} \exp W\left(\frac{x}{1+e^{-3}}\, e^{2/(1+e^{-3})}\right) - 1. \tag{14}$$

This last of these corollaries, $(\epsilon = e^{-3})$, exhibits somewhat poorer bounding performance at intermediate values of x, but eventually overtakes corollary 5, $(\epsilon = e^{-1})$, once $x \approx e^{2e+3} \approx 4600$, and then asymptotically provides a better bound. Numerous variations on this theme can also be constructed, amounting to different ways of approximating the logarithms appearing in Equation (5).

In summary, we have used the Lambert W function to obtain a number of bounds, and some general classes of bounds, on the prime counting function $\pi(x)$ in terms of the Lambert W function $W(x)$. We shall now turn attention to the n'th prime p_n.

3. The *n*'th Prime

Theorem 4. *The n'th prime* p_n *satisfies*

$$p_n < -n\, W_{-1}\left(-\frac{1}{n}\right); \qquad (n \geq 4). \tag{15}$$

Here $W_{-1}(x)$ *is the second real branch of the Lambert W function, which is a strictly negative function defined on the domain* $x \in [-1/e, 0)$*.*

Proof. We start from the fact that $n \geq p_n / \ln p_n$, this inequality certainly being valid for $p_n \geq 7$, corresponding to $n \geq 4$ [12]. Inverting, (and appealing to the monotonicity of $x/\ln x$), we have $p_n < -n\, W_{-1}(-1/n)$, certainly for $n \geq 4$. Explicitly inspecting $n \in \{1, 2, 3\}$ shows that the actual domain of validity is indeed $n \geq 4$. □

Corollary 7.

$$p_n < -n\, W_{-1}\left(-\frac{1}{n+e}\right); \qquad (n \geq 1). \tag{16}$$

Proof. Note that the function $-W_{-1}(x)$ is monotone increasing on $[-e^{-1}, 0)$. So we see that $-W_{-1}(-1/[n+e]) > -W_{-1}(-1/n)$, and the claimed inequality certainly holds for $n \geq 4$. For $n \in \{1, 2, 3\}$ verify the claimed inequality by explicit computation. □

The virtue of this specific corollary is that it now holds for all positive integers.
There are many other variations on this theme that one could construct.

Theorem 5. *The prime number theorem, which can be written in the form* $p_n \sim n \ln n$*, is equivalent to the statement*

$$p_n \sim -n\, W_{-1}\left(-\frac{1}{n}\right); \qquad (n \to \infty). \tag{17}$$

Proof. Trivial. Consider the asymptotic result [1]

$$W_{-1}(x) = \ln(-x) - \ln(-\ln(-x)) + o(1) \qquad (x \to 0^-). \tag{18}$$

Then

$$-\, n\, W_{-1}\left(-\frac{1}{n}\right) = n\{\ln n + \ln \ln n + o(1)\}. \tag{19}$$

□

Comment: Note that use of the Lambert W function, simply because its asymptotic expansion contains both $\ln(x)$ and $\ln(\ln(x))$ terms, automatically yields the first two terms of the Cesàro–Cippola asymptotic expansion [13,14]:

$$p_n = n\{\ln n + \ln \ln n - 1 + o(1)\}. \tag{20}$$

We can even obtain the first *three* terms of the Cesàro–Cippola asymptotic expansion by refining the prime number theorem slightly as follows:

Theorem 6.

$$p_n \sim -n\, W_{-1}\left(-\frac{e}{n}\right); \qquad (n \to \infty). \tag{21}$$

We note that use of the Lambert W function yields *both* a strict upper bound *and* an asymptotic result.

In counterpoint, to obtain a lower bound on p_n we start with an upper bound on $\pi(x)$. Consider for instance the standard result [15]:

$$\pi(x) < \frac{x}{\ln x - \frac{3}{2}}\,; \qquad (x > e^{3/2}) \tag{22}$$

Note that the RHS of this inequality is monotone increasing for $x > e^{5/2}$. Now we always have $p_{\pi(x)} \leq x < p_{\pi(x)+1}$, so

$$n < \frac{p_{n+1}}{\ln p_{n+1} - \frac{3}{2}}\,. \tag{23}$$

This holds at the very least for $p_n > e^{5/2}$, corresponding to $n \geq 6$, but an explicit check shows that it in fact holds for $n \geq 2$. This is perhaps more clearly expressed as

$$n - 1 < \frac{p_n}{\ln p_n - \frac{3}{2}}; \qquad (n \geq 3). \tag{24}$$

Inverting, and noting the constraint arising from the domain of definition of W_{-1}, we now obtain:

Theorem 7.

$$p_n > -(n-1)\, W_{-1}\left(-\frac{e^{3/2}}{n-1}\right); \qquad (n \geq 14). \tag{25}$$

There are many other variations on this theme that one could construct.

To find some implicit bounds we use the old result (see for example Rosser [11]) that $\forall \epsilon > 0,\ \exists N(\epsilon) : \forall n \geq N(\epsilon)$

$$\frac{x}{\ln x - 1 + \epsilon} < \pi(x) < \frac{x}{\ln x - 1 - \epsilon}\,. \tag{26}$$

Without an explicit calculation of $N(\epsilon)$ these bounds are qualitative, rather than quantitative. Nevertheless it may be of interest to point out that a minor variant of the arguments above immediately yields:

Theorem 8. $\forall \epsilon > 0,\ \exists M(\epsilon) : \forall n \geq M(\epsilon)$

$$-\,n\, W_{-1}\left(-\frac{e^{1-\epsilon}}{n}\right) > p_n > -(n-1)\, W_{-1}\left(-\frac{e^{1+\epsilon}}{n-1}\right). \tag{27}$$

It is now "merely" a case of estimating $M(\epsilon)$ to turn these into explicit bounds. We have already seen that $\epsilon = 1$ provides a widely applicable upper bound, and $\epsilon = 1/2$ a widely applicable lower bound. Taking $\epsilon \to 0$ now makes it clear why

$$p_n \sim -n\, W_{-1}\left(-\frac{e}{n}\right); \qquad (n \to \infty), \tag{28}$$

is such a good asymptotic estimate for p_n.

4. Discussion

While the calculations carried out above are very straightforward, almost trivial, it is perhaps the shift of viewpoint that is more interesting. The Lambert W function provides (in this context) a "new" special function to work with, one which may serve to perhaps simplify and unify many otherwise disparate results. It is perhaps worth noting that the infamous "$\ln \ln x$" terms that infest the analytic theory of prime numbers will automatically appear as the sub-leading terms in asymptotic expansions of the Lambert W function. Whether there is anything "deeper" at play will be deferred for future investigation.

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Appendix A. The Lambert *W* Function

The Lambert W function is a multi-valued complex function defined implicitly by [1]

$$W(x)\, e^{W(x)} = x. \tag{A1}$$

There are two real branches: $W_0(x)$ defined for $x \in [-e^{-1}, \infty)$, and $W_{-1}(x)$ defined for $x \in [e^{-1}, 0)$. These two branches meet at the common point $W_0(-e^{-1}) = W_{-1}(-e^{-1}) = -1$. It is common to use $W(x)$ in place of $W_0(x)$ when there is no risk of confusion.

Asymptotic expansions are [1]:

$$W_0(x) = \ln x - \ln\ln x + o(1); \qquad (x \to \infty); \tag{A2}$$

$$W_{-1}(x) = \ln(-x) - \ln(-\ln(-x)) + o(1); \qquad (x \to 0^-). \tag{A3}$$

More details, and a Taylor expansion for $|x| < e^{-1}$, can be found in Corless et al. [1]. A key identity is:

$$\ln(a + bx) + cx = \ln d \qquad \Longrightarrow \qquad x = \frac{1}{c}\, W\left(\frac{cd}{b}\, \exp\left(\frac{ac}{b}\right)\right) - \frac{a}{b}\,. \tag{A4}$$

References

1. Corless, R.M.; Gonnet, G.H.; Hare, D.E.G.; Jeffrey, D.J.; Knuth, D.E. On the Lambert W function. *Adv. Comput. Math.* **1996**, *5*, 329–359, doi:10.1007/BF02124750.
2. Valluri, S.R.; Jeffrey, D.J.; Corless, R.M. Some Applications of the Lambert W Function to Physics. *Can. J. Phys.* **2000**, *78*, 823–831, doi:10.1139/p00-065.
3. Vial, A. Fall with linear drag and Wien's displacement law: Approximate solution and Lambert function. *Eur. J. Phys.* **2012**, *33*, 751, doi:10.1088/0143-0807/33/4/751.
4. Stewart, S.M. Wien Peaks and the Lambert W Function. *Rev. Bras. Ensino Física* **2011**, *33*, 3308. Available online: www.sbfisica.org.br (accessed on 5 April 2018).
5. Stewart, S.M. Spectral Peaks and Wien's Displacement Law. *J. Therm. Heat Transfer* **2012**, *26*, 689–692, doi:10.2514/1.T3789.
6. Valluri, S.R.; Gil, M.; Jeffrey, D.J.; Basu, S. The Lambert W function and quantum statistics. *J. Math. Phys.* **2009**, *50*, 102103, doi:10.1063/1.3230482.
7. Boonserm, P.; Visser, M. Bounding the greybody factors for Schwarzschild black holes. *Phys. Rev. D* **2008**, *78*, 101502, arXiv:0806.2209.
8. Barkley Rosser, J. The n'th Prime is Greater than $n \ln n$. *Proc. Lond. Math. Soc.* **1938**, *45*, 21–44.
9. Ribenboim, P. *The Little Book of Big Primes*; Springe: New York, NY, USA, 1991.
10. Ribenboim, P. *The New Book of Prime Number Records*; Springer: New York, NY, USA, 1996.
11. Barkley Rosser, J. Explicit Bounds for some functions of prime numbers. *Am. J. Math.* **1941**, *64*, 211–232.
12. Dusart, P. The k^{th} prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$. *Math. Comput.* **1999**, *68*, 411–415.
13. Cesàro, E. Sur une formule empirique de M. Pervouchine. *Comptes rendus* **1894**, *119*, 848–849. (In French)
14. Cipolla, M. La determinazione assintotica dell'n^{imo} numero primo. *Mat. Nap.* **1902**, *3*, 132–166. (In Italian)
15. Barkley Rosser, J.; Schoenfeld, L. Approximate Formulas for Some Functions of Prime Numbers. *Ill. J. Math.* **1962**, *6*, 64–97.