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Explicit Baker–Campbell–Hausdorff Expansions

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Abstract: The Baker–Campbell–Hausdorff (BCH) expansion is a general purpose tool of use in many branches of mathematics and theoretical physics. Only in some special cases can the expansion be evaluated in closed form. In an earlier article we demonstrated that whenever $[X, Y] = uX + vY + cI$, BCH expansion reduces to the tractable closed-form expression

$$Z(X, Y) = \ln(e^X e^Y) = X + Y + f(u, v)\ [X, Y],$$

where $f(u, v) = f(v, u)$ is explicitly given by the the function

$$f(u, v) = \frac{(u - v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)} = \frac{(u - v) - (ue^{-v} - ve^{-u})}{uv(e^{-v} - e^{-u})}.$$

This result is much more general than those usually presented for either the Heisenberg commutator, $[P, Q] = -i\hbar I$, or the creation-destruction commutator, $[a, a^\dagger] = I$. In the current article, we provide an explicit and pedagogical exposition and further generalize and extend this result, primarily by relaxing the input assumptions. Under suitable conditions, to be discussed more fully in the text, and taking $L_A B = [A, B]$ as usual, we obtain the explicit result

$$\ln(e^X e^Y) = X + Y + \frac{I}{e^{-L_X} - e^{+L_Y}} \left(\frac{I - e^{-L_X}}{L_X} + \frac{I - e^{+L_Y}}{L_Y} \right) [X, Y].$$

We then indicate some potential applications.

Keywords: Lie algebras; matrix exponentials; matrix logarithms; Baker–Campbell–Hausdorff (BCH) formula; commutators; creation-destruction algebra; Heisenberg commutator

MSC: 16W25 (derivations, actions of Lie algebras); 16S20 (centralizing and normalizing extensions); 15A16 (matrix exponential and similar functions of matrices)

1. Introduction

The Baker–Campbell–Hausdorff (BCH) expansion for $Z(X, Y) = \ln(e^X e^Y)$ when X and Y are non-commutative quantities is a general multi-purpose result of considerable interest in not only both pure and applied mathematics [1–14], but also within the fields of theoretical physics, physical chemistry, the theory of numerical integration, and other disciplines [11–20]. Applications include topics as apparently remote and unconnected as the embedding problem for stochastic matrices. For our current purposes, the general BCH expansion can best be written as [11]

$$\ln(e^X e^Y) = X + Y + \int_0^1 \mathrm{d}t \sum_{n=1}^{\infty} \frac{(I - e^{L_X} e^{tL_Y})^{n-1}}{n(n+1)}\ \frac{(e^{L_X} - I)}{L_X}\ [X, Y]. \tag{1}$$

Here, as usual, $L_A B = [A, B]$. If one makes no further simplifying assumptions, then this expression expands to an infinite series of nested commutators with the first few well-known terms being [1,2]

$$\ln(e^X e^Y) \quad = \quad X + Y + \frac{1}{2}\,[X,Y] + \frac{1}{12} L_{X-Y}[X,Y] - \frac{1}{24} L_Y L_X [X,Y] + \ldots \tag{2}$$

Higher-order terms in the expansion quickly become very unwieldy (see, for instance, references [1–14].)

In contrast, by making specific simplifying assumptions about the commutator $[X,Y]$, one can sometimes obtain a terminating series, or develop other ways of simplifying the expansion. The most common terminating series results are as follows:

- If $[X,Y] = 0$, then: $\ln(e^X e^Y) = X + Y$.
- If $[X,Y] = cI$, then: $\ln(e^X e^Y) = X + Y + \frac{1}{2} cI$.
- If $[X,Y] = vY$, then

$$\ln(e^X e^Y) \quad = \quad X + \frac{vY}{1 - e^{-v}} = X + Y + \frac{ve^v - e^v + 1}{v(e^v - 1)}\,[X,Y]. \tag{3}$$

Observe that $[X,Y] = vY$ implies that X acts as a "shift operator", a "ladder operator", for Y, thus allowing one to invoke the techniques of Sack [6]. This particular result can also be extracted from Equation (7.9) of Wilcox [7]; but only after some nontrivial manipulations.

Considerably more subtle is our recent result [11]:

- If $[X,Y] = uX + vY + cI$, then

$$\ln(e^X e^Y) = X + Y + f(u,v)\,[X,Y], \tag{4}$$

where, explicitly, we have

$$f(u,v) = f(u,v) = \frac{(u-v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)}. \tag{5}$$

It is often more useful to write this as

$$f(u,v) = \frac{(u-v) - (ue^{-v} - ve^{-u})}{uv(e^{-v} - e^{-u})}. \tag{6}$$

Sometimes, the structure is more clearly brought out by writing this in the form

$$f(u,v) = \frac{1}{e^{-u} - e^{-v}} \left(\frac{1 - e^{-u}}{u} - \frac{1 - e^{-v}}{v} \right). \tag{7}$$

In a series of recent articles, Matone [21–23] generalized this result in various ways. Matone and Pasti also applied somewhat related ideas to the "covariantization" of differential operators [24], while Bravetti et al. developed a variant for the contact Heisenberg algebra [25]. In the current article, we also develop several generalizations but work towards a rather different goal by instead seeking to weaken the conditions under which this simplified form of the BCH formula applies.

2. Strategy

In reference [11] our strategy was to use the commutator

$$[X,Y] = uX + vY + cI \tag{8}$$

to first deduce

$$L_X[X,Y] = v[X,Y]; \qquad L_Y[X,Y] = -u[X,Y]. \tag{9}$$

Once Equation (9) is established, then Equation (1) collapses to

$$\ln(e^X e^Y) \to X + Y + \int_0^1 \mathrm{d}t \sum_{n=1}^{\infty} \frac{(1-e^v e^{-tu})^{n-1}}{n(n+1)} \frac{(e^v-1)}{v} [X,Y]. \tag{10}$$

This implies

$$f(u,v) = \frac{(e^v-1)}{v} \int_0^1 \mathrm{d}t \sum_{n=1}^{\infty} \frac{(1-e^v e^{-tu})^{n-1}}{n(n+1)}. \tag{11}$$

Performing the sum and evaluating the integral is straightforward, (if a little tedious), with the result given in Equations (5)–(7) (see reference [11] for details). However to obtain this final result, the key step involves the two subsidiary commutators in Equation (9), not the original commutator in Equation (8). This suggests it might be more useful to focus attention on Equation (9), since that is a less restrictive result that does not require Equation (8). The question is whether there are situations where we can get Equation (9) to hold with Equation (8) being violated.

3. Structure Constants

For pedagogical purposes, it is advantageous to consider finite-dimensional Lie algebras with explicit conditions imposed on the structure constants; this pedagogical choice is often particularly useful when communicating with the physics and engineering communities. This is not really a restriction on the mathematics, since once one has found a result, it is easy to extend the discussion to infinite dimensionality. Let us work in some Lie algebra with basis T_a, and define the structure constants $f_{ab}{}^c$ by taking $[T_a, T_b] = f_{ab}{}^c\, T_c$. Then, setting

$$X = x^a\, T_a \qquad \text{and} \qquad Y = y^a\, T_a \tag{12}$$

implies

$$[X,Y] = (x^a\, y^b\, f_{ab}{}^c)\, T_c. \tag{13}$$

We now systematically build up to deriving our most general result in several incremental stages. This is, again, a pedagogical choice aimed at usefully communicating with as wide of a scientific community as possible. We see that the art lies in choosing structure constants appropriately.

3.1. Case 1: Reproducing the Special Commutator

Let us first choose

$$f_{ab}{}^c = m_{[a}\, \delta_{b]}{}^c, \tag{14}$$

then

$$[X,Y] = \frac{1}{2}\left\{(x^a\, m_a)Y - (y^a\, m_a)X\right\}. \tag{15}$$

If we now define

$$X = \hat{X} + \alpha I; \qquad Y = \hat{Y} + \beta I, \tag{16}$$

then

$$[\hat{X},\hat{Y}] \;=\; \frac{1}{2}\left\{(x^a\, m_a)\hat{Y} - (y^a\, m_a)\hat{X}\right\} + \frac{1}{2}\left\{(x^a\, m_a)\beta - (y^a\, m_a)\alpha\right\} I. \tag{17}$$

This is our special commutator of Equation (8) under the identifications

$$u = -\frac{(y^a\, m_a)}{2}; \quad v = \frac{(x^a\, m_a)}{2} \quad \text{with} \quad c = \frac{1}{2}\left\{(x^a\, m_a)\beta - (y^a\, m_a)\alpha\right\}. \tag{18}$$

Thus, this particular set of structure constants has not actually generalized our previous result. Instead it has provided an explicit and quite natural way in which the specific commutator (8) will automatically arise.

3.2. Case 2: Commutator Algebras of Dimension Unity

Let us now choose

$$f_{ab}{}^{c} = \omega_{ab}\, n^{c}. \tag{19}$$

Note that the special commutator of Equation (8) can certainly be put into this form. Specifically, by taking $T_a = \{X, Y, I\}$ we have

$$\omega_{ab} = \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad n^c = (u, v, c). \tag{20}$$

However, we shall now work with completely arbitrary n^c and ω_{ab}, thereby generalizing our previous result. Let us define

$$u = -y^a\, \omega_{ab}\, n^b; \qquad v = x^a\, \omega_{ab}\, n^b \tag{21}$$

and observe

$$[X, Y] = (\omega_{ab}\, x^a\, y^b)\ (n^c\, T_c). \tag{22}$$

Now, we compute

$$\begin{aligned} L_X[X, Y] &= (\omega_{ab}\, x^a\, y^b)\ L_X(n^c\, T_c) \\ &= (\omega_{ab}\, x^a\, y^b)\ (\omega_{ab}\, x^a\, n^b)\ (n^c\, T_c) \\ &= (\omega_{ab}\, x^a\, n^b)\ (\omega_{ab}\, x^a\, y^b)\ (n^c\, T_c) \\ &= v[X, Y]. \end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned} L_Y[X, Y] &= (\omega_{ab}\, x^a\, y^b)\ L_Y(n^c\, T_c) \\ &= (\omega_{ab}\, x^a\, y^b)\ (\omega_{ab}\, y^a\, n^b)\ (n^c\, T_c) \\ &= (\omega_{ab}\, y^a\, n^b)\ (\omega_{ab}\, x^a\, y^b)\ (n^c\, T_c) \\ &= -u[X, Y]. \end{aligned} \tag{24}$$

This establishes Equation (9) *without* requiring Equation (8). More formally, this condition can be phrased as the statement that the commutator $[X, Y]$ is a simultaneous eigenvector of the two adjoint operators L_X and L_Y. Consequently,

$$\ln(e^X e^Y) = X + Y + f(u, v)[X, Y] \tag{25}$$

for the *same* function ($f(u, v)$) as previously encountered. More explicitly, we now have

$$\ln(e^X e^Y) \ = \ X + Y + f\left(x^a\, \omega_{ab}\, n^b, -y^a\, \omega_{ab}\, n^b\right)\ \left(\omega_{ab}\, x^a\, y^b\right)\ (n^c\, T_c)\,. \tag{26}$$

We can also write this as

$$\ln(e^X e^Y) = \left\{x^c + y^c + f\left(x^a\, \omega_{ab}\, n^b, -y^a\, \omega_{ab}\, n^b\right)\ \left(\omega_{ab}\, x^a\, y^b\right)\ n^c\right\}\ T_c. \tag{27}$$

That is, at least in this particular class of Lie algebras, the BCH formula can be viewed as a generalized notion of "addition". By defining the generalized "addition" operator $\oplus$ via $(x \oplus y)^c\, T_c = \ln(e^X e^Y)$, we explicitly have

$$(x \oplus y)^c = x^c + y^c + f\left(x^a \omega_{ab} n^b, -y^a\, \omega_{ab}\, n^b\right)\, \left(\omega_{ab}\, x^a\, y^b\right)\, n^c. \tag{28}$$

Now, $f_{ab}{}^c = \omega_{ab}\, n^c$ can be rephrased as the statement that the commutator sub-algebra $[\mathfrak{g}, \mathfrak{g}]$, (the sub-algebra formed from the commutators of the ambient Lie algebra $\mathfrak{g}$), is of dimension unity.

We typically take $\mathfrak{g}$ to be some arbitrary but fixed ambient Lie algebra, with both $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$. Alternatively, we might initially take $\mathfrak{g}$ to be the minimal free Lie algebra generated by X and Y, but then might add some constraints (e.g., nilpotency, solvability, etc.) to modify that free algebra. Note that the object $[\mathfrak{g}, \mathfrak{g}]$ is also called the first derived sub-algebra, or the first lower central sub-algebra, (i.e., the first descending central sub-algebra), though these two series of sub-algebras will differ once one goes to higher levels.

We observe the following:

- If the commutator sub-algebra $[\mathfrak{g}, \mathfrak{g}]$ is of dimension zero, then the Lie algebra is Abelian, and the BCH result is trivial: $\ln(e^X e^Y) = X + Y$.
- If the commutator sub-algebra $[\mathfrak{g}, \mathfrak{g}]$ is of dimension one, then $[T_a, T_b] \propto N$ for some fixed N. Now, we write $N = n^c\, T_c$, then $[T_a, T_b] \propto (n^c\, T_c)$, thereby implying $[T_a, T_b] = \omega_{ab}\, n^c\, T_c$.
- We can naturally split this into two sub-cases:

$$\omega_{ab}\, n^b = 0 \qquad \text{and} \qquad \omega_{ab}\, n^b \neq 0. \tag{29}$$

- If $\omega_{ab}\, n^b = 0$, then both

$$u = y^a\, \omega_{ab}\, n^b = 0; \qquad \text{and} \qquad v = -x^a\, \omega_{ab}\, n^b = 0. \tag{30}$$

  Therefore, $L_X[X, Y] = 0 = L_Y[X, Y]$, and so

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0. \tag{31}$$

  That is, the second lower central sub-algebra is trivial, and, in particular, the original Lie algebra is nilpotent. (For example, the Heisenberg algebra $[P, Q] = -i\hbar I$ and the creation-destruction algebra $[a, a^\dagger] = I$ are very commonly occurring Lie algebras of this type.)
- If $\omega_{ab}\, n^b \neq 0$ then u and v are nontrivial, and $f(u, v)$ is also nontrivial. The Lie algebra is now not nilpotent but satisfies the more subtle condition that

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \mathfrak{g}]. \tag{32}$$

  That is, the second lower central sub-algebra, (and so all the higher-order lower central sub-algebras), equals the first lower central sub-algebra. This can also be phrased as the demand that the commutator sub-algebra be an ideal of the underlying Lie algebra.

In short, the explicit BCH Formula (26) holds whenever the commutator sub-algebra $[\mathfrak{g}, \mathfrak{g}]$ is of dimension unity.

3.3. Case 3: Nilpotent Lie Algebras

Now we consider the higher terms in the lower central series, defined iteratively by

$$\mathfrak{g}_0 = \mathfrak{g}; \qquad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]; \qquad \mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]. \tag{33}$$

If, for some n, we have $\mathfrak{g}_n = 0$, then the Lie algebra $\mathfrak{g}$ is said to be "nilpotent". In this case, all nth-order and higher commutators vanish and the BCH series truncates, but this result has previously been (implicitly) used when developing the Reinsch algorithm [9], and our own simplified variant thereof [14]. That algorithm works by utilizing a faithful representation for the first n nested commutators of a free Lie algebra in terms of strictly upper triangular $(n+1) \times (n+1)$ matrices with entires only on the first super-diagonal. That is, working with a level-n nilpotent Lie algebra "merely" reproduces the first n terms in the BCH formula, and gives zeros thereafter. So, while certainly being useful, this is not really new [9,14]. In terms of the structure constants, nilpotency is achieved if at some stage

$$f_{ab}{}^{i}\, f_{ic}{}^{j}\, f_{jd}{}^{k}\, f_{ke}{}^{m} \cdots = 0. \tag{34}$$

3.4. Case 4: Abelian Commutator Algebras

Can the discussion above be generalized even further? Note that, in all generality,

$$[[T_a, T_b], [T_c, T_d]] = f_{ab}{}^{m} f_{cd}{}^{n} f_{mn}{}^{e}\, T_e. \tag{35}$$

So, whenever $f_{ab}{}^{c} = \omega_{ab}\, n^c$, we have

$$[[T_a, T_b], [T_c, T_d]] = 0, \tag{36}$$

or more abstractly,

$$[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0. \tag{37}$$

That is, the commutator sub-algebra is Abelian. This is a specific special case of a "solvable" Lie algebra. Can anything be done for more general solvable Lie algebras?

Let us now consider the situation where the the commutator sub-algebra is Abelian, but we do *not* demand that the commutator algebra is one dimensional. The Jacobi identity leads to

$$\begin{aligned} L_X L_Y W &= [X, [Y, W]] && (38)\\ &= -[Y, [W, X]] - [W, [X, Y]] && (39)\\ &= [Y, [X, W]] + [[X, Y], W] && (40)\\ &= L_Y L_X W + L_{[X,Y]} W. && (41) \end{aligned}$$

That is,

$$[L_X, L_Y] W = L_{[X,Y]} W. \tag{42}$$

However, if W is itself a commutator $W = [U, V]$, and if the commutator algebra is Abelain, $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, then we have

$$[L_X, L_Y][U, V] = 0. \tag{43}$$

That is, in this situation, and *when acting on commutators*, L_X and L_Y commute. However, this is exactly the situation in theBCH expansion of Equation (1)—L_X and L_Y are always acting on commutators. So, as long as the commutator sub-algebra is itself Abelian we can rearrange Equation (1) to write

$$\ln(e^X e^Y) = X + Y + \frac{(e^{L_X} - I)}{L_X} \int_0^1 \mathrm{d}t \sum_{n=1}^{\infty} \frac{(I - e^{L_X} e^{tL_Y})^{n-1}}{n(n+1)}\, [X, Y] \tag{44}$$

and treat the L_X and L_Y *as though they commute* with each other. However, then, summing and integrating as previously, we have

$$\ln(e^X e^Y) = X + Y + f(L_X, -L_Y)[X, Y] \tag{45}$$

for exactly the same function ($f(u,v)$) as before, but now subject only to the condition ($[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]=0$) that the commutator algebra be Abelian. Note that this last formula is still an operator equation, which still contains an infinite set of nested commutators—albeit in a relatively explicit manner. Indeed, under the stated conditions from Equation (7), we see

$$\ln(e^X e^Y) = X + Y + \frac{I}{e^{-L_X} - e^{+L_Y}} \left(\frac{I - e^{-L_X}}{L_X} + \frac{I - e^{+L_Y}}{L_Y} \right) [X,Y]. \tag{46}$$

With some hindsight, this operator expression can be seen to be closely related to the meta-Abelian analysis of Kurlin [26]. Note that in terms of the structure constants, the condition $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]=0$ is equivalent to the explicit constraint,

$$f_{ab}{}^{m}\, f_{cd}{}^{n}\, f_{mn}{}^{e} = 0. \tag{47}$$

Furthermore, we note the series expansion

$$f(u,v) \quad = \quad \frac{1}{2} + \frac{u+v}{12} + \frac{uv}{24} - \frac{(u+v)(u^2-5uv+v^2)}{720} - \frac{uv(u^2-4uv+v^2)}{1440} + \dots \tag{48}$$

This verifies (as it should) that the operator $f(L_X,-L_Y)$ contains only non-negative powers of L_X and L_Y.

3.5. Case 5: $[X,Y]$ is in the Centre of the Commutator Algebra

Let us now relax the conditions for the validity of this result even further. The key step is to realize that

$$[L_X, L_Y][U,V] = L_{[X,Y]}[U,V], \tag{49}$$

so that L_X (effectively) commutes with L_Y as long as $[[X,Y],[U,V]] = 0$. As previously noted, this certainly holds as long as the commutator algebra is Abelian, $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 0$, but it is quite sufficient to demand the weaker condition that the *specific* commutator $[X,Y]$ is an element of the centre ($Z_{[\mathfrak{g},\mathfrak{g}]}$) of the commutator algebra $[\mathfrak{g},\mathfrak{g}]$. Note this is already a weaker condition than the meta-Abelian condition considered by Kurlin [26]. That is,

$$[[X,Y],[\mathfrak{g},\mathfrak{g}]] = 0. \tag{50}$$

In terms of the structure constants, this is equivalent to the weakened constraint:

$$x^a\, y^b\, f_{ab}{}^{m}\, f_{cd}{}^{n}\, f_{mn}{}^{e} = 0. \tag{51}$$

Under this milder condition, we still have (effective) commutativity of L_X with L_Y, thereby allowing us to treat the L_X and L_Y appearing in the BCH Formula (1) *as though* they commute. By integrating and summing the series, we again see

$$\ln(e^X e^Y) = X + Y + f(L_X, -L_Y)[X,Y] \tag{52}$$

again for exactly the same function ($f(u,v)$), now subject only to the weaker condition that $[[X,Y],[\mathfrak{g},\mathfrak{g}]] = 0$. Under the stated conditions that the *specific* commutator $[X,Y]$ be an element of the centre of the commutator algebra ($[\mathfrak{g},\mathfrak{g}]$), we again find

$$\ln(e^X e^Y) = X + Y + \frac{I}{e^{-L_X} - e^{+L_Y}} \left(\frac{I - e^{-L_X}}{L_X} + \frac{I - e^{+L_Y}}{L_Y} \right) [X,Y]. \tag{53}$$

3.6. Case 6: $[X,Y]$ is in the Centralizer of $\{L_X^m L_Y^n [X,Y]\}$

As our final weakening of the input assumptions, (while still keeping the same strength conclusions), an arbitrary but fixed ambient Lie algebra $\mathfrak{g}$ is taken and the following set is considered:

$$\mathcal{S} = \{L_X^m L_Y^n [X,Y];\ m \geq 0, n \geq 0\}. \tag{54}$$

The construction of this set is inspired by considering the form of the terms which appear in the BCH expansion of Equation (1). If, in contrast, we were to take $\mathfrak{g}$ as the minimal free algebra generated by X and Y, then this would not be a weakening of case 5; it would merely be a restatement of case 5.

If we now demand merely that $[X,Y]$ commute with all the elements of $\mathcal{S}$ (that is, $[[X,Y],\mathcal{S}] = 0$ or equivalently $L_{[X,Y]}\mathcal{S} = 0$, so that $[X,Y]$ is in the so-called centralizer of the set $\mathcal{S}$), then the Jacobi identity (in the form (42)), implies

$$[L_X, L_Y] L_X^m L_Y^n [X,Y] = 0, \tag{55}$$

which we could also write as

$$[L_X, L_Y]\mathcal{S} = 0. \tag{56}$$

Note this is again a weaker condition than the meta-Abelian condition considered by Kurlin [26]. Then, in particular,

$$\begin{aligned} L_Y L_X^m L_Y^n [X,Y] &= L_X L_Y L_X^{m-1} L_Y^n [X,Y] \\ &= L_X^2 L_Y L_X^{m-2} L_Y^n [X,Y] \\ &= \quad \ldots \\ &= L_X^m L_Y^{n+1} [X,Y]. \end{aligned} \tag{57}$$

That is, under these conditions, L_X and L_Y can still be treated *as though* they commute in the BCH expansion. Under these conditions, all of the terms appearing in the BCH expansion of Equation (1) can now be reduced to elements of the set $\mathcal{S}$. By integrating and summing the series, we again see

$$\ln(e^X e^Y) = X + Y + f(L_X, -L_Y)[X,Y] \tag{58}$$

again for exactly the same function ($f(u,v)$), but now subject only to the even weaker condition ($[[X,Y],\mathcal{S}] = 0$) that the specific commutator $[X,Y]$ be an element of the centralizer of $\mathcal{S}$. To be explicit about this, under the stated conditions

$$\ln(e^X e^Y) = X + Y + \frac{I}{e^{-L_X} - e^{+L_Y}} \left(\frac{I - e^{-L_X}}{L_X} + \frac{I - e^{+L_Y}}{L_Y} \right) [X,Y]. \tag{59}$$

Careful inspection of the above quickly verifies that the *only* terms present when one expands the above are of the form $L_X^m L_Y^n [X,Y]$, (the elements of the set $\mathcal{S}$), and that our simplifying assumption has eliminated all terms, such as $L_{[X,Y]} L_X^m L_Y^n [X,Y]$ and variants thereof. By summing over the integers m and n, the centralizer condition can also be restated as

$$[[X,Y], e^{sL_X} e^{tL_Y} [X,Y]] = 0; \qquad \forall s, t. \tag{60}$$

This is as far as we have currently been able to weaken the input assumptions that we originally started with while still keeping a reasonably close analogue of our initial result involving the function $f(u,v)$.

4. Discussion

The BCH formula is a general purpose tool that has found many applications both in pure and applied mathematics [1–11,14], and generally in the physical sciences [11,14–20]. Via the study of the embeddability problem for stochastic matrices (Markov processes), there are even potential applications in the social sciences and financial sector. Explicit closed-form results are relatively rare (see the Introduction for examples). In this present article, we significantly extended our previous results reported in reference [11] by systematically weakening the input assumptions. In a number of increasingly general situations, we showed that the BCH expansion can be written in closed form as

$$\ln(e^X e^Y) = X + Y + f(u,v)[X,Y], \tag{61}$$

where $f(u,v)$ is the symmetric function

$$f(u,v) = f(v,u) = \frac{(u-v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)}. \tag{62}$$

This was first demonstrated in reference [11] for the very explicit commutator $[X,Y] = uX + vY + cI$. Herein, (with suitable expressions for u and v), a structurally identical result was established for Lie algebras with a one-dimensional commutator sub-algebra. More generally, whenever the commutator sub-algebra is Abelian, one has

$$\ln(e^X e^Y) = X + Y + f(L_X, -L_Y)[X,Y]. \tag{63}$$

More specifically,

$$\ln(e^X e^Y) = X + Y + \frac{I}{e^{-L_X} - e^{+L_Y}} \left(\frac{I - e^{-L_X}}{L_X} + \frac{I - e^{+L_Y}}{L_Y} \right) [X,Y]. \tag{64}$$

This result furthermore extends to the weaker input condition $[X,Y] \in Z_{[g,g]}$, that is, $[X,Y]$ is an element of the centre of the commutator algebra. Even more generally, this result extends to $[X,Y]$ which is an element of the centralizer of those Lie brackets that appear in the BCH expansion. Overall, we find it quite remarkable just how far we have been able to push this result. There are of course many other directions that one might also wish to explore—we have concentrated our efforts on directions in which it seems that relatively concrete and explicit results might be readily extractable.

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