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Edge Irregular Reflexive Labeling for the Disjoint Union of Gear Graphs and Prism Graphs

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Abstract: In graph theory, a graph is given names—generally a whole number—to edges, vertices, or both in a chart. Formally, given a graph $G = (V, E)$, a vertex naming is a capacity from V to an arrangement of marks. A diagram with such a capacity characterized defined is known as a vertex-marked graph. Similarly, an edge naming is a mapping of an element of E to an arrangement of marks. In this case, the diagram is called an edge-marked graph. We consider an edge irregular reflexive k-labeling for the disjoint association of wheel-related diagrams and deduce the correct estimation of the reflexive edge strength for the disjoint association of m copies of some wheel-related graphs, specifically gear graphs and prism graphs.

Keywords: edge irregular reflexive labeling; reflexive edge strength; gear graphs; prism graphs

MSC: 05C12, 05C90

1. Introduction.

All diagrams considered in this paper are basic, limited and undirected. Chartrand et al. [1] proposed the following: relegate a positive whole number mark from the set $\{1, 2, ..., k\}$ to the edges of a straightforward associated graph with no less than three in such a path, so that the diagram is sporadic, i.e., the weight (name entirety) at every vertex are particular. The question is then what is the base estimation of the biggest k over all such unpredictable assignments. This parameter for a chart G is the irregularity strength of the graph G.

A comprehensive overview on the irregularity strength is given by Lahel [2]. For further analysis, see the papers by Amar and Togni [3], Dimitz et al. [4], Gyarfas [5] and Nierhoff [6]. Following from these papers, an edge irregular k-labeling as a vertex naming $\xi : V(G) \to \{1, 2, ..., k\}$ was characterized so that for each two distinct edges hg and $h^{'}g^{'}$ there is $wt_{\phi}(hg) \neq w_{\xi}(h^{'}g^{'})$, where the heaviness of an edge $hg \in E(G)$ is $wt_{\xi}(hg) = \xi(h) + \xi(g)$. The base k for which the graph G has an edge abnormality quality k-labeling is called the edge irregularity strength of the diagram G, represented as $es(G)$. In [7] the limits of the parameters $es(G)$, are evaluated and the correct estimation of the edge irregularity strength for a few groups of diagrams are resolved, in particular for ways, stars, and twofold stars, and the Cartesian result of two paths.

Baca et al. [8] called the aggregate marking $\xi : V(G) \cup E(G) \to \{1, 2, ..., k\}$ an edge irregular total k-labeling of the chart G if for each two distinctive edge qr and $q^{'}r^{'}$ of G, $wt_{\xi}(qr) = \xi(q) + \xi(qr) + \xi(r) \neq wt_{\xi}(q^{'}r^{'}) = \xi(q^{'}) + \xi(q^{'}r^{'}) + \xi(r^{'})$. The edge abnormality quality, $tes(G)$, is the base k for which G has an edge irregular, total k-labeling. Evaluation of these parameters can be obtained, giving exact estimations of the aggregate edge abnormality quality for ways, cycles, stars, and haggle charts. Further details about the aggregate abnormality quality can be found in [9–14].

Issue related to unpredictable marking emerge from thinking of graphs with particular degree. In a straightforward graph, it is not possible to develop a diagram in which each vertex has a one-of-a-kind degree; be that as it may, it is possible in multigraphs (graphs in which we permit various edges between the neighboring vertices). The inquiry at this point moves toward: what is the smallest number of parallel edges between two vertices required to guarantee that the diagram shows vertex abnormality? This issue is equivalent to the marking issue as portrayed toward the start of this section. Ryan et al. [15] asserted that the vertex marks ought to represent circles at the vertex. The outcome was two-fold: first, every vertex mark was required to be a whole number, since each circle added two to the vertex degree; and second, dissimilar to absolute unpredictable marking, the mark 0 was allowed to represent a loopless vertex. Edges were then named by whole numbers from 1 to k. In this manner, they defined the marking $\chi_e : E(G) \to \{1, 2, ..., k_e\}$ and $\chi_v : V(G) \to \{0, 2, ..., 2k_v\}$, and termed χ as an aggregate k-labeling of G, with $\chi(x) = \chi_v(x)$ if $x \in V(G)$. Further, $\chi(x) = \chi_e(x)$ if $x \in E(G)$, where $k = max\{k_e, 2k_v\}$.

The total k-labeling χ is called an edge irregular reflexive k-labeling of the diagram G if for each two unique edges rs and $r^{'}s^{'}$ of G, $wt(rs) = \chi_v(r) + \chi_e(rs) + \chi_v(s) \neq wt(r^{'}s^{'}) = \chi_v(r^{'}) + \chi_e(r^{'}s^{'}) + \chi_v(s^{'})$. The smallest estimation of k for which such a marking exists is known as the reflexive edge strength of the diagram G and is termed $res(G)$. For ongoing outcomes see [16,17].

The effect of this variety is not generally visible in the marking quality, but rather produced some imperative outcomes:

$$tes(K_5) = 5 \mathrm{whereasres}(\mathrm{K}_5) = 4$$

The impact of this change leads to the following conjecture which can be used to evaluate some problematic special cases (see [18]).

Conjecture 1. *Any graph G with the most extreme degree $\Delta(G)$ other than K_5 fulfills:*

$$\mathrm{tes}(G) = \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil\right\}.$$

Baca et al. [19] proposed the following conjecture and proved Theorem 1.

Conjecture 2. *Any graph G with most extreme degree $\Delta(G)$ fulfills:*

$$\mathrm{res}(G) = \max\left\{\left\lceil \frac{|E(G)|}{3} + r \right\rceil, \left\lfloor \frac{\Delta+2}{2} \right\rfloor\right\},$$

where $r = 1$ for $|E(G)| \equiv 3(mod \;\; 6)$, $|E(G)| \equiv 2(mod \;\; 6)$ and zero otherwise.

Theorem 1. *For every graph G, $res(C_n) =$* $\begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } n \not\equiv 2, 3(mod \;\; 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } n \equiv 2, 3(mod \;\; 6). \end{cases}$

2. Constructing an Edge Irregular Reflexive Labeling

Let us recall the following lemma:

Lemma 1. *For every graph G, $res(G) \geq$* $\begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } n \not\equiv 2, 3(mod \;\; 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } n \equiv 2, 3(mod \;\; 6). \end{cases}$

The lower bound for $res(G)$ follows from the insignificant edge weight under an edge irregular reflexive labeling in one. The basis of the maximal edge weight is that $|E(G)|$ can be accomplished as the total of only three numbers, no less than two of which are indeed.

In this paper, we explore the reflexive edge irregularity strength for disjoint association of m duplicates of gear and prism graphs.

3. The Gear Graph

The Jahangir graph is denoted $J_{n,m}, n \geq 3, m \geq 1$, and can be obtained from a wheel graph by adding vertices in between the vertices that lie on the rim. I. It was introduced by Tomescu in [20]. For $m = 2$ it is known as the *gear graph* (see [21]). In the next theorem, we determined the edge irregular reflexive strength of disjoint union of consecutive, non-isomorphic m copies of a gear graph $J^j_{n_j,2}$, $n_j \geq 3$. The vertex set and the edge set of $\left(\bigcup\limits_{j=1}^{m} J^j_{n_j,2}\right)$ are defined as follows:

$$V\Big(\bigcup_{j=1}^{m} J^j_{n_j,2}\Big) = \Big\{x_i^j, y_i^j : 1 \leq i \leq n_j, \;\; 1 \leq j \leq m\Big\} \cup \bigcup_{j=1}^{m}\{c^j\},$$

$$E\Big(\bigcup_{j=1}^{m} J^j_{n_j,2}\Big) = \Big\{x_i^j y_i^j, c^j x_i^j, x_{i+1}^j y_i^j : 1 \leq i \leq n_j, \;\; 1 \leq j \leq m\Big\}.$$

Also it is easy to see that:

$$\Big|V\Big(\bigcup_{j=1}^{m} J^j_{n_j,2}\Big)\Big| = 2 \cdot \sum_{j=1}^{m} n_j + j$$

$$\Big|E\Big(\bigcup_{j=1}^{m} J^j_{n_j,2}\Big)\Big| = 3 \cdot \sum_{j=1}^{m} n_j$$

Theorem 2. *Let* $(\bigcup\limits_{j=1}^{m} J^j_{n_j,2})$ *be the disjoint union of m consecutive, non-isomorphic copies of gear graphs with* $n_j \geq 3, \quad 1 \leq j \leq m$ *and* $n_{j+1} = n_j + 1$, *with* $n_1 = 3$, *and i is to be taken modulo* n_j. *Then*

$$res(\bigcup_{j=1}^{m} J^j_{n_j,2}) = \begin{cases} \sum\limits_{j=1}^{m} n_j + 1, & \text{if } \sum\limits_{j=1}^{m} n_j \equiv 2,3(mod\ \ 6), \\ \sum\limits_{j=1}^{m} n_j, & \text{if } \sum\limits_{j=1}^{m} n_j \not\equiv 2,3(mod\ \ 6). \end{cases}$$

Proof. From Lemma 1, we get

$$res(\bigcup_{j=1}^{m} J^j_{n_j,2}) \geq \begin{cases} \sum\limits_{j=1}^{m} n_j + 1, & \text{if } \sum\limits_{j=1}^{m} n_j \equiv 2,3(mod\ \ 6), \\ \sum\limits_{j=1}^{m} n_j, & \text{if } \sum\limits_{j=1}^{m} n_j \not\equiv 2,3(mod\ \ 6). \end{cases}$$

Next, we will show that:

$$res(\bigcup_{j=1}^{m} J^j_{n_j,2}) \leq \begin{cases} \sum\limits_{j=1}^{m} n_j + 1, & \text{if } \sum\limits_{j=1}^{m} n_j \equiv 2,3(mod\ \ 6), \\ \sum\limits_{j=1}^{m} n_j, & \text{if } \sum\limits_{j=1}^{m} n_j \not\equiv 2,3(mod\ \ 6). \end{cases}$$

For this we define an f-labeling on $(\bigcup\limits_{j=1}^{m} J^j_{n_j,2})$ as follow:

For $j = 1$ and $n_1 = 3$, Figure 1 shows labelings of vertices and edges along with their weights.

For $j \geq 2$, we have the following labelings and their weights as follows:

$$f(x_i^j) = \begin{cases} \sum_{j=1}^{m} n_j, & \text{if } i = 1,2 \ \ and \ \ j \equiv 0(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 1 + 2\lceil \frac{i}{2} \rceil, & \text{if } 3 \leq i < n_j \ \ and \ \ j \equiv 0(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 2 + 2\lceil \frac{i}{2} \rceil, & \text{if } 1 \leq i < n_j \ \ and \ \ j \equiv 1(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 1 + 2\lceil \frac{i}{2} \rceil, & \text{if } 1 \leq i < n_j \ \ and \ \ j \equiv 2(mod\ \ 4) \\ \sum_{j=1}^{m} n_j, & \text{if } i = 1 \ \ and \ \ j \equiv 3(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 2 + 2\lceil \frac{i}{2} \rceil, & \text{if } 2 \leq i < n_j \ \ and \ \ j \equiv 3(mod\ \ 4) \end{cases}$$

$$f(y_i^j) = \begin{cases} \sum_{j=1}^{m-1} n_j - 2 + 2\lceil \frac{i}{2} \rceil, & \text{if } 1 \leq i \leq n_j \ \ and \ \ j \equiv 0,1,3(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 1 + 2\lceil \frac{i}{2} \rceil, & \text{if } 1 \leq i \leq n_j \ \ and \ \ j \equiv 2(mod\ \ 4) \end{cases}$$

$$f(c^j) = \begin{cases} \sum_{j=1}^{m} n_j, & \text{if } j \equiv 0,3(mod\ \ 4) \\ \sum_{j=1}^{m} n_j + 1, & \text{if } j \equiv 1,2(mod\ \ 4) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \sum_{j=1}^{m-1} n_j - n_j - 1 + 2i, & \text{if } i = 1,2 \ \ and \ \ j \equiv 0(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j + 1, & \text{if } 3 \leq i \leq n_j - 1 \ \ and \ \ (i \ \ is \ \ odd) \ \ j \equiv 0(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j + 1, & \text{if } 1 \leq i \leq n_j \ \ and \ \ (i \ \ is \ \ odd) \ \ j \equiv 1(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 1, & \text{if } 1 \leq i \leq n_j - 1 \ \ and \ \ (i \ \ is \ \ odd) \ \ j \equiv 2(mod\ \ 4) \\ \sum_{j=1}^{m-2} n_j - 1, & \text{if } i = 1 \ \ and \ \ j \equiv 3(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j - 1, & \text{if } 3 \leq i \leq n_j, \ \ (i \ \ is \ \ odd) \ \ and \ \ j \equiv 3(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j + 3, & \text{if } 4 \leq i \leq n_j, \ \ (i \ \ is \ \ even) \ \ and \ \ j \equiv 0(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j + 3, & \text{if } 2 \leq i \leq n_j - 1, \ \ (i \ \ is \ \ even) \ \ and \ \ j \equiv 1(mod\ \ 4) \\ \sum_{j=1}^{m-1} n_j + 1, & \text{if } 2 \leq i \leq n_j, \ \ (i \ \ is \ \ even) \ \ and \ \ j \equiv 2,3(mod\ \ 4) \end{cases}$$

$$f(x_{i+1}^j y_i^j) = \begin{cases} \sum\limits_{j=1}^{m-1} n_j - n_j + 2, & \text{if } i = 1 \ \ and \quad j \equiv 0 (mod \ \ 4) \\ \sum\limits_{j=1}^{m-1} n_j + 2, & \text{if } 2 \leq i \leq n_j - 1 \ \ and \quad j \equiv 0 (mod \ \ 4) \\ \sum\limits_{j=1}^{m-1} n_j + 2, & \text{if } 1 \leq i \leq n_j - 1 \ \ and \quad j \equiv 1 (mod \ \ 4) \\ \sum\limits_{j=1}^{m} n_j + 1, & \text{if } i = n_j \ \ and \quad j \equiv 1 (mod \ \ 4) \\ \sum\limits_{j=1}^{m-1} n_j, & \text{if } 1 \leq i \leq n_j \ \ and \quad j \equiv 2,3 (mod \ \ 4) \end{cases}$$

$$f(c^j x_i^j) = \begin{cases} \sum\limits_{j=1}^{m} n_j - 1, & \text{if } 1 \leq i \leq n_j - 1, \ \ (i \ \ is \ \ odd) \quad j \equiv 0,2 (mod \ \ 4) \\ \sum\limits_{j=1}^{m} n_j, & \text{if } 1 \leq i \leq n_j, \ \ (i \ \ is \ \ odd) \quad j \equiv 1 (mod \ \ 4) \\ \sum\limits_{j=1}^{m} n_j, & \text{if } i = 1 \ \ and \quad j \equiv 3 (mod \ \ 4) \\ \sum\limits_{j=1}^{m} n_j - 1, & \text{if } 3 \leq i \leq n_j, \ \ (i \ \ is \ \ odd) \quad j \equiv 3 (mod \ \ 4) \\ \sum\limits_{j=1}^{m} n_j, & \text{if } 2 \leq i \leq n_j, \ \ (i \ \ is \ \ even) \quad j \equiv 0,1,2,3 (mod \ \ 4) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} 3\sum\limits_{j=1}^{m-1} n_j - 3 + 2i + 2\lceil \frac{i}{2} \rceil, & \text{if } i = 1,2 \ \ and \quad j \equiv 0 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 3 + 4\lceil \frac{i}{2} \rceil, & \text{if } 3 \leq i \leq n_j - 1 \ \ (i \ \ is \ \ odd) \ \ and \quad j \equiv 0 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 3 + 4\lceil \frac{i}{2} \rceil, & \text{if } 1 \leq i \leq n_j, \ \ (i \ \ is \ \ odd) \ \ and \quad j \equiv 1 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 3 + 4\lceil \frac{i}{2} \rceil, & \text{if } 1 \leq i \leq n_j - 1, \ \ (i \ \ is \ \ odd) \ \ and \quad j \equiv 2 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j + n_j - 1, & \text{if } i = 1 \ \ and \quad j \equiv 3 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 5 + 4\lceil \frac{i}{2} \rceil, & \text{if } 3 \leq i \leq n_j, \ \ (i \ \ is \ \ odd) \ \ and \quad j \equiv 3 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 1 + 4\lceil \frac{i}{2} \rceil, & \text{if } 4 \leq i \leq n_j, \ \ (i \ \ is \ \ even) \ \ and \quad j \equiv 0 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 1 + 4\lceil \frac{i}{2} \rceil, & \text{if } 2 \leq i \leq n_j - 1, \ \ (i \ \ is \ \ even) \ \ and \quad j \equiv 1 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 1 + 4\lceil \frac{i}{2} \rceil, & \text{if } 2 \leq i \leq n_j, \ \ (i \ \ is \ \ even) \ \ and \quad j \equiv 2 (mod \ \ 4) \\ 3\sum\limits_{j=1}^{m-1} n_j - 3 + 4\lceil \frac{i}{2} \rceil, & \text{if } 2 \leq i \leq n_j - 1, \ \ (i \ \ is \ \ even) \ \ and \quad j \equiv 3 (mod \ \ 4) \end{cases}$$

$$
wt(x^j_{i+1}y^j_i) = \begin{cases}
3\sum\limits_{j=1}^{m-1} n_j + 2, & \text{if } i = 1 \ \ and \ \ j \equiv 0(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j - 2 + 4\lceil \frac{i}{2} \rceil, & \text{if } 2 \le i \le n_j - 1 \ \ and \ \ j \equiv 0(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j - 2 + 4\lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n_j - 1 \ \ and \ \ j \equiv 1(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j - 3 + n_j + 4\lceil \frac{n_j}{2} \rceil, & \text{if } i = n_j \ \ and \ \ j \equiv 1(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j - 2 + 4\lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n_j \ \ and \ \ j \equiv 2(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j - 4 + 4\lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n_j \ \ and \ \ j \equiv 3(mod\ 4)
\end{cases}
$$

$$
wt(c^j x^j_i) = \begin{cases}
3\sum\limits_{j=1}^{m} n_j - 1, & \text{if } i = 1 \ \ and \ \ j \equiv 0(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j, & \text{if } i = 2 \ \ and \ \ j \equiv 0(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j - 3 + 2\lceil \frac{i}{2} \rceil, & \text{if } 3 \le i \le n_j - 1 \ \ (i \ is \ odd) \ \ and \ \ j \equiv 0(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j - 1 + 2\lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n_j, \ \ (i \ is \ odd) \ \ and \ \ j \equiv 1(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j - 1 + 2\lceil \frac{i}{2} \rceil, & \text{if } 1 \le i \le n_j - 1, \ \ (i \ is \ odd) \ \ and \ \ j \equiv 2(mod\ 4) \\
3\sum\limits_{k=1}^{j} n_k, & \text{if } i = 1 \ \ and \ \ j \equiv 3(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j - 3 + 2\lceil \frac{i}{2} \rceil, & \text{if } 3 \le i \le n_j - 1, \ \ (i \ is \ odd) \ \ and \ \ j \equiv 3(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j - 2 + 2\lceil \frac{i}{2} \rceil, & \text{if } 4 \le i \le n_j - 1, \ \ (i \ is \ even) \ \ and \ \ j \equiv 0(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j + 2\lceil \frac{i}{2} \rceil, & \text{if } 2 \le i \le n_j - 1, \ \ (i \ is \ even) \ \ and \ \ j \equiv 1(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j + 2\lceil \frac{i}{2} \rceil, & \text{if } 2 \le i \le n_j, \ \ (i \ is \ even) \ \ and \ \ j \equiv 2(mod\ 4) \\
3\sum\limits_{j=1}^{m-1} n_j + 2n_j - 2 + 2\lceil \frac{i}{2} \rceil, & \text{if } 2 \le i \le n_j - 1, \ \ (i \ is \ even) \ \ j \equiv 3(mod\ 4)
\end{cases}
$$

It is easy to check that no two edges have the same weight. Therefore, f is an edge irregular reflexive labeling of $(\bigcup\limits_{j=1}^{m} J^j_{n_j,2})$ for $n_j \ge 3$, which completes the proof. □

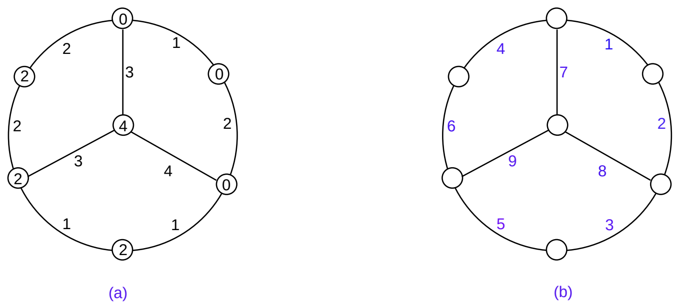

Figure 1. (**a**) A reflexive irregular 4-labeling of $J^1_{3,2}$ and (**b**) The edge weights of $J^1_{3,2}$.

4. The Prism Graph

Let D_n be a prism graph with

$$V(D_n) = \{x_i, y_i : i = 1, 2, ..., n\},$$

$$E(H_n) = \{x_i x_{i+1}, x_i y_i, x_i x_{i+1} : i = 1, 2, ..., n\}.$$

In the next theorem, we determine the edge irregular reflexive k-labeling for a disjoint union of consecutive, non-isomorphic m copies of prism graphs $D^j_{n_j}$ for $n_j \geq 3$. The vertex set and the edge set of $\left(\bigcup\limits_{j=1}^{m} D^j_{n_j}\right)$ is given as follows:

$$V\Big(\bigcup_{j=1}^{m} D^j_{n_j}\Big) = \Big\{x^j_i, y^j_i : 1 \leq i \leq n_j, \quad 1 \leq j \leq m\Big\},$$

$$E\Big(\bigcup_{j=1}^{m} D^j_{n_j}\Big) = \Big\{x^j_i x^j_{i+1}, x^j_i y^j_i, y^j_i y^j_{i+1} : 1 \leq i \leq n_j, \quad 1 \leq j \leq m\Big\}.$$

It is easy to see that

$$\Big|V\Big(\bigcup_{j=1}^{m} D^j_{n_j}\Big)\Big| = 2 \cdot \sum_{j=1}^{m} n_j$$

$$\Big|E\Big(\bigcup_{j=1}^{m} D^j_{n_j}\Big)\Big| = 3 \cdot \sum_{j=1}^{m} n_j$$

Theorem 3. *Let* $(\bigcup\limits_{j=1}^{m} D^j_{n_j})$ *be the disjoint union of consecutive, non-isomorphic* m *copies of prism graphs with* $n_j \geq 3, \quad 1 \leq j \leq m$ *and* $n_{j+1} = n_j + 1$, $n_1 = 3$, i *is to be taken modulo* n_j. *Then*

$$res(\bigcup_{j=1}^{m} D^j_{n_j}) = \begin{cases} \sum\limits_{j=1}^{m} n_j + 1, & \text{if } \sum\limits_{j=1}^{m} n_j \equiv 2, 3 (mod \;\; 6), \\ \sum\limits_{j=1}^{m} n_j, & \text{if } \sum\limits_{j=1}^{m} n_j \not\equiv 2, 3 (mod \;\; 6). \end{cases}$$

Proof. From Lemma 1, we get

$$res(\bigcup_{j=1}^{m} D^j_{n_j}) \geq \begin{cases} \sum\limits_{j=1}^{m} n_j + 1, & \text{if } \sum\limits_{j=1}^{m} n_j \equiv 2, 3 (mod \;\; 6), \\ \sum\limits_{j=1}^{m} n_j, & \text{if } \sum\limits_{j=1}^{m} n_j \not\equiv 2, 3 (mod \;\; 6). \end{cases}$$

Next, we will show that

$$res(\bigcup_{j=1}^{m} D^j_{n_j}) \leq \begin{cases} \sum\limits_{j=1}^{m} n_j + 1, & \text{if } \sum\limits_{j=1}^{m} n_j \equiv 2, 3 (mod \;\; 6), \\ \sum\limits_{j=1}^{m} n_j, & \text{if } \sum\limits_{j=1}^{m} n_j \not\equiv 2, 3 (mod \;\; 6). \end{cases}$$

For this we define a f-labeling on $(\bigcup\limits_{j=1}^{m} D^j_n)$ as follow:

For $j = 1$, Figure 2 shows the labeling of vertices and edges along with their weights.

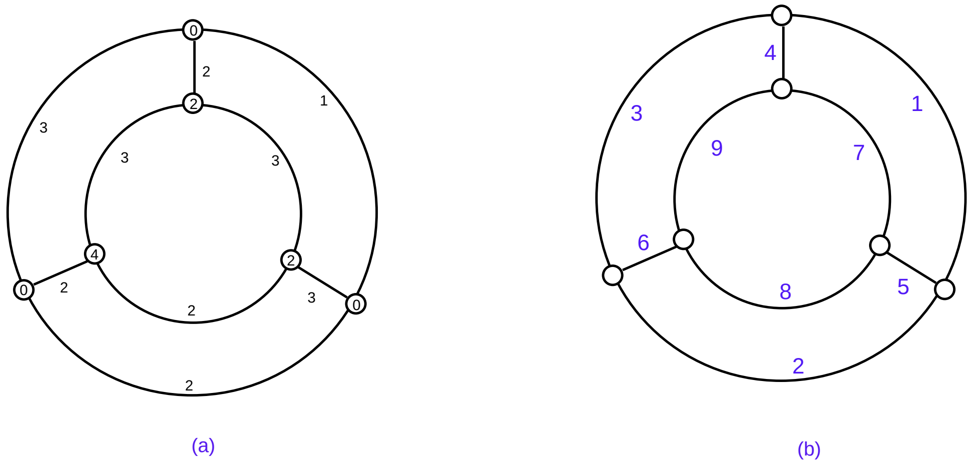

Figure 2. (**a**) A reflexive irregular 4-labeling for D_3^1 and (**b**) The edge weights for D_3^1.

For $j \geq 2$, we have the following labeling and their weights as follows:

$$f(x_i^j) = \begin{cases} 1 + \sum_{j=1}^{m-1} n_j, & \text{if } 1 \leq i < n_j, \quad j \equiv 2,3 (mod \ 4) \\ \sum_{j=1}^{m-1} n_j, & \text{if } 1 \leq i < n_j, \quad j \equiv 0,1 (mod \ 4) \end{cases}$$

$$f(y_i^j) = \begin{cases} 1 + \sum_{j=1}^{m} n_j, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 1,2 (mod \ 4) \\ \sum_{j=1}^{m} n_j, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 0,3 (mod \ 4) \end{cases}$$

$$f(x_i^j x_{i+1}^j) = \begin{cases} \sum_{j=1}^{m-1} n_j - 3 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 2 (mod \ 4) \\ \sum_{j=1}^{m-1} n_j - 1 + i, & \text{if } 2 \leq i \leq n_j, \quad j \equiv 1 (mod \ 4) \\ \sum_{j=1}^{m-1} n_j - 2 + i, & \text{if } 2 \leq i \leq n_j, \quad j \equiv 3 (mod \ 4) \\ \sum_{j=1}^{m-1} n_j + i, & \text{if } 2 \leq i \leq n_j, \quad j \equiv 0 (mod \ 4) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \sum_{j=1}^{m-1} n_j - 3 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 2 (mod \ 4) \\ \sum_{j=1}^{m} n_j - 2 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 3 (mod \ 4) \\ \sum_{j=1}^{m} n_j - 1 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 1 (mod \ 4) \\ \sum_{j=1}^{m-1} n_j + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 0 (mod \ 4) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \sum_{j=1}^{m-1} n_j - 3 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 2(mod\ 4) \\ \sum_{j=1}^{m-1} n_j - 1 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 3(mod\ 4) \\ \sum_{j=1}^{m-1} n_j + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 0(mod\ 4) \\ \sum_{j=1}^{m-1} n_j - 2 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 1(mod\ 4) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} 3\sum_{j=1}^{m-1} n_j - 3 + 2\lceil \frac{i}{2} \rceil + 2\lceil \frac{i+1}{2} \rceil, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 1,2,3(mod\ 4) \\ 3\sum_{k=1}^{j-1} n_k + 1, & \text{if } i = 1, \quad j \equiv 0(mod\ 4) \\ 3\sum_{j=1}^{m-1} n_j - 3 + 2\lceil \frac{i}{2} \rceil + 2\lceil \frac{i+}{2} \rceil, & \text{if } 2 \leq i \leq n_j, \quad j \equiv 0(mod\ 4) \end{cases}$$

$$wt(x_i^j x_{i+1}^j) = \begin{cases} 3\sum_{j=1}^{m-1} n_j - 1 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 2,3(mod\ 4) \\ 3\sum_{j=1}^{m-1} n_j + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 0,1(mod\ 4) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} 3\sum_{j=1}^{m-1} n_j + n_j - 1 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 2(mod\ 4) \\ 3\sum_{j=1}^{m-1} n_j + n_j - 2 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 3(mod\ 4) \\ 3\sum_{j=1}^{m-1} n_j + n_j + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 0(mod\ 4) \\ 3\sum_{j=1}^{m-1} n_j + n_j + 1 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 1(mod\ 4) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} 3\sum_{j=1}^{m-1} n_j + 2n_j - 1 + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 2,3(mod\ 4) \\ 3\sum_{j=1}^{m-1} n_j + 2n_j + i, & \text{if } 1 \leq i \leq n_j, \quad j \equiv 0,1(mod\ 4) \end{cases}$$

It is easy to check that no two edges have the same weight. Therefore, f is an edge irregular reflexive labeling of $\bigcup_{j=1}^{m} D_{n_j}^j$ for $n_j \geq 3$, which completes the proof. □

5. Conclusions

In this paper we discuss the union of m consecutive copies of gear graphs and prism graphs. We also determined the exact value of the reflexive edge strength of $\left(\bigcup_{j=1}^{m} J_{n_j,2}^j\right)$ and $\left(\bigcup_{j=1}^{m} D_{n_j}^j\right)$. We conclude this paper with two open problems for future work.

6. Open Problems

1. Determine the exact value of the reflexive edge strength for arbitrary union of gear graphs.
2. Determine the exact value of the reflexive edge strength for arbitrary union of prism graphs.

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