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On Magnifying Elements in E-Preserving Partial Transformation Semigroups

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Abstract: Let S be a semigroup. An element a of S is called a right [left] magnifying element if there exists a proper subset M of S satisfying $S = Ma$ [$S = aM$]. Let E be an equivalence relation on a nonempty set X. In this paper, we consider the semigroup $P(X, E)$ consisting of all E-preserving partial transformations, which is a subsemigroup of the partial transformation semigroup $P(X)$. The main propose of this paper is to show the necessary and sufficient conditions for elements in $P(X, E)$ to be right or left magnifying.

Keywords: magnifying elements; transformation semigroups; equivalence relations

1. Introduction

An element a of a semigroup S is a right [left] magnifying element in S if there exists a proper subset M of S satisfying $S = Ma$ [$S = aM$]. The concepts of right and left magnifying elements of a semigroup were first introduced in 1963 by Ljapin [1]. Many initially significant results were later published by Migliorini in [2,3], where he also introduced the notion of minimal subset related to a magnifying element of S. In [4], Catino and Migliorini determined the existence of strong magnifying elements in a semigroup and the existence of magnifying elements in simple and bisimple semigroups as well as regular semigroups. Semigroups with strong and nonstrong magnifying elements were investigated by Gutan [5]. A year later, he showed in [6] that every semigroup which contains magnifying elements is factorizable; this solved a problem raised by Catino and Migliorini. Gutan also established in [7] the method for obtaining semigroups having good left magnifying elements such that none of those is very good.

Let X be a nonempty set. The full transformation semigroup on X is the set

$$T(X) = \{f : X \to X \mid f \text{ is a function}\}$$

of all transformations from X into itself, which is a semigroup under the composition of functions. In [8], Magill, Jr. characterized transformation semigroups with identities containing magnifying elements. Gutan and Kisielewicz solved in [9] a long-standing open problem by showing the existence of semigroups containing both good and bad magnifying elements.

Interesting properties, especially regularity and Green's relations, on semigroups of transformations preserving relations have been widely conducted; see, e.g., [10–14]. In 2013, Huisheng and Weina [15] studied naturally orderd semigroups of partial transformations preserving an equivalence relation. Chinram and Baupradist have lately investigated right and left magnifying elements in some generalized transformation semigroups in [16,17].

Let E be an equivalence relation of a nonempty set X. We conventionally set $P(X) = \{\alpha : A \rightarrow X \mid A \subseteq X\}$. All functions will be written from the right, $(x)\alpha$ rather than $\alpha(x)$, and composed as $(x)(\alpha\beta)$ rather than $(\beta \circ \alpha)(x)$, for $\alpha, \beta \in P(X)$. The semigroup of partial transformations preserving the equivalence relation E

$$P(X, E) = \{\alpha \in P(X) \mid (x, y) \in E \text{ implies } ((x)\alpha, (y)\alpha) \in E\}$$

is exactly a subsemigroup of $P(X)$. Furthermore, if $E = X \times X$, then $P(X, E) = P(X)$. In this paper, we study right and left magnifying elements in $P(X, E)$ and conclude necessary and sufficient conditions for elements of $P(X, E)$ to be left or right magnifying.

2. Right Magnifying Elements

Lemma 1. *If α is a right magnifying element in $P(X, E)$, then α is onto.*

Proof. Suppose α is a right magnifying element in $P(X, E)$. According to the definition of right magnifying element, there exists a proper subset M of $P(X, E)$ with $M\alpha = P(X, E)$. Clearly, the identity map id_X on X belongs to $P(X, E)$. Thus, there exists $\beta \in M$ such that $\beta\alpha = id_X$. This shows that α is onto. □

Lemma 2. *Let α be a right magnifying element in $P(X, E)$. For any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$.*

Proof. Suppose α is a right magnifying element in $P(X, E)$. Again, by definition, we obtain a proper subset M of $P(X, E)$ satisfying $M\alpha = P(X, E)$. Since $id_X \in P(X, E)$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. Let $x, y \in X$ be such that $(x, y) \in E$. It follows that $(x)\beta\alpha = x$ and $(y)\beta\alpha = y$. Since $\beta \in P(X, E)$, we have $((x)\beta, (y)\beta) \in E$. We then choose $a = (x)\beta$ and $b = (y)\beta$. Therefore, the proof is complete. □

Lemma 3. *If $dom\ \alpha = X$ and α is bijective in $P(X, E)$, then α is not right magnifying.*

Proof. Assume that $dom\ \alpha = X$ and α is bijective in $P(X, E)$. By Lemma 2, $\alpha^{-1} \in P(X, E)$ such that $dom\ \alpha^{-1} = X$. Suppose that α is right magnifying. By definition, there is a proper subset M of $P(X, E)$ with $M\alpha = P(X, E)$. Consequently, $M\alpha = P(X, E)\alpha$. Then $M = M\alpha\alpha^{-1} = P(X, E)\alpha\alpha^{-1} = P(X, E)$, which is a contradiction since M is a proper subset of $P(X, E)$. Hence α is not right magnifying. □

Lemma 4. *If $\alpha \in P(X, E)$ is onto but not one-to-one, $dom\ \alpha = X$ and, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$, then α is right magnifying.*

Proof. Let $\alpha \in P(X, E)$ be onto but not one-to-one and $dom\ \alpha = X$. For any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$. Let $M = \{\beta \in P(X, E) \mid \beta \text{ is not onto}\}$. Then, $M \neq P(X, E)$.

Let γ be any function in $P(X, E)$. Since α is onto, we have for each $x \in dom\ \gamma$, there exists $a_x \in dom\ \alpha$ such that $(a_x)\alpha = (x)\gamma$ (if $(a_1)\gamma = (a_2)\gamma$, we must choose $a_{x_1} = a_{x_2}$ and if $((x)\gamma, (y)\gamma) \in E$, we must choose $(a_x, a_y) \in E$). Define $\beta \in P(X)$ by $(x)\beta = a_x$ for all $x \in dom\ \gamma$. To show that $\beta \in P(X, E)$, let $x, y \in X$ be such that $(x, y) \in E$. Since $\gamma \in P(X, E)$, $((x)\gamma, (y)\gamma) \in E$. By assumption, we obtain $(a_x, a_y) \in E$ such that $(x)\gamma = (a_x)\alpha$ and $(y)\gamma = (a_y)\alpha$. Hence, $((x)\beta, (y)\beta) \in E$. Since α is not one-to-one, β is not onto either. Thus, $\beta \in M$ and we obtain, for all $x \in X$, that

$$(x)\beta\alpha = ((x)\beta)\alpha = (a_x)\alpha = (x)\gamma.$$

Then, $\beta\alpha = \gamma$, hence $M\alpha = P(X, E)$. Therefore, α is right magnifying. □

Example 1. *Let $X = \mathbb{N}$. Define a relation E on X by*

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Consider $X/E = \{\{1,2\}\} \cup \{\{x, x+1, x+2\} \mid x \in 3X\} = \{\{1,2\}, \{3,4,5\}, \{6,7,8\}, \ldots\}$. It is clear that E is an equivalence relation on X. Let $\alpha \in P(X, E)$ be defined by $(x)\alpha = x$ for all positive integers $x \leq 5$ and $(x)\alpha = x - 3$ for all positive integers $x > 5$, that is,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

Then, α is onto but not one-to-one and, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$. Let $M = \{\beta \in P(X, E) \mid \beta \text{ is not onto}\}$. For any function $\gamma \in P(X, E)$, Lemma 4 ensures that there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate these ideas by considering the element γ of $P(X, E)$, which is defined by $(1)\gamma = 1$, $(2)\gamma = 2$ and $(x)\gamma = x - 3$ for positive integers $x > 5$, that is,

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

Define a function $\beta : dom\ \gamma \to X$ by $(x)\beta = x$ for all $x \in dom\ \gamma$, that is,

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 6 & 7 & 8 & \cdots \end{pmatrix}.$$

Thus $\beta \in M$ and we have

$$\begin{aligned} \beta\alpha &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 6 & 7 & 8 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 3 & 4 & 5 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

Lemma 5. *If $\alpha \in P(X, E)$ is onto, $dom\ \alpha \neq X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$, then α is right magnifying.*

Proof. Let $\alpha \in P(X, E)$ be onto and $dom\ \alpha \neq X$. For any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$. We follow the method of proof used in Lemma 4 and define $\beta : dom\ \gamma \longrightarrow X$ by $(x)\beta = a_x$ for all $x \in dom\ \gamma$. To show that $\beta \in P(X, E)$, let $x, y \in X$ be such that $(x, y) \in E$. Since $\gamma \in P(X, E)$, we have $((x)\gamma, (y)\gamma) \in E$. By assumption, there exists $(a_x, a_y) \in E$ such that $(x)\gamma = (a_x)\alpha$ and $(y)\gamma = (a_y)\alpha$. Hence, $((x)\beta, (y)\beta) \in E$. Since $dom\ \alpha \neq X$, β is not onto either.

Thus, $\beta \in M$ and we obtain, for all $x \in X$, that

$$(x)\beta\alpha = ((x)\beta)\alpha = (a_x)\alpha = (x)\gamma.$$

Then, $\beta\alpha = \gamma$, hence $M\alpha = P(X, E)$. Therefore, α is right magnifying. □

Example 2. *Let $X = \mathbb{N}$. Define an equivalence relation E on X by*

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

We obtain $X/E = \{\{1,2\}\} \cup \{\{x, x+1, x+2\} \mid x \in 3X\} = \{\{1,2\}, \{3,4,5\}, \{6,7,8\}, \ldots\}$. *Let* $\alpha \in P(X,E)$ *be defined by* $(3)\alpha = 1, (4)\alpha = 2$ *and* $(x)\alpha = x - 3$ *for all positive integers* $x > 5$, *that is,*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 1 & 2 & - & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

Then, α *is onto, dom* $\alpha \neq X$ *and for any* $(x,y) \in E$, *there exists* $(a,b) \in E$ *such that* $x = (a)\alpha, y = (b)\alpha$. *Let* $M = \{\beta \in P(X,E) \mid \beta \text{ is not onto}\}$. *For any function* $\gamma \in P(X,E)$, *Lemma 5 ensures that there exists* $\beta \in M$ *such that* $\beta\alpha = \gamma$.

We will illustrate these ideas by considering the element γ *of* $P(X,E)$, *which is defined by* $(x)\gamma = \lfloor \frac{x+3}{3} \rfloor$ *for all* $x > 2$, *that is,*

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 2 & 2 & 2 & 3 & 3 & 3 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta : \text{dom } \gamma \to X$ *by* $(x)\beta = \lfloor \frac{x+9}{3} \rfloor$ *if* $x = 3,4,5$ *and* $(x)\beta = \lfloor \frac{x+12}{3} \rfloor$ *for all positive integers* $x \geq 6$, *that is,*

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 4 & 4 & 4 & 6 & 6 & 6 & \cdots \end{pmatrix}.$$

Thus, $\beta \in M$ *and we have*

$$\begin{aligned} \beta\alpha &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 4 & 4 & 4 & 6 & 6 & 6 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 1 & 2 & - & 3 & 4 & 5 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 2 & 2 & 2 & 3 & 3 & 3 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

We summarize those lemmas in a theorem as follows.

Theorem 1. α *is right magnifying in* $P(X,E)$ *if and only if* α *is onto, for any* $(x,y) \in E$, *there exists* $(a,b) \in E$ *such that* $x = (a)\alpha, y = (b)\alpha$ *and either*

1. *dom* $\alpha \neq X$ *or*
2. *dom* $\alpha = X$ *and* α *is not one-to-one.*

Proof. It follows by Lemmas 1–5. □

As a consequence, the following result holds for $E = X \times X$.

Corollary 1. *Let* $\alpha \in P(X)$. *Then,* α *is right magnifying in a semigroup* $P(X)$ *if and only if* α *is onto and either*

1. *dom* $\alpha \neq X$ *or*
2. *dom* $\alpha = X$ *and* α *is not one-to-one.*

Proof. It follows immediately from Theorem 1. □

3. Left Magnifying Elements

Lemma 6. *If* α *is a left magnifying element in* $P(X,E)$, *then* α *is one-to-one.*

Proof. Using the same argument as in Lemma 1, we obtain a proper subset M of $P(X,E)$ with $\alpha M = P(X,E)$ and there exists $\beta \in M$ such that $\alpha\beta = id_X$. This shows that α is one-to-one. □

Lemma 7. *If α is a left magnifying element in $P(X,E)$, then dom $\alpha = X$.*

Proof. Suppose α is a left magnifying in $P(X,E)$. Again, by definition, there is a proper subset M of $P(X,E)$ with $\alpha M = P(X,E)$. Let $\gamma \in P(X,E)$ be such that *dom* $\gamma = X$. Then, $\gamma = \alpha\beta$ for some $\beta \in M$. Since *dom* $\gamma = X$, we obtain *dom* $\alpha = X$. $\square$

Lemma 8. *Let α be a left magnifying element in $P(X,E)$. For any $x,y \in X$, if $((x)\alpha,(y)\alpha) \in E$, then $(x,y) \in E$.*

Proof. Suppose α is a left magnifying element in $P(X,E)$. By Lemma 7, *dom* $\alpha = X$. There exists a proper subset M of $P(X,E)$ satisfying $\alpha M = P(X,E)$. Since $id_X \in P(X,E)$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Let $x,y \in X$ be such that $((x)\alpha,(y)\alpha) \in E$. It follows that $(x)\alpha\beta = x$ and $(y)\alpha\beta = y$. Then, we obtain $(x,y) = ((x)\alpha\beta,(y)\alpha\beta) \in E$ because $\beta \in P(X,E)$. $\square$

Lemma 9. *If $\alpha \in P(X,E)$ is bijective and dom $\alpha = X$, then α is not left magnifying.*

Proof. As in Lemma 3, the result holds by applying Lemma 8. $\square$

Lemma 10. *If $\alpha \in P(X,E)$ is one-to-one but not onto, dom $\alpha = X$ and for any $x,y \in X$, $((x)\alpha,(y)\alpha) \in E$ implies $(x,y) \in E$, then α is a left magnifying element in $P(X,E)$.*

Proof. Assume that $\alpha \in P(X,E)$ is one-to-one but not onto, *dom* $\alpha = X$ and for any $x,y \in X$, $((x)\alpha,(y)\alpha) \in E$ implies $(x,y) \in E$. Let $M = \{\beta \in P(X,E) \mid dom\ \beta \subseteq ran\ \alpha\}$. Claim that $\alpha M = P(X,E)$. Let $\gamma \in P(X,E)$. For $x \in ran\ \alpha$, then there exists $a_x \in dom\ \gamma$ such that $(a_x)\alpha = x$. Define $\beta \in P(X)$ by $(x)\beta = (a_x)\gamma$ if $x \in ran\ \alpha$ and $a_x \in dom\ \gamma$. We see that $dom\ \beta = \{x \in ran\ \alpha \mid a_x \in dom\ \gamma\} \subseteq ran\ \alpha$. To show that $\beta \in P(X,E)$, assume $x,y \in ran\ \alpha$, $(a_x)\alpha = x, (a_y)\alpha = y$, $a_x, a_y \in dom\ \gamma$ and $(x,y) \in E$. Then, $((a_x)\alpha,(a_y)\alpha) \in E$ and hence, by assumption, we obtain $(a_x,a_y) \in E$. Thus, $((x)\beta,(y)\beta) = ((a_x)\gamma,(a_y)\gamma) \in E$ because $\gamma \in P(X,E)$. Then, $\beta \in M$ and for $x \in X$, we have

$$(x)\alpha\beta = ((x)\alpha)\beta = (x)\gamma.$$

This gives us that $\alpha\beta = \gamma$ and $\alpha M = P(X,E)$. Hence, α is a left magnifying element in $P(X,E)$. $\square$

Example 3. *Let $X = \mathbb{N}$. Define a relation E on X by*

$$(x,y) \in E \text{ if and only if } \lfloor \frac{x}{2} \rfloor \equiv \lfloor \frac{y}{2} \rfloor \pmod 2.$$

It is obvious that E is an equivalence relation on X and, in addition, $X/E = \{\{1,4,5,8,9,\ldots\},\{2,3,6,7,\ldots\}\}$. Let $\alpha \in E$ be defined by $(x)\alpha = x+2$ for all positive integers $x \in X$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \\ 3 & 6 & 7 & 10 & 11 & 14 & 15 & \cdots & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \end{pmatrix}.$$

We now obtain that α is one-to-one but not onto and for any $x,y \in X$, $((x)\alpha,(y)\alpha) \in E$ implies $(x,y) \in E$. Let $M = \{\beta \in P(X,E) \mid dom\ \beta \subseteq ran\ \alpha\}$ and $\gamma \in P(X,E)$ be any function. By Lemma 10, there exists $\beta \in M$ such that $\alpha\beta = \gamma$.

We will illustrate these ideas by considering the element γ of $P(X,E)$, which is defined by $(x)\gamma = x-2$ for all positive integers $x > 3$, that is,

$$\gamma = \begin{pmatrix} 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \\ - & 2 & 3 & 6 & 7 & 10 & 11 & \cdots & - & - & 4 & 5 & 8 & 9 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in P(X, E)$ by $(x)\beta = x - 4$ for all $x > 5$, that is,

$$\beta = \begin{pmatrix} 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \\ - & - & - & 4 & 5 & 8 & 9 & \cdots & - & - & 2 & 3 & 6 & 7 & \cdots \end{pmatrix}.$$

Thus, $\beta \in M$ and we have

$$\alpha\beta = \begin{pmatrix} 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \\ 3 & 6 & 7 & 10 & 11 & 14 & 15 & \cdots & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \\ - & - & - & 4 & 5 & 8 & 9 & \cdots & - & - & 2 & 3 & 6 & 7 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \\ - & 2 & 3 & 6 & 7 & 10 & 11 & \cdots & - & - & 4 & 5 & 8 & 9 & \cdots \end{pmatrix} = \gamma.$$

Example 4. *Let $X = \mathbb{Z} \times \mathbb{Z}$. Define a relation E on X by*

$$((a, b), (c, d)) \in E \text{ if and only if } a = c.$$

Let $\alpha \in P(X, E)$ be defined by $(a, b)\alpha = (2a, 2b)$ for all $a, b \in \mathbb{Z}$. Then, α is one-to-one but not onto and for any $(a, b), (c, d) \in X$, $((a, b)\alpha, (c, d)\alpha) \in E$ implies $((a, b), (c, d)) \in E$. Let $M = \{\beta \in P(X, E) \mid dom\ \beta \subseteq ran\ \alpha\}$ and $\gamma \in P(X, E)$ be any function. By Lemma 10, there exists $\beta \in M$ such that $\alpha\beta = \gamma$. We will illustrate these ideas by considering the element γ of $P(X, E)$, which is defined by $(a, b)\gamma = (a + 1, b + 2)$ for all $a, b \in \mathbb{Z}$.

To get the required result, define a function $\beta \in P(X, E)$ by $(a, b)\beta = (k + 1, l + 2)$ if $a = 2k$ and $b = 2l$ for some $k, l \in \mathbb{Z}$. Thus, $\beta \in M$ and we have $(a, b)\alpha\beta = ((a, b)\alpha)\beta = (2a, 2b)\beta = (a + 1, b + 2) = (a, b)\gamma$ for all $a, b \in \mathbb{Z}$, which shows that $\alpha\beta = \gamma$.

We summarize those lemmas in a theorem as follows.

Theorem 2. *α is left magnifying in $P(X, E)$ if and only if α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $((x)\alpha, (y)\alpha) \in E$ implies $(x, y) \in E$.*

Proof. It follows from Lemmas 6–10. □

As a consequence, the following result holds for $E = X \times X$.

Corollary 2. *α is a left magnifying element in $P(X)$ if and only if α is one-to-one but not onto and dom $\alpha = X$.*

Proof. It follows from Theorem 2. □

4. Conclusions

In this paper, necessary and sufficient conditions for elements in $P(X, E)$ to be right or left magnifying are established as follows:

1. α is right magnifying in $P(X, E)$ if and only if α is onto, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$ and either
   1. *dom* $\alpha \neq X$ or
   2. *dom* $\alpha = X$ and α is not one-to-one.
2. Let $\alpha \in P(X)$. Then, α is right magnifying in a semigroup $P(X)$ if and only if α is onto and either

1. $dom\ \alpha \neq X$ or
2. $dom\ \alpha = X$ and α is not one-to-one.

3. α is left magnifying in $P(X, E)$ if and only if α is one-to-one but not onto, $dom\ \alpha = X$ and for any $x, y \in X$, $((x)\alpha, (y)\alpha) \in E$ implies $(x, y) \in E$.
4. α is a left magnifying element in $P(X)$ if and only if α is one-to-one but not onto and $dom\ \alpha = X$.

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