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Convergence Ball and Complex Geometry of an Iteration Function of Higher Order

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Abstract: Higher-order derivatives are used to determine the convergence order of iterative methods. However, such derivatives are not present in the formulas. Therefore, the assumptions on the higher-order derivatives of the function restrict the applicability of methods. Our convergence analysis of an eighth-order method uses only the derivative of order one. The convergence results so obtained are applied to some real problems, which arise in science and engineering. Finally, stability of the method is checked through complex geometry shown by drawing basins of attraction of the solutions.

Keywords: iterative methods; fast methods; local convergence; Banach space; complex dynamics

MSC: 65H10; 65J10; 41A25; 49M15

1. Introduction

Let $F : \Omega \subseteq B_1 \to B_2$ be differentiable continuously according to Fréchet between the Banach spaces B_1 and B_2 and Ω be a convex set. Let $B(\mu, h) = \{\nu \in B_1 : \|\mu - \nu\| < h\}$ for $h > 0$. Denote by $\bar{B}(\mu, h)$ the closure of $B(\mu, h)$. Let also $\mathcal{L}(B_1, B_2)$ stand for the set of bounded linear operators from B_1 to B_2.

In this study, we locate p by solving equation

$$F(x) = 0. \tag{1}$$

Many problems look like (1) [1–3]. The solutions of such equations are rarely attainable in closed form. That is why most methods for solving such equations are usually iterative. Convergence analysis is an important part in the development of an iterative method. In general, the convergence domain is narrow. Without additional hypotheses, it is important to enlarge the convergence domain. Knowledge of initial guesses requires the convergence radius. Other studies are found in [1,2,4–9].

The most well-known method is Newton's method, which is written as

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2, \ldots, \tag{2}$$

Many higher orders of convergence, modified Newton's, or Newton-like methods have been appeared in the literature, e.g., [1–3,5–7,9–21] and references therein. In particular, Cordero et al. [11] studied eighth-order method for finding approximate solution of $F(x) = 0$ defined for each $n = 0, 1, 2, \ldots$ by

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
z_n &= y_n - \Big(\frac{1}{4}I + \frac{1}{2}F'(y_n)^{-1}F'(x_n) + \frac{1}{4}\big(F'(y_n)^{-1}F'(x_n)\big)^2\Big)F'(x_n)^{-1}F(y_n), \\
x_{n+1} &= z_n - \Big(\frac{1}{2}I + \frac{1}{2}(F'(y_n)^{-1}F'(x_n))^2\Big)F'(x_n)^{-1}F(z_n).
\end{aligned} \tag{3}
$$

They considered the method (3) for solving system of equations, when $B_1 = B_2 = \mathbb{R}^i$ $(i \in \mathbb{N})$. The method was compared favorably to existing methods. They proved the eighth order of convergence of the method but using Taylor series as well as eighth-order derivatives. The convergence order of the other methods mentioned in [11] also use higher-order derivatives. Therefore, they can be handled with the same technique. We simply picked (3) to work with which seems to be the best to study among the rest.

It can be clearly seen that the assumptions on the higher-order Fréchet derivatives of the operator F limit the applicability of method (3). As a motivational example, we consider the following:

Let $B_1 = B_2 = \mathcal{C}[0,1]$ and $\Omega = \bar{B}(p,1)$. Consider the Hammerstein-type equation [1,4] defined by

$$x(s) = \int_0^1 D(s,t)\Big(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\Big)dt, \tag{4}$$

where D is defined on $[0,1]\times[0,1]$ as

$$D(s,t) = \begin{cases} (1-s)t, \ t \le s, \\ s(1-t), \ s \le t. \end{cases}$$

Clearly, $p(x) = 0$ solves

$$[H(x)](s) = x(s) - \int_0^1 D(s,t)\Big(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\Big)dt = 0 \tag{5}$$

Then, we have that the Fréchet derivative is given by

$$[H'(x)y](s) = y(s) - \int_0^1 D(s,t)\Big(\frac{3}{2}x(t)^{\frac{1}{2}} + x(t)\Big)dt, \tag{6}$$

where the prime denotes derivative with respect to x. We have

$$H'(p(s)) = I, \quad \Big\| \int_0^1 D(s,t)dt \Big\| \le \frac{1}{8}, \quad \text{and}$$

$$\|H'(p)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}\Big(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\Big).$$

Boundary value problems of order two can be found in many disciplines: In Physics, many problems can be expressed in this way, e.g., Newton's laws; calculating concentrations of various chemicals in a reaction; computing modes in biology etc. If we assume kinetic plus potential energy is constant. Then, the mechanical system is called conservative. Consider the conservative system defined by the Boundary Value Problem (BVP)

$$\frac{d^2x(s)}{ds^2} + \eta(x(s)) = 0, \ x(0) = x(1) = 0,$$

where $\eta(x)$ is differentiable at least one time. Then, solving BVP reduced to finding a solution of an integral equation like (4) [15].

In this work, our approach is to weaken the assumptions in [11]. We work with Banach space valued operators which constitute a more general setting and use only first-order derivatives.

We summarize the contents of the paper. The local convergence of (3) is given in Section 2. Experiments on some problems of the applied sciences are performed to verify the theoretical results in Section 3. Then, in Section 4 we check the convergence domain of the iterative technique geometrically by means of drawing basin of attractors. Concluding remarks are given in Section 5.

2. Local Convergence

Let $w_0 : [0,\infty) \to [0,\infty)$ be a continuous and increasing function with $w_0(0) = 0$. Assume a minimal positive solution

$$w_0(t) = 1, \tag{7}$$

has ρ_1. Let also $w : [0,\rho_1) \to [0,\infty)$, $w_1 : [0,\rho_1) \to [0,\infty)$ be continuous and increasing functions with $w(0) = 0$, and functions $q_1(t)$, $\bar{q}_1(t)$ on the interval $[0,\rho_1)$ are given as

$$q_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)}$$

and

$$\bar{q}_1(t) = q_1(t) - 1.$$

Clearly, $\bar{q}_1(0) = -1$ and $\bar{q}_1(t) \to \infty$ as $t \to \rho_1^-$. By the intermediate value theorem, equation $\bar{q}_1(t) = 0$ has at least one solution in the interval $(0,\rho_1)$, denote by R_1. Assume

$$w_0(\phi_1(t)t) = 1, \tag{8}$$

has a minimal positive solution ρ_2 and let $\rho_0 = \min\{\rho_1,\rho_2\}$. Define the functions $q_2(t)$ and $\bar{q}_2(t)$ on interval $[0,\rho_0)$ by

$$\begin{aligned} q_2(t) = \Bigg(&\frac{\int_0^1 w((1-\theta)q_1(t)t)d\theta}{1-w_0(q_1(t)t)} + \frac{(w_0(q_1(t)t)+w_0(t))\int_0^1 w_1(\theta q_1(t)t)d\theta}{(1-w_0(t))(1-w_0(q_1(t)t))} \\ &+ \frac{1}{4}\Big(\frac{(w_0(t)+w_0(q_1(t)t))^2}{(1-w_0(q_1(t)t))^2} + \frac{2(w_0(t)+w_0(q_1(t)t))}{1-w_0(q_1(t)t)}\Big)\frac{\int_0^1 w_1(\theta q_1(t)t)d\theta}{1-w_0(t)}\Bigg) q_1(t) \end{aligned}$$

and

$$\bar{q}_2(t) = q_2(t) - 1.$$

Since, $\bar{q}_2(0) = -1$ and $\bar{q}_2(t) \to \infty$ as $t \to \rho_0^-$, let R_2 be the minimal solution of equation $\bar{q}_2(t) = 0$ in $(0,\rho_0)$. Assume

$$w_0(q_3(t)t) = 1, \tag{9}$$

has a minimal solution ρ_3 and let $\rho = \min\{\rho_2,\rho_3\}$. Define the functions $q_3(t)$ and $\bar{q}_3(t)$ on interval $[0,\rho)$ by

$$\begin{aligned} q_3(t) = \Bigg(&\frac{\int_0^1 w((1-\theta)q_2(t)t)d\theta}{1-w_0(q_2(t)t)} + \frac{(w_0(q_2(t)t)+w_0(q_1(t)t))\int_0^1 w_1(\theta q_2(t)t)d\theta}{(1-w_0(q_2(t)t))(1-w_0(t))} \\ &+ \frac{1}{2}\Big(\frac{(w_0(q_1(t)t)+w_0(t))^2}{(1-w_0(q_1(t)t))^2} + \frac{2(w_0(q_1(t)t)+w_0(t))^2}{(1-w_0(t))(1-w_0(q_1(t)t))}\Big)\frac{\int_0^1 w_1(\theta q_2(t)t)d\theta}{1-w_0(t)}\Bigg) q_2(t) \end{aligned}$$

and

$$\bar{q}_3(t) = q_3(t) - 1.$$

However, $\bar{q}_3(0) = -1 < 0$ and $\bar{q}_3(t) \to \infty$ as $t \to \rho^-$, let R_3 be the minimal solution of equation $\bar{q}_3 = 0$.

Define radius of convergence

$$R = \min\{R_m\}, \quad m = 1, 2, 3. \tag{10}$$

Then, for all $t \in [0, R)$

$$0 \le w_0(t) \le 1, \tag{11}$$

$$0 \le w_0(q_1(t)t) \le 1, \tag{12}$$

$$0 \le w_0(q_2(t)t) \le 1 \tag{13}$$

and

$$0 \le q_m(t) \le 1. \tag{14}$$

We shall use the conditions (C) in the local convergence analysis of method (3) given below:

(C_1) $F : \Omega \to B_2$ is differentiable in the sense of Fréchet, $p \in \Omega$ with $F(p) = 0$ and $F'(p)^{-1} \in \mathcal{L}B(B_2, B_1)$.

(C_2) $w_0 : [0, \infty) \to [0, \infty)$ continuous and increasing, $w_0(0) = 0$ and for each $x \in \Omega$

$$\|F'(p)^{-1}(F'(p) - F'(x))\| \le w_0(\|p - x\|).$$

Set: $\Omega_0 = \Omega \cap B(p, \rho_1)$, where ρ_1 is given in (7).

(C_3) $w : [0, \rho_1) \to [0, +\infty)$, $w_1 : [0, \rho_1) \to [0, +\infty)$ are continuous, increasing, $w(0) = 0$ and for each $x, y \in \Omega_0$

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \le w(\|x - y\|)$$

and

$$\|F'(p)^{-1}F'(x)\| \le w_1(\|x - p\|).$$

(C_4) $\bar{B}(p, R) \subset \Omega$, where R is defined in (10).

(C_5) There exists $R_1 \ge R$ such that $\int_0^1 w_0(\theta R_1)d\theta < 1$.

Set: $\Omega_1 = \Omega \cap \bar{B}(p, R_1)$.

Next, the convergence analysis of method (3) follows using the preceding notations and the conditions (C).

Theorem 1. *Assume that the conditions (C) hold. Then, the sequence $\{x_n\}$ starting at $x_0 \in B(p, R) - \{p\}$ converges to p, and the following inequalities hold*

$$\|y_n - p\| \le q_1(\|x_n - p\|)\|x_n - p\| \le \|x_n - p\| < R, \tag{15}$$

$$\|z_n - p\| \le q_2(\|x_n - p\|)\|x_n - p\| \le \|x_n - p\| \tag{16}$$

and

$$\|x_{n+1} - p\| \le q_3(\|x_n - p\|)\|x_n - p\| \le \|x_n - p\|, \tag{17}$$

where the "q" functions are given previously and R is defined in (10). Furthermore, p is the only solution of equation $F(x) = 0$ in Ω_1.

Proof. A mathematical induction-based proof is used. If $x_0 \in B(p, R) - \{p\}$, using (10) and (C_2), we get that

$$\|F'(p)^{-1}(F'(p) - F'(x_0))\| \le w_0(\|p - x_0\|) < w_0(R) < 1, \tag{18}$$

so, by the Banach Lemma on invertible operators [2,22], we have that $F'(x_0)^{-1} \in \mathcal{L}B(B_2, B_1)$ and

$$\|F'(x_0)^{-1}F'(p)\| \leq \frac{1}{1 - w_0(\|x_0 - p\|)}. \tag{19}$$

This also shows that y_0 is well defined. Using (10), (14) (for $m = 1$), (C_3) and (19), we get in turn that

$$\begin{aligned}
\|y_0 - p\| &\leq \|x_0 - p - F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(p)\| \Big\| \int_0^1 F'(p)^{-1}\big(F'(p + \theta(x_0 - p)) - F'(x_0)\big)(x_0 - p)d\theta \Big\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|x_0 - p\|)d\theta}{1 - w_0(\|x - p\|)} \\
&= q_1(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\| < R,
\end{aligned} \tag{20}$$

so (15) holds for $n = 0$ and $y_0 \in B(p, R)$, so z_0 and x_1 are well defined.

We can have by (C_1) that

$$F(x_0) = F(x_0) - F(p) = \int_0^1 F'(p + \theta(x_0 - p))d\theta(x_0 - p). \tag{21}$$

Then, by the second condition in (C_3), we get that

$$\|F'(p)^{-1}F(x_0)\| \leq \int_0^1 w_1(\theta\|x_0 - p\|)d\theta\|x_0 - p\|. \tag{22}$$

By second substep of method (3) for $n = 0$, we have

$$\begin{aligned}
z_0 - p = \;& y_0 - p - F'(y_0)^{-1}F(y_0) + F'(y_0)^{-1}(F'(x_0) - F'(y_0))F'(x_0)^{-1}F(y_0) \\
& + \tfrac{1}{4}\Big(\big(F'(y_0)^{-1}(F'(x_0) - F'(y_0)\big)^2 - 2(F'(y_0)^{-1}(F'(x_0) - F'(y_0)))\Big)F'(x_0)^{-1}F(y_0).
\end{aligned} \tag{23}$$

By using (10), (14) (for $m = 2$), (19) (for $x_0 = y_0$), (20), (22) (for $x_0 = y_0$) and (23), we obtain in turn that

$$\begin{aligned}
\|z_0 - p\| \leq\;& \|y_0 - p - F'(y_0)^{-1}F(y_0)\| \\
& + \|F'(y_0)^{-1}F'(p)\|\big(\|F'(p)^{-1}(F'(y_0) - F'(p))\| + \|F'(p)^{-1}(F'(x_0) - F'(p))\|\big) \\
& \times \|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F(y_0)\| + \frac{1}{4}\Big(\|F'(y_0)^{-1}F'(p)\|^2\big(\|F'(p)^{-1}(F'(x_0) - F'(p))\| \\
& + \|F'(p)^{-1}(F'(y_0) - F'(p))\|\big)^2 + 2\|F'(y_0)^{-1}F'(p)\|\big(\|F'(p)^{-1}(F'(x_0) - F'(p))\| \\
& + \|F'(p)^{-1}(F'(y_0) - F'(p))\|\big)\Big)\|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F(y_0)\| \qquad (24) \\
\leq & \Bigg(\frac{\int_0^1 w((1-\theta)\|y_0 - p\|)d\theta}{1 - w_0(\|y_0 - p\|)} + \frac{(w_0(\|y_0 - p\|) + w_0(\|x_0 - p\|)\int_0^1 w_1(\theta\|y_0 - p\|)d\theta}{(1 - w_0(\|x_0 - p\|))(1 - w_0(\|y_0 - p\|)} \\
& + \frac{1}{4}\bigg(\frac{(w_0(\|x_0 - p\|) + w_0(\|y_0 - p\|))^2}{(1 - w_0(\|x - p\|))^2} + \frac{2(w_0(\|x_0 - p\|) + w_0(\|y_0 - p\|))}{1 - w_0(\|x - p\|)}\bigg) \\
& \times \frac{\int_0^1 w_1(\theta\|y_0 - p\|)d\theta}{1 - w_0(\|x - p\|)}\Bigg)\|y_0 - p\| \\
\leq & q_2(\|x_0 - p\|)\|x_0 - p\| \leq \|x_0 - p\| < R,
\end{aligned}$$

so (17) holds for $n = 0$, and $z_0 \in B(p, R)$. Moreover, by the third sub step of method (3), we have that

$$\begin{aligned} x_1 - p =& z_0 - p - F'(z_0)^{-1}F(z_0) + (F'(z_0)^{-1} - F'(x_0)^{-1})F(z_0) \\ &+ \frac{1}{2}\Big(I - (F'(y_0)^{-1}F'(x_0)^2\Big)F'(x_0)^{-1}F(z_0) \\ =& (z_0 - p - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1}F(z_0) \\ &+ \frac{1}{2}\Big((I - F'(y_0)^{-1}F'(x_0))^2 + 2(I - F'(y_0)^{-1}F'(x_0))F'(y_0)^{-1}F'(x_0)\Big)F'(x_0)^{-1}F(z_0). \end{aligned} \tag{25}$$

Using (10), (14) (for $m = 3$), (19) (for $x_0 = y_0, z_0$), (20), (22) (for $x_0 = z_0$), (24) and (25), we get in turn that

$$\begin{aligned} \|x_1 - p\| \le& \|z_0 - p - F'(z_0)^{-1}F(z_0)\| + \|F'(z_0)^{-1}F'(p)\|(\|F'(p)^{-1}(F'(z_0) - F'(p))\| \\ &+ \|F'(p)^{-1}(F'(x_0) - F'(p))\|)\|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F'(z_0)\| \\ &+ \frac{1}{2}\Big(\|F'(y_0)^{-1}(F'(y_0) - F'(x_0))\|^2 + 2\|F'(y_0)^{-1}(F'(y_0) - F'(x_0))\| \\ &\times \|F'(y_0)^{-1}F'(p)\|\|F'(p)^{-1}F'(x_0)\|\Big)\|F'(x_0)^{-1}F'(p)\|\|F'(p)^{-1}F(z_0)\| \\ \le& \Bigg(\frac{\int_0^1 w((1-\theta)\|z_0 - p\|)d\theta}{1 - w_0(\|z_0 - p\|)} + \frac{(w_0(\|z_0 - p\|) + w_0(\|x_0 - p\|))\int_0^1 w_1(\theta\|z_0 - p\|)d\theta}{(1 - w_0(\|z_0 - p\|))(1 - w_0(\|x_0 - p\|))} \\ &+ \frac{1}{2}\Bigg(\frac{(w_0(\|y_0 - p\|) + w_0(\|x_0 - p\|))^2}{(1 - w_0(\|y_0 - p\|))^2} + \frac{2(w_0(\|y_0 - p\|) + w_0(\|x_0 - p\|))}{1 - w_0(\|x_0 - p\|)} \\ &\times \frac{w_1(\|x_0 - p\|)}{1 - w_0(\|y_0 - p\|)}\Bigg)\frac{\int_0^1 w(\theta\|z_0 - p\|)d\theta}{1 - w_0(\|x_0 - p\|)}\Bigg)\|z_0 - p\| \\ \le& q_3(\|x_0 - p\|)\|x_0 - p\| \le \|x_0 - p\| < R, \end{aligned} \tag{26}$$

so (17) holds for $n = 0$ and $x_1 \in B(p, R)$. The induction for (17) is completed if x_0, y_0, z_0, x_1 are replaced by x_j, y_j, z_j, x_{j+1} in the preceding calculations. Then, in view of the inequality

$$\|x_{j+1} - p\| \le \lambda\|x_j - p\| < R, \tag{27}$$

where $\lambda = q_3(\|x_0 - p\|) \in [0, 1)$, we get that $\lim_{j\to\infty} x_j = p$ and $x_{j+1} \in B(p, R)$.

Furthermore, for the uniqueness part, let $Q = \int_0^1 F'(p + \theta(p_* - p))d\theta$ for some $p_* \in \Omega_1$ such that $F(p_*) = 0$. Using (C_5), we get that

$$\begin{aligned} \|F'(p)^{-1}(Q - F'(p))\| &\le \int_0^1 w_0(\theta\|p_* - p\|)d\theta \\ &\le \int_0^1 w_0(\theta R)d\theta < 1, \end{aligned}$$

so $Q^{-1} \in \mathcal{L}B(B_2, B_1)$. Finally, $0 = F(p_*) - F(p) = Q(p_* - p)$, we get $p_* = p$. □

Remark 1.

(i) *In view of* (C_2)

$$\begin{aligned} \|F'(p)^{-1}F'(x)\| &= \|F'(p)^{-1}(F'(x) - F'(p)) + I\| \\ &= 1 + \|F'(p)^{-1}(F'(p) - F'(x))\| \\ &\le 1 + w_0(\|p - x\|) \\ &\le 2 \end{aligned}$$

Then, we can set $w_1(t) = 1 + w_0(t)$, and condition (c_3) can be removed, condition (C_3) can be dropped and w_1 can be replaced by $w_1(t) = 2$.

(ii) *Let $\{w_n\}$ be any iterative method. Then, we define the computational order of convergence (COC) [21] by*

$$COC = log\left\|\frac{w_{n+2}-p}{w_{n+1}-p}\right\| \Big/ log\left\|\frac{w_{n+1}-p}{w_n-p}\right\|, \text{ for all } n = 1,2,.... \tag{28}$$

and the approximate computational order of convergence (ACOC) [13], by

$$ACOC = log\left\|\frac{w_{n+2}-w_{n+1}}{w_{n+1}-w_n}\right\| \Big/ log\left\|\frac{w_{n+1}-w_n}{w_n-w_{n-1}}\right\|, \text{ for all } n = 1,2,.... \tag{29}$$

The order of convergence is derived.

3. Numerical Experiments

To show the applicability of our theory, we consider the following problems:

Example 1. *The Van der Waals equation of state for a vapor is (see [23])*

$$\left(P + \frac{a}{V^2}\right)(V-b) = RT,$$

Then, we must solve equation

$$PV^3 - (Pb + RT)V^2 + aV - ab = 0 \tag{30}$$

in V, where all constants have a physical meaning whose values can be found in [23]. We solve this problem when $P = 10,000$ kPa and $T = 800$ K. The solution p of resulting equation is $36.9167\ldots$. Then, we can choose $w_0(t) = w(t) = 0.386121\ t$ and $w_1(t) = 2$, and by using (C) conditions the parameters are given by

$$R_1 = 1.72657, \qquad R_2 = 1.00118 \qquad and \qquad R_3 = 0.695478.$$

So,

$$R = 0.695478.$$

Thus, the convergence of the method (3) to $p = 36.9167\ldots$ is guaranteed, provided that $x_0 \in B(p, R)$.

Example 2. *The following equation appears in the study of fractional conversation to ammonia from nitrogen-hydrogen [24,25]. In particular, for 250 atm, 500 ^{o}C and the equation is*

$$f(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674.$$

Figure 1 shows the conversion process. Then, for $p = 0.27776\ldots$ we have $w_0(t) = 2.11111\ t$, $w(t) = 3.28224\ t$ and $w_1(t) = 2$. The parameters by using (C) conditions are computed as

$$R_1 = 0.266509, \qquad R_2 = 0.155662 \qquad and \qquad R_3 = 0.111387.$$

So,

$$R = 0.111387.$$

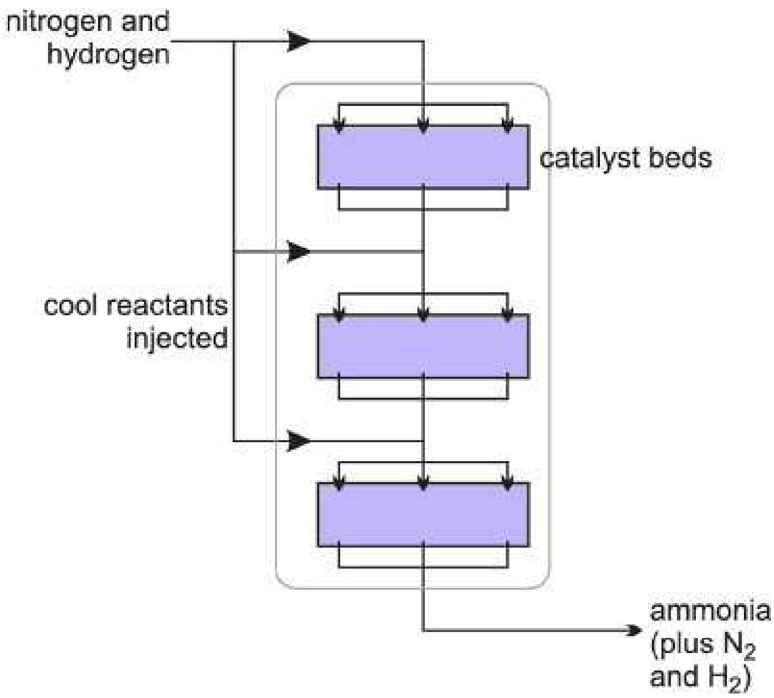

Figure 1. Ammonia process.

Example 3. *Consider the three-mode feedback control of a stirred-tank heater system (Figure 2). The measured output variable is the feed stream temperature [26]. Using standard methods [26], we get the control system defined by*

$$\frac{\bar{T}}{\bar{T}_i} = \frac{(\tau_I s)(\tau_v s+1)(\tau_m s+1)}{(\tau_I s)(\tau_P s+1)(\tau_v s+1)(\tau_m s+1) + K(\tau_I s + 1 + \tau_D \tau_I s^2)} \tag{31}$$

where, the constants appearing in (31) have a physical meaning [26].

To study stability, we first specialize constants and then set the denominator in (31) equal to zero, and solve

$$\begin{gathered}\tau_I \tau_P \tau_m \tau_v s^4 + (\tau_I \tau_P \tau_m + \tau_I \tau_P \tau_v + \tau_I \tau_m \tau_v)s^3 + (K\tau_I \tau_D + \tau_I \tau_P + \tau_I \tau_v + \tau_I \tau_m)s^2 + \\ (\tau_I + K\tau_I)s + K = 0.\end{gathered} \tag{32}$$

We solve the characteristic polynomial when K_c is equal to its "critical" value that is 0.9396 *using*

$$\tau_I = 10, \quad \tau_D = 1, \quad \tau_P = 10, \quad \tau_m = 5, \quad \tau_v = 5, K_P = 10, \quad K_v = 2, \quad K_m = 0.09.$$

By substituting the above parameters in (32), we get that

$$f(s) = 2500s^4 + 1250s^3 + 209.396s^2 + 19.396s + 0.9396, \tag{33}$$

so $p = -0.285837\ldots$. Then, we have that $w_0(t) = 15.2501\, t$, $w(t) = 15.2501\, t$, $w_1(t) = 2$ and by using the (C) conditions, we obtain the parameter values

$$R_1 = 0.0437156, \qquad R_2 = 0.025349 \qquad \textit{and} \qquad R_3 = 0.017609,$$

which implies that

$$R = 0.017609.$$

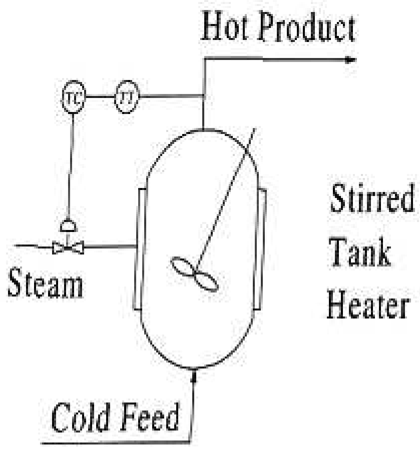

Figure 2. Stirred-tank heater.

Example 4. *Let $B_1 = B_2 = C[0,1]$ and $\Omega = \bar{B}(0,1)$. Define function Q on Ω by*

$$Q(\varphi)(x) = \xi(x) - 5\int_0^1 x\tau\varphi(\tau)^3 d\theta.$$

We have that

$$Q'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in \Omega.$$

Then for $p = 0$ we have that $w_0(s) = 7.5\, s$, $w(s) = 15\, s$ and $w_1(s) = 2$. So, by (C) conditions, we obtain the parameters

$$R_1 = 0.666667, \quad R_2 = 0.0395822 \quad \text{and} \quad R_3 = 0.0288681.$$

So,

$$R = 0.028861.$$

Example 5. *In the example, of introduction, we can choose $w_0(s) = w(s) = \frac{1}{8}\left(\frac{3}{2}\sqrt{s} + s\right)$ and $v(s) = 2$, so*

$$\begin{aligned} R &= \min\{R_1, R_2, R_3\} \\ &= \min\{2.630300,\ 1.633190,\ 1.341759\} = 1.341759. \end{aligned}$$

Thus, the convergence of the method (3) to $p = 0$ is guaranteed, provided that $x_0 \in B(p, R)$.

4. Complex Dynamics of Method

The convergence and stability of iterative methods use complex dynamics of rational functions [18,27,28]. A more complete study can be found, for example, in [29]. Consider mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ is a Riemann sphere, the set of its iterates can be considered as a discrete dynamical system. The set

$$\{z_0, g(z_0), g^2(z_0), \ldots, g^m(z_0), \ldots\}$$

defines the orbit of $z \in \widehat{\mathbb{C}}$.

The dynamical behavior of the orbit of a point of $\widehat{\mathbb{C}}$ can be categorized on its asymptotic behavior. We need the standard definitions

- attractor if $|g^k(z_0)| < 1$,
- superattractor if $|g^k(z_0)| = 0$,

- repuslor if $|g^k(z_0)| > 1$,
- parabolic if $|g^k(z_0)| = 1$.

The basin of attraction $\mathcal{A}(z_0)$ of an attracting point z_0 consists of the set of points $z \in \widehat{\mathbb{C}}$ that accumulate on z_0 under iteration of $g(z)$, that is

$$\mathcal{A}(z_0) = \{z \in \widehat{\mathbb{C}} : g^m(z) \to z_0, m \to \infty\}. \tag{34}$$

The Fatou set contains elements with orbits converging to a fixed point. Moreover, the Juila set is the closure of a set containing fixed points that are repelling.

We take the initial point as $z_0 \in \Omega$, where Ω is a rectangular region in complex plane containing all the roots of $f(z) = 0$. The iterative methods beginning at point z_0 in a rectangle can converge to the zero of $f(z)$ or not converge. We consider the stopping criterion for convergence as 10^{-3} up to a maximum of 25 iterations. If we have not obtained the desired tolerance in 25 iterations, we do not continue and decide that the iterative method starting at z_0 does not converge to any root. The approach taken into account is following: A color is allotted to each starting point z_0 in the basin of attraction of a zero. If the iteration starting from the initial point z_0 converges then it represents the basins of attraction with that particular color assigned to it and if it fails to converge in 25 iterations then it shows the black color. In this way, we discriminate the attraction basins by their colors for the method.

Next, basin of attraction is analyzed.

Test problem 1. Consider the polynomial $P_1(z) = z^2 - 1$ having two simple zeros $\{-1, 1\}$. The basin of attractors for this polynomial are shown in Figure 3. From this figure, it can be observed that method (3) has very stable behavior. In addition, the method does not exhibit chaotic behavior on the boundary points.

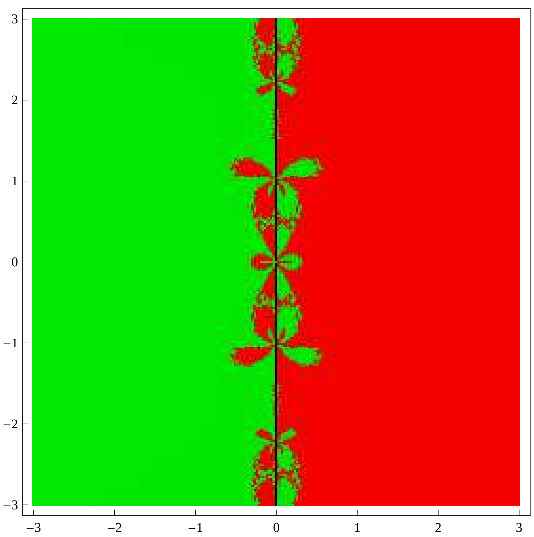

Figure 3. Basins of attraction of method (3) for test problem 1.

Test problem 2. Consider $P_2(z) = z^3 - z$ having three simple zeros $\{-1, 0, 1\}$. The basin of attractors for this polynomial are shown in Figure 4. From this figure, we observe the stable behavior of method (3). Moreover, the method does not show chaotic behavior on the boundary of basins.

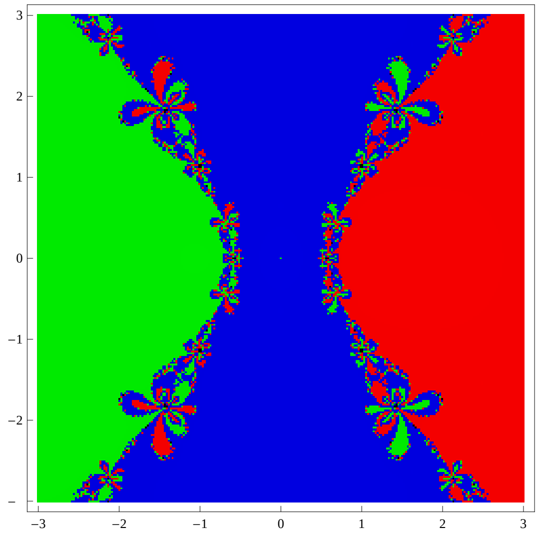

Figure 4. Basins of attraction of method (3) for test problem 2.

Test problem 3. Consider the polynomial $P_4(z) = z^4 - z$ having four simple zeros $\{-0.5 - 0.86602i, 0, 1, -0.5 + 0.86602i\}$. The basin of attractors is shown in Figure 5. In this case, we also observe the beautiful shape of the basins of attraction of different roots. At the boundaries, however, a few small black points show that the method is divergent at such points.

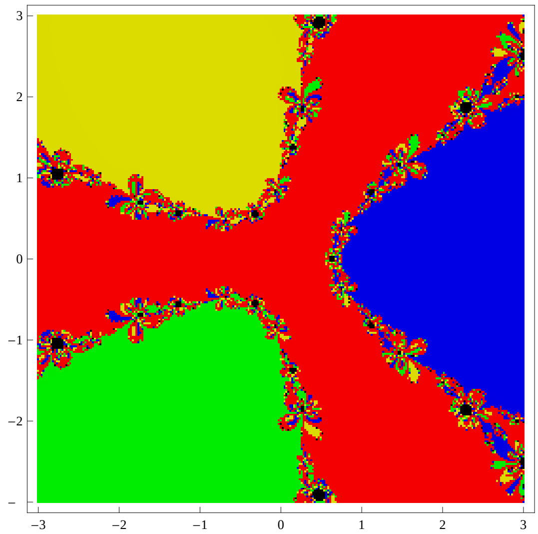

Figure 5. Basins of attraction of method (3) for test problem 3.

5. Conclusions

In this study, we have extended the usage of method (3) by presenting its convergence analysis and complex dynamics. In contrast to other techniques relying on higher derivative order as well as Taylor series, we have used only derivative of order one, since this actually appears in the method. Another advantage of our approach is the computation of balls, uniqueness balls where the iterates lie as well as estimates on $\|x_n - x^*\|$. These goals are achieved using our Lipschitz-like conditions. Theoretical results so derived are verified on some practical problems. Finally, we have checked the stability of the method by means of using complex dynamics tool, namely, basin of attraction.

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