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Article

Some New Classes of Preinvex Functions and Inequalities

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Abstract: In this article, we introduce some new class of preinvex functions involving two arbitrary auxiliary functions. We derive some new integral inequalities for these classes of preinvex functions. We also discuss some special cases which can be deduced from our main results.

Keywords: convex function; preinvex function; Hermite–Hadamard inequality; Holder's inequality and power-mean inequality

MSC: 26D15; 26D10; 90C23

1. Introduction

Several branches of mathematical and engineering science have been developed by using the crucial and significant concepts of convex analysis. Inequalities present a very active and fascinating field of research. In recent years, a wide class of integral inequalities is being derived via different concepts of convexity. These integral inequalities are useful in Physics, where upper bounds for natural phenomena described by integrals such as mechanical work (virtual work) are required. Integral inequalities are closely related to the convex functions and their variant forms.

Convexity theory is an effective and powerful technique for studying a wide class of problems which arise in various branches of pure and applied sciences. Several new classes of convex functions and convex sets have been introduced and investigated. Various new inequalities related to these new classes of convex functions have been derived by researchers—see, for example, Refs. [1–13] and the reference therein.

However, it is amazing that convexity allows many diversified applications in every branch of pure and applied sciences. It is said that $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, if and only if it satisfies the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

which is called the Hermite–Hadamard inequality for convex function (see [12–16]). For the novel applications of the Hermite-Hadamard Inequality (1) (see [7]). Hanson [8] introduced and investigated another class of generalized convex functions, which is called invex functions. Ben-Israel and Mond [17] introduced the concepts of invex sets and preinvex functions. They have shown that the differentiable preininvex functions are invex functions, but the converse may not be true. These preinvex functions are not convex functions, but they enjoy some nice properties, which convex functions have. It is known that the invex functions and preinvex functions are equivalent under some suitable conditions, see [10]. Noor [11,13] has shown that a function f is a preinvex function on $\Omega = [a, a + \eta(b,a)]$, if and only if, it satisfies the inequality

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{2}$$

which is called Hermite-Hadamard-Noor type inequality. Several integral inequalities for various type of preinvex functions have been obtained in recent years. For more details, see [11–15,18–31] and the references therein.

In this paper, we introduce a new class of preinvex functions with respect to two nonnegative arbitrary functions h_1 and h_2, which is called (h_1, h_2)-preinvex function. We establish some new Hermite-Hadamarad inequality for (h_1, h_2)-preinvex function. Some special cases are also discussed which can be obtained from our results.

2. Preliminaries

We now define some new classes of preinvex functions involving two arbitrary functions. Let $\Omega \subset \mathbb{R}$ be a set and $\eta(\cdot,\cdot) : \Omega \times \Omega \longrightarrow \mathbb{R}$ be a continuous bifunction.

First of all, we recall the following well known concepts and results.

Definition 1 ([9,17,19])**.** *A set $\Omega \subset \mathbb{R}$ is said to be invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if and only if*

$$x + t\eta(y,x) \in \Omega, \quad \forall x, y \in \Omega, \; t \in [0,1].$$

The invex set Ω is also called η-connected set. Note that, if $\eta(b,a) = b - a$, this means that every convex set is an invex set, but the converse is not true (see [9]).

From now onward, the set Ω is an invex set, unless otherwise it is specified.

We now consider a new class of preinvex function with respect to two arbitrary functions h_1 and h_2.

Definition 2. *Let $h_1, h_2 : (0,1) \subseteq J \longrightarrow \mathbb{R}$ be two nonnegative functions and Ω be an invex set. A function $f : \Omega \longrightarrow \mathbb{R}$ is said to be a (h_1, h_2)- preinvex function, if*

$$f(x + t\eta(y,x)) \leq h_1(1-t)h_2(t)f(x) + h_1(t)h_2(1-t)f(y), \forall x, y \in \Omega, t \in [0,1]. \tag{3}$$

Note that for $t = \frac{1}{2}$, we have Jensen type (h_1, h_2)-preinvex function, that is,

$$f\left(\frac{2x+\eta(y,x)}{2}\right) \leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [f(x) + f(y)]. \tag{4}$$

We now discuss several special cases.

(I). If $h_1(t) = t^s$ and $h_2(t) = t^s$ in Definition 2, then we have a new class of s-preinvex functions.

Definition 3. *Let $s \in [0,1]$ be a real number and Ω be an invex set. We say that $f : \Omega \longrightarrow \mathbb{R}$ is a s-preinvex function, if*

$$f(x + t\eta(y,x)) \leq t^s(1-t)^s[f(x) + f(y)], \quad \forall x, y \in \Omega, t \in [0,1].$$

Definition 4. *Let $s_1, s_2 \in [0,1]$ be two real numbers and Ω be an invex set. We say that $f : \Omega \longrightarrow R$ is an (s_1, s_2)-preinvex functions, if*

$$f(x + t\eta(y,x)) \leq t^{s_1}(1-t)^{s_2}f(x) + (1-t)^{s_1}t^{s_2}f(y), \quad \forall x, y \in \Omega, t \in [0,1].$$

(II). If $h_1(t) = t^{-s}$ and $h_2(t) = t^{-s}$ in Definition 2, then we have a new class of preinvex functions, which is called Godunova–Levin (s_1, s_2)-preinvex functions.

Definition 5. *Let $s_1, s_2 \in [0,1]$ be two real numbers and Ω be an invex set. We say that $f : \Omega \longrightarrow R$ is a (s_1, s_2)-preinvex functions, if*

$$f(x + t\eta(y,x)) \leq \frac{1}{t^{s_1}(1-t)^{s_2}} f(x) + \frac{1}{(1-t)^{s_1} t^{s_2}} f(y), \quad \forall x, y \in \Omega, t \in [0,1].$$

For appropriate and suitable choice of functions h_1, h_2 and the bifunction, one can obtain several new and known classes of preinvex functions and convex functions as special cases. See, for example, [1,3,4,11,21,22,24,27–31]. This shows that the concept of (h_1, h_2)-preinvex function is quite a general and unifying one.

We need the following result.

Lemma 1. *Let $f : \Omega = [a, a + \eta(b,a)] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then, for some fixed $\alpha, \beta > 0$,*

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a + \eta(b,a) - u)^\beta f(u) \mathrm{d}u = (\eta(b,a))^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(a + t\eta(b,a)) \mathrm{d}t.$$

Proof. Using the change of variables, $u = a + t\eta(b,a)$, we have

$$\begin{aligned} &\int_a^{a+\eta(b,a)} (u-a)^\alpha (a + \eta(b,a) - u)^\beta f(u) \mathrm{d}u \\ &= \int_0^1 \eta(b,a)(t\eta(b,a))^\alpha ((1-t)\eta(b,a))^\beta f(a + t\eta(b,a)) \mathrm{d}t \\ &= (\eta(b,a))^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(a + t\eta(b,a)) \mathrm{d}t, \end{aligned}$$

which is the required result. □

We recall the special functions which are known as Gamma function and Beta function, respectively:

$$\begin{aligned} \Gamma(x) &= \int_0^\infty e^{-t} t^{x-1} dt, \\ \mathbb{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0. \end{aligned}$$

In addition, we recall the well known fact about the bifunction $\eta(.,.)$:

$$\eta(a + t_2\eta(b,a), a + t_1\eta(b,a)) = (t_2 - t_1)\eta(b,a), \quad \forall a, b \in \Omega, \ t_1, t_2 \in [0,1]. \tag{5}$$

3. Main Results

In this section, we establish several new Hermite–Hadamard type inequalities for (h_1, h_2)-preinvex functions.

Theorem 1. *Let $f : \Omega = [a, a + \eta(b, a)] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be (h_1, h_2)-preinvex function with $\eta(b, a) > 0$ and $h_1\left(\frac{1}{2}\right) \neq 0$, $h_2\left(\frac{1}{2}\right) \neq 0$. If $f \in \mathcal{L}[a, a + \eta(b, a)]$ and Equation (5) holds, then we have*

$$\begin{aligned}\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{2a + \eta(b, a)}{2}\right) &\leq \frac{1}{\eta(b, a)} \int\limits_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\ &\leq f(a) \int\limits_0^1 h_1(t)h_2(1 - t)\mathrm{d}t + f(b) \int\limits_0^1 h_2(t)h_1(1 - t)\mathrm{d}t.\end{aligned}$$

Proof. Let f be an (h_1, h_2)-preinvex function. Then, from inequality (2), we have

$$f\left(\frac{2x + \eta(y, x)}{2}\right) \leq \quad h_1\left(\tfrac{1}{2}\right) h_2\left(\tfrac{1}{2}\right) [f(x) + f(y)], \ \forall x, y \in \Omega. \tag{6}$$

Substituting $x = a + (1 - t)\eta(b, a)$ and $y = a + t\eta(b, a)$ in (6) and using (5), we have

$$\begin{aligned}& f\left(\frac{2x + \eta(y, x)}{2}\right) \\ &= f\left(\frac{2\big(a + (1 - t)\eta(b, a)\big) + \eta\big(a + t\eta(b, a), a + (1 - t)\eta(b, a)\big)}{2}\right) \\ &\leq f\left(\frac{2\big(a + (1 - t)\eta(b, a)\big) + (2t - 1)\eta(b, a)\big)}{2}\right).\end{aligned}$$

From inequality (6), we have

$$\begin{aligned}f\left(\frac{2a + \eta(b, a)}{2}\right) &\leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [f(x) + f(y)] \\ &= h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [f(a + (1 - t)\eta(b, a)) + f(a + t\eta(b, a))].\end{aligned}$$

Integrating the above inequality with respect to t over [0,1], we have

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq h_1(\frac{1}{2})h_2(\frac{1}{2})\Big[\int\limits_0^1 f(a + (1 - t)\eta(b, a))dt + \int\limits_0^1 f(a + t\eta(b, a))dt\Big].$$

Using the change of variable technique, $x = a + t\eta(b, a), \quad w = a + (1 - t)\eta(b, a)$, we have

$$\begin{aligned}f\left(\frac{2a + \eta(b, a)}{2}\right) &= h_1\Big(\frac{1}{2}\Big)h_2\Big(\frac{1}{2}\Big)\frac{1}{\eta(b, a)}\Big[\int\limits_a^{a+\eta(b,a)} f(x)dx - \int\limits_{a+\eta(b,a)}^{a} f(w)dw\Big] \\ &= 2h_1\Big(\frac{1}{2}\Big)h_2\Big(\frac{1}{2}\Big)\frac{1}{\eta(b, a)} \int\limits_a^{a+\eta(b,a)} f(x)dx.\end{aligned}$$

Thus,

$$
\begin{aligned}
\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq & \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx. \\
\leq & \int\limits_0^1 \left(h_1(t)h_2(1-t)f(a) + h_1(1-t)h_2(t)f(b)\right) dt \\
= & f(a)\int\limits_0^1 h_1(t)h_2(1-t)dt + f(b)\int\limits_0^1 h_1(1-t)h_2(t)dt,
\end{aligned}
$$

which is the required result. □

We now discuss the new special cases of Theorem 1.

(III). If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then under the assumptions of Theorem 1, then we have a new result for Breckner type of (s_1, s_2)-preinvex functions:

$$
\begin{aligned}
\frac{1}{2^{1-s_1-s_2}} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq & \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \\
\leq & [f(a)+f(b)]\mathbb{B}(s_1+1, s_2+1),
\end{aligned}
$$

where $\mathbb{B}(x, y)$ denotes the beta as a special function.

(IV). If $h(t_1) = t^{-s_1}$ and $h(t_2) = t^{-s_2}$, under the assumptions of Theorem 1, then we have a new result for Godunova–Levin type of (s_1, s_2)-preinvex function:

$$
\begin{aligned}
\frac{1}{2^{1+s_1+s_2}} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq & \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx \\
\leq & [f(a)+f(b)]\mathbb{B}(1-s_1, 1-s_2).
\end{aligned}
$$

Theorem 2. *Let $f : \Omega = [a, a+\eta(b,a)] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. If f is a (h_1, h_2)-preinvex function, then, for some fixed $\alpha, \beta > 0$,*

$$
\int\limits_a^{a+\eta(b,a)} (u-a)^{\alpha}(a+\eta(b,a)-u)^{\beta} f(u)du \leq (\eta(b,a))^{\alpha+\beta+1}[\Psi_1(t)f(a) + \Psi_2(t)f(b)],
$$

where

$$
\begin{aligned}
\Psi_1(t) := & \int\limits_0^1 t^{\alpha}(1-t)^{\beta} h_1(1-t)h_2(t)dt, \\
\Psi_2(t) := & \int\limits_0^1 t^{\alpha}(1-t)^{\beta} h_1(t)h_2(1-t)dt.
\end{aligned}
$$

Proof. Using Lemma 1 and the fact that f is a (h_1, h_2)-preinvex function, we have

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \\
&= (\eta(b,a))^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(a+t\eta(b,a))dt \\
&\leq (\eta(b,a))^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta \left[h_1(1-t)h_2(t)f(a) + h_2(1-t)h_1(t)f(b)\right] dt \\
&= (\eta(b,a))^{\alpha+\beta+1} \left[\Psi_1(t)f(a) + \Psi_2(t)f(b)\right],
\end{aligned}$$

which completes the proof. □

The next results are special cases of Theorem 2.

(V). If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ under the assumption of Theorem 2, then we have a new result for Breckner type of (s_1, s_2)-preinvex functions, we have

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \leq (\eta(b,a))^{\alpha+\beta+1} \left[\gamma_1(t)f(a) + \gamma_2(t)f(b)\right],$$

where

$$\begin{aligned}
\gamma_1(t) &:= \mathbb{B}(\alpha+s_2+1, \beta+s_1+1), \\
\gamma_2(t) &:= \mathbb{B}(\alpha+s_1+1, \beta+s_2+1).
\end{aligned}$$

(VI). If $h(t_1) = t^{-s_1}$ and $h(t_2) = t^{-s_2}$, under the assumption of Theorem 2, then we have a new result for Godunova–Levin type of (s_1, s_2)-preinvex functions, we have

$$\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \leq (\eta(b,a))^{\alpha+\beta+1} \left[\delta_1(t)f(a) + \delta_2(t)f(b)\right],$$

where

$$\begin{aligned}
\delta_1(t) &:= \mathbb{B}(\alpha-s_2+1, \beta-s_1+1), \\
\delta_2(t) &:= \mathbb{B}(\alpha-s_1+1, \beta-s_2+1).
\end{aligned}$$

Theorem 3. *Let $f : \Omega = [a, a+\eta(b,a)] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. If $|f|^{\frac{r}{r-1}}$ is a (h_1, h_2)-preinvex function, then for some fixed $\alpha, \beta > 0$,*

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \\
&\leq (\eta(b,a))^{\alpha+\beta+1} (\mathbb{B}(r\alpha+1, r\beta+1) \left[|f(a)|^{\frac{r}{r-1}}\theta_1(t) + |f(b)|^{\frac{r}{r-1}}\theta_2(t)\right]^{\frac{r-1}{r}},
\end{aligned}$$

where

$$\theta_1(t) = \int_0^1 h_1(1-t)h_2(t)dt, \qquad \theta_2(t) = \int_0^1 h_1(t)h_2(1-t)dt.$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^{\frac{r}{r-1}}$ is a (h_1, h_2)-preinvex function, we have

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \\
&= (\eta(b,a))^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f(a+t\eta(b,a))dt \\
&\le (\eta(b,a))^{\alpha+\beta+1} \left[\int_0^1 t^{r\alpha}(1-t)^{r\beta}dt\right]^{\frac{1}{r}} \left[\int_0^1 |f(a+t\eta(b,a))|^{\frac{r}{r-1}}dr\right]^{\frac{r-1}{r}} \\
&\le (\eta(b,a))^{\alpha+\beta+1} (\mathbb{B}(r\alpha+1, r\beta+1))^{\frac{1}{r}} \Bigg[\int_0^1 [h_1(1-t)h_2(t)|f(a)|^{\frac{r}{r-1}} \\
&\quad + h_1(t)h_2(1-t)|f(b)|^{\frac{r}{r-1}}]dt\Bigg]^{\frac{r-1}{r}} \\
&\le (\eta(b,a))^{\alpha+\beta+1} (\mathbb{B}(r\alpha+1, r\beta+1))^{\frac{1}{r}} \Bigg[|f(a)|^{\frac{r}{r-1}} \left(\int_0^1 h_1(1-t)h_2(t)dt\right) \\
&\quad + |f(b)|^{\frac{r}{r-1}} \left(\int_0^1 h_1(t)h_2(1-t)dt\right)\Bigg]^{\frac{r-1}{r}} \\
&= (\eta(b,a))^{\alpha+\beta+1} (\mathbb{B}(r\alpha+1, r\beta+1))^{\frac{1}{r}} \left[|f(a)|^{\frac{r}{r-1}}\theta_1(t) + |f(b)|^{\frac{r}{r-1}}\theta_2(t)\right]^{\frac{r-1}{r}},
\end{aligned}
$$

which completes the proof. □

The following results are some special cases of Theorem 3.

(VII). If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 3, then we have a Breckner type of (s_1, s_2)-preinvex function:

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \\
&\le \eta(b,a)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1)^{\frac{1}{r}} \left[\mathbb{B}(s_1+1, s_2+1)\right]^{\frac{r-1}{r}} \left[|f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}}\right]^{\frac{r-1}{r}}.
\end{aligned}
$$

(VIII). If $h(t_1) = t^{-s_1}$ and $h(t_2) = t^{-s_2}$ in Theorem 3, then we have Godunova–Levin type (s_1, s_2)-preinvex function:

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} (u-a)^\alpha (a+\eta(b,a)-u)^\beta f(u)du \\
&\le \eta(b,a)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1)^{\frac{1}{r}} \left[\mathbb{B}(1-s_1, 1-s_2)\right]^{\frac{r-1}{r}} \left[|f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}}\right]^{\frac{r-1}{r}}.
\end{aligned}
$$

We now derive some some Hermite-Hadamard type inequalities for differentiable (h_1, h_2)-preinvex function. For this, we need the following result which can be proved using integration by parts. For the sake of completeness, we include its proof.

Lemma 2. *Let $f : \Omega = [a, a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable preinvex function on the interior Ω° of Ω with $\eta(b, a) > 0$. If $f' \in \mathcal{L}[a, a + \eta(b, a)]$ is (h_1, h_2)-preinvex function and $\lambda \in [0, 1]$, then*

$$
\begin{aligned}
&(1-\lambda) f\left(\frac{2a+\eta(b,a)}{2}\right) + \lambda \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \\
&= \frac{\eta(b,a)}{2} \int_0^1 \mu(t) f'(a + t\eta(b,a))dt,
\end{aligned}
$$

where

$$
\mu(t) = \begin{cases} 2t - \lambda, & t \in [0, \frac{1}{2}), \\ 2t - 2 + \lambda, & t \in [\frac{1}{2}, 1]. \end{cases} \tag{7}
$$

Proof. Consider

$$
\begin{aligned}
I &= \frac{\eta(b,a)}{2} \int_0^1 \mu(t) f'(a + t\eta(b,a))dt \\
&= \frac{\eta(b,a)}{2} \left[\int_0^{\frac{1}{2}} (2t - \lambda) f'(a + t\eta(b,a))dt + \int_{\frac{1}{2}}^1 (2t - 2 + \lambda) f'(a + t\eta(b,a))dt \right] \\
&= I_1 + I_2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
I_1 &= \frac{\eta(b,a)}{2} \int_0^{\frac{1}{2}} (2t - \lambda) f'(a + t\eta(b,a))dt \\
&= \frac{1}{2} \left| (2t - \lambda) f(a + t\eta(b,a)) \right|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f(a + t\eta(b,a))dt \\
&= \frac{1-\lambda}{2} f\left(\frac{2a+\eta(b,a)}{2}\right) + \frac{\lambda}{2} f(a) - \int_0^{\frac{1}{2}} f(a + t\eta(b,a)).dt.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \frac{\eta(b,a)}{2} \left[\int_{\frac{1}{2}}^1 (2t - 2 + \lambda)(f'(a + t\eta(b,a))dt \right] \\
&= \frac{1}{2} \left| (2t - 2 + \lambda)(f(a + t\eta(b,a)) \right|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 f(a + t\eta(b,a))dt \\
&= \frac{1-\lambda}{2} f\left(\frac{2a+\eta(b,a)}{2}\right) + \frac{\lambda}{2} f(a + \eta(b,a) - \int_{\frac{1}{2}}^1 f(a + t\eta(b,a)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& I_1 + I_2 \\
&= (1-\lambda) f\left(\frac{2a+\eta(b,a)}{2}\right) + \lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right) - \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx
\end{aligned}
$$

the required result. □

Theorem 4. *Let $f : \Omega = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable preinvex function on the interior Ω° of Ω with $\eta(b,a) > 0$. If $f' \in \mathcal{L}[a, a+\eta(b,a)]$ and $|f'|^q$ is a (h_1, h_2)-preinvex function on Ω for $q \geq 1$ and $\lambda \in [0,1]$, then*

$$
\begin{aligned}
& (1-\lambda) f\left(\frac{2a+\eta(b,a)}{2}\right) + \lambda \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx \\
&= \frac{\eta(b,a)}{2}\Bigg[(\zeta_1(a,b;\lambda))^{1-\frac{1}{q}} \Big[\zeta_2(a,b;\lambda,h_1,h_2)|f'(a)|^q \\
&\quad + \zeta_3(a,b;\lambda,h_1,h_2)|f'(b)|^q\Big]^{\frac{1}{q}} + (\zeta_4(a,b;\lambda))^{1-\frac{1}{q}} \Big[\zeta_5(a,b;\lambda,h_1,h_2)|f'(a)|^q \\
&\quad + \zeta_6(a,b;\lambda,h_1,h_2)|f'(b)|^q\Big]^{\frac{1}{q}} \Bigg],
\end{aligned}
$$

where $\mu(t)$ is defined by (7) and

$$\zeta_1(a,b;\lambda) = \int\limits_0^{\frac{1}{2}} |\mu(t)|dt, \tag{8}$$

$$\zeta_2(a,b;\lambda,h_1,h_2) = \int\limits_0^{\frac{1}{2}} h_1(1-t)h_2(t)|\mu(t)|dt, \tag{9}$$

$$\zeta_3(a,b;\lambda,h_1,h_2) = \int\limits_0^{\frac{1}{2}} h_1(t)h_2(1-t)|\mu(t)|dt, \tag{10}$$

$$\zeta_4(a,b;\lambda,h) = \int\limits_{\frac{1}{2}}^{1} |\mu(t)|dt, \tag{11}$$

$$\zeta_5(a,b;\lambda,h_1,h_2) = \int\limits_{\frac{1}{2}}^{1} h_1(1-t)h_2(t)|\mu(t)|dt, \tag{12}$$

$$\zeta_6(a,b;\lambda,h_1,h_2) = \int\limits_{\frac{1}{2}}^{1} h_1(t)h_2(1-t)|\mu(t)|dt. \tag{13}$$

Proof. Using Lemma 2 and the power mean inequality, we have

$$
\begin{aligned}
&(1-\lambda)f\left(\frac{2a+\eta(b,a)}{2}\right)+\lambda\frac{f(a)+f(a+\eta(b,a)}{2}-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)}f(x)dx \\
&\leq \frac{\eta(b,a)}{2}\left[\int\limits_{0}^{\frac{1}{2}}|2t-\lambda||f^{'}(a+t\eta(b,a))|dt+\int\limits_{\frac{1}{2}}^{1}|2t-2+\lambda||f^{'}(a+t\eta(b,a))|dt\right] \\
&\leq \frac{\eta(b,a)}{2}\left[\left(\int\limits_{0}^{\frac{1}{2}}|2t-\lambda|dt\right)^{1-\frac{1}{q}}\left(\int\limits_{0}^{\frac{1}{2}}|2t-\lambda||(f^{'}(a+t\eta(b,a))|)^{q}dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int\limits_{\frac{1}{2}}^{1}|2t-2+\lambda|dt\right)^{1-\frac{1}{q}}\left(\int\limits_{\frac{1}{2}}^{1}|2t-2+\lambda||(f^{'}(a+t\eta(b,a))|)dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\eta(b,a)}{2}\left[\left(\int\limits_{0}^{\frac{1}{2}}|2t-\lambda|dt\right)^{1-\frac{1}{q}}\left(\int\limits_{0}^{\frac{1}{2}}|2t-\lambda|\Big[h_1(1-t)h_2(t)|f^{'}(a)|^{q}\right.\right. \\
&\quad\left.+h_2(1-t)h_1(t)|f^{'}(b)|^{q}\Big]dt\right)^{\frac{1}{q}}+\left(\int\limits_{\frac{1}{2}}^{1}|2t-2+\lambda|dt\right)^{1-\frac{1}{q}}\left(\int\limits_{\frac{1}{2}}^{1}|2t-2+\lambda|\right. \\
&\quad\left.\left.\Big[h_1(1-t)h_2(t)|f^{'}(a)|^{q}+h_2(1-t)h_1(t)|f^{'}(b)|^{q}\Big]dt\right)^{\frac{1}{q}}\right] \\
&=\frac{\eta(b,a)}{2}\left[(\zeta_1(a,b;\lambda))^{1-\frac{1}{q}}\Big[\zeta_2(a,b;\lambda,h_1,h_2)|f^{'}(a)|^{q}+\zeta_3(a,b;\lambda,h_1,h_2)|f^{'}(b)|^{q}\Big]^{\frac{1}{q}}\right. \\
&\quad\left.+(\zeta_4(a,b;\lambda))^{1-\frac{1}{q}}\Big[\zeta_5(a,b;\lambda,h_1,h_2)|f^{'}(a)|^{q}+\zeta_6(a,b;\lambda,h_1,h_2)|f^{'}(b)|^{q}\Big]^{\frac{1}{q}}\right]
\end{aligned}
$$

the required result. □

For appropriate and suitable choice of q and λ, we obtain several new results for the example midpoint, Trapezoidal, three point Trapezoidal rule and Simpson's rule.

Corollary 1. *If $q=1$, then, under the assumption of Theorem 4, we have*

$$
\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a+\eta(b,a)}{2}\right)+\lambda\frac{f(a)+f(a+\eta(b,a)}{2}-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)}f(x)dx\right| \\
&\leq \frac{\eta(b,a)}{2}\left[\Big[\zeta_2(a,b;\lambda,h_1,h_2)|f^{'}(a)|+\zeta_3(a,b;\lambda,h_1,h_2)|f^{'}(b)|\Big]\right. \\
&\quad\left.+\Big[\zeta_5(a,b;\lambda,h_1,h_2)|f^{'}(a)|+\zeta_6(a,b;\lambda,h_1,h_2)|f^{'}(b)|\Big]\right].
\end{aligned}
$$

Corollary 2. *If $\lambda = 0$, then under the assumption of Theorem 4, we have two points midpoint, which is*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right|$$
$$\leq \frac{\eta(b,a)}{2} \left[\left[\zeta_2(a,b;0,h_1,h_2)|f'(a)|^q + \zeta_3(a,b;0,h_1,h_2)|f'(b)|^q \right] + \left[\zeta_5(a,b;0,h_1,h_2)|f'(a)|^q + \zeta_6(a,b;0,h_1,h_2)|f'(b)|^q \right] \right]^{\frac{1}{q}},$$

where $\zeta_2(a,b;0,h_1,h_2), \zeta_3(a,b;0,h_1,h_2), \zeta_5(a,b;0,h_1,h_2)$ and $\zeta_6(a,b;0,h_1,h_2)$ are given by (9), (10), (12) and (13), respectively.

Corollary 3. *If $\lambda = 1$, then, under the assumption of Theorem 4, we have Trapezoidal rule, which is*

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right|$$

$$\leq \frac{\eta(b,a)}{2} \left[(\zeta_1(a,b;1))^{1-\frac{1}{q}} \left[\zeta_2(a,b;1,h_1,h_2)|f'(a)|^q + \zeta_3(a,b;1,h_1,h_2)|f'(b)|^q \right]^{\frac{1}{q}} + (\zeta_4(a,b;1))^{1-\frac{1}{q}} \left[\zeta_5(a,b;1,h_1,h_2)|f'(a)|^q + \zeta_6(a,b;1,h_1,h_2)|f'(b)|^q \right] \right]^{\frac{1}{q}},$$

where $\zeta_1(a,b;1), \zeta_2(a,b;1,h_1,h_2), \zeta_3(a,b;1,h_1,h_2), \zeta_4(a,b;1,h), \zeta_5(a,b;1,h_1,h_2)$ and $\zeta_6(a,b;1,h_1,h_2)$ are given by (8)–(13), respectively.

Corollary 4. *If $\lambda = \frac{1}{2}$, then, under the assumption of Theorem 4, we have the three points Trapezoidal rule*

$$\left| \frac{1}{4}\left[f(a) + 2f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right|$$
$$\leq \eta(b,a) \left[\left(\zeta_1(a,b;\frac{1}{2})\right)^{1-\frac{1}{q}} \left[\zeta_2(a,b;\frac{1}{2},h_1,h_2)|f'(a)|^q + \zeta_3(a,b;\frac{1}{2},h_1,h_2)|f'(b)|^q \right]^{\frac{1}{q}} + \left(\zeta_4(a,b;\frac{1}{2})\right)^{1-\frac{1}{q}} \left[\zeta_5(a,b;\frac{1}{2},h_1,h_2)|f'(a)|^q + \zeta_6(a,b;\frac{1}{2},h_1,h_2)|f'(b)|^q \right]^{\frac{1}{q}} \right],$$

where $\zeta_1(a,b;\frac{1}{2}), \zeta_2(a,b;\frac{1}{2},h_1,h_2), \zeta_3(a,b;\frac{1}{2},h_1,h_2), \zeta_4(a,b;\frac{1}{2},h), \zeta_5(a,b;\frac{1}{2},h_1,h_2)$ and $\zeta_6(a,b;\frac{1}{2},h_1,h_2)$ are given by (8)–(13), respectively.

Corollary 5. *If $\lambda = \frac{1}{3}$, then, under the assumption of Theorem 4, we have Simpson's rule*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{\eta(b,a)}{2}\left[\left(\zeta_1(a,b;\frac{1}{3})\right)^{1-\frac{1}{q}}\left[\zeta_2(a,b;\frac{1}{3},h_1,h_2)|f^{'}(a)|^q+\zeta_3(a,b;\frac{1}{3},h_1,h_2)|f^{'}(b)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left(\zeta_4(a,b;\frac{1}{3})\right)^{1-\frac{1}{q}}\left[\zeta_5(a,b;\frac{1}{3},h_1,h_2)|f^{'}(a)|^q+\zeta_6(a,b;\frac{1}{3},h_1,h_2)|f^{'}(b)|^q\right]^{\frac{1}{q}}\right],$$

where $\zeta_1(a,b;\frac{1}{3}), \zeta_2(a,b;\frac{1}{3},h_1,h_2), \zeta_3(a,b;\frac{1}{3},h_1,h_2), \zeta_4(a,b;\frac{1}{3},h), \zeta_5(a,b;\frac{1}{3},h_1,h_2)$ and $\zeta_6(a,b;\frac{1}{3},h_1,h_2)$ are given by (8)–(13), respectively.

Theorem 5. *Let $f : \Omega = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable preinvex function on the interior Ω° of Ω with $\eta(b,a) > 0$. If $f^{'} \in \mathcal{L}[a, a+\eta(b,a)]$ and $|f^{'}|^q$ is a (h_1,h_2)-preinvex function on Ω for $p,q \geq 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\lambda \in [0,1]$, then*

$$(1-\lambda)f\left(\frac{2a+\eta(b,a)}{2}\right)+\lambda\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)dx$$
$$\leq \frac{\eta(b,a)}{2}\left[(\zeta_7(a,b,p;\lambda))^{\frac{1}{p}}\left(\frac{|f^{'}(a)|^q+\left|f^{'}\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q}{2}\int\limits_{0}^{1} h_1(1-t)h_2(t)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+(\zeta_8(a,b,p;\lambda))^{\frac{1}{p}}\left(\frac{\left|f^{'}\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+|f^{'}(a+\eta(b,a))|^q}{2}\int\limits_{0}^{1} h_2(1-t)h_1(t)dt\right)^{\frac{1}{q}}\right],$$

where

$$\zeta_7(a,b,p;\lambda) = \int\limits_{0}^{\frac{1}{2}} |\mu(t)|^p dt, \qquad \zeta_8(a,b,p;\lambda) = \int\limits_{\frac{1}{2}}^{1} |\mu(t)|^p dt.$$

Proof. Using Lemma 2 and the Holder's integral inequality, we have

$$
\begin{aligned}
&(1-\lambda) f\left(\frac{2a+\eta(b,a)}{2}\right)+\lambda \frac{f(a)+f(a+\eta(b,a)}{2}-\frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x) d x \\
&\leq \frac{\eta(b,a)}{2}\left[\left(\int_{0}^{\frac{1}{2}}|2t-\lambda||(f^{\prime}(a+t\eta(b,a))| d t\right)\right. \\
&\quad \left.+\left(\int_{\frac{1}{2}}^{1}|2t-2+\lambda||(f^{\prime}(a+t\eta(b,a))| d t\right)\right] \\
&\leq \frac{\eta(b,a)}{2}\left[\left(\int_{0}^{\frac{1}{2}}|2t-\lambda|^{p} d t\right)^{\frac{1}{p}}\left(|f^{\prime}(a+t\eta(b,a))|^{q} d t\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_{\frac{1}{2}}^{1}|2t-2+\lambda|\right)^{p} d t\Big]^{\frac{1}{p}}\left(|(f^{\prime}(a+t\eta(b,a))|^{q} d t\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\eta(b,a)}{2}\left[\left(\int_{0}^{\frac{1}{2}}|2t-\lambda|^{p} d t\right)^{\frac{1}{p}}\left[\frac{1}{\eta(b,a)}\left(\int_{a}^{\frac{2a+\eta(b,a)}{2}}|f^{\prime}(x)|^{q} d x\right)\right]^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_{\frac{1}{2}}^{1}|2t-2+\lambda|^{p} d t\right)^{\frac{1}{p}}\left[\frac{1}{\eta(b,a)}\left(\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)}|f^{\prime}(x)|^{q} d x\right)\right]^{\frac{1}{q}}\right].
\end{aligned}
$$

Using the definition of (h_1, h_2)-preinvex function of $|f^{'}|^q$, we obtain the inequality (3):

$$
\begin{aligned}
\frac{2}{\eta(b,a)} \int_{a}^{\frac{2a+\eta(b,a)}{2}}|f^{\prime}(x)|^{q} d x \leq & \left[|f^{\prime}(a)|^{q}\left(\int_{0}^{1} h_{1}(1-t) h_{2}(t) d t\right)\right. \\
& \left.+\left|f^{\prime}\left(\frac{2a+\eta(b,a)}{2}\right)\right|^{q}\left(\int_{0}^{1} h_{2}(1-t) h_{1}(t) d t\right)\right]
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
\frac{2}{\eta(b,a)} \int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)}|f^{\prime}(x)|^{q} d x \leq & \left[\left|f^{\prime}\left(\frac{2a+\eta(b,a)}{2}\right)\right|^{q}\left(\int_{0}^{1} h_{1}(1-t) h_{2}(t) d t\right)\right. \\
& \left.+|f^{\prime}(a+\eta(b,a))|^{q}\left(\int_{0}^{1} h_{2}(1-t) h_{1}(t) d t\right)\right]
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&=\frac{\eta(b,a)}{2}\left[(\zeta_{7}(a,b,p;\lambda))^{\frac{1}{p}}\left(\frac{|f^{\prime}(a)|^{q}+\left|f^{\prime}\left(\frac{2a+\eta(b,a)}{2}\right)\right|^{q}}{2} \int_{0}^{1} h_{1}(1-t) h_{2}(t) d t\right)^{\frac{1}{q}}\right. \\
&\left.+(\zeta_{8}(a,b,p;\lambda))^{\frac{1}{p}}\left(\frac{\left|f^{\prime}\left(\frac{2a+\eta(b,a)}{2}\right)\right|^{q}+|f^{\prime}(a+\eta(b,a))|^{q}}{2} \int_{0}^{1} h_{2}(1-t) h_{1}(t) d t\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{16}
$$

A combination of (14)–(15) gives the inequality (16). □

For appropriate and suitable choice of λ, we obtain several new results for two points midpoint, Trapezoidal rule, three point Trapezoidal rule and Simpson's rule.

Theorem 6. *Let $f : \Omega = [a, a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable preinvex function on the interior Ω° of Ω with $\eta(b, a) > 0$. If $f' \in \mathcal{L}[a, a + \eta(b, a)]$ and $|f'|^q$ is a (h_1, h_2)-preinvex function on Ω for $p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1$ and $\lambda \in [0, 1]$, then*

$$\begin{aligned}&\left|(1-\lambda)f\left(\frac{2a+\eta(b,a)}{2}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\&\leq\frac{\eta(b,a)}{2}\times(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)})^{\frac{1}{p}}\left[\left(\zeta_9(a,b;q,h_1,h_2)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right.\right.\\&\quad\left.+\left(\zeta_{10}(a,b;q,h_1,h_2)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\zeta_9(a,b;\lambda,h_1,h_2)=\int_0^{\frac{1}{2}}h_1(t)h_2(1-t)dt,\qquad\zeta_{10}(a,b;\lambda,h_1,h_2)=\int_{\frac{1}{2}}^1h_1(t)h_2(1-t)dt.$$

Proof. Using Lemma 2 and the Holder's integral inequality, we have

$$\begin{aligned}&(1-\lambda)f\left(\tfrac{2a+\eta(b,a)}{2}\right)+\lambda\tfrac{f(a)+f(a+\eta(b,a))}{2}-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\\&\leq\tfrac{\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}|2t-\lambda||f'(a+t\eta(b,a)|)dt+\int_{\frac{1}{2}}^1|2t-2+\lambda||f'(a+t\eta(b,a))|dt\right]\\&\leq\tfrac{\eta(b,a)}{2}\left[\left(\int_0^{\frac{1}{2}}|2t-\lambda|^pdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(a+t\eta(b,a))|^qdt\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\int_{\frac{1}{2}}^1|2t-2+\lambda|^pdt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(a+t\eta(b,a))|^qdt\right)^{\frac{1}{q}}\right]\\&\leq\tfrac{\eta(b,a)}{2}\left[\left(\int_0^{\frac{1}{2}}|2t-\lambda|^pdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}[h_1(1-t)h_2(t)|f'(a)|^q+h_2(1-t)h_1(t)|f'(b)|^q]dt\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\int_{\frac{1}{2}}^1|2t-2+\lambda|^pdt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1[h_1(1-t)h_2(t)|f'(a)|^q+h_2(1-t)h_1(t)|f'(b)|^q]dt\right)^{\frac{1}{q}}\right]\\&\leq\tfrac{\eta(b,a)}{2}\times\left(\tfrac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\zeta_9(a,b;q,h_1,h_2)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right.\right.\\&\quad\left.+\left(\zeta_{10}(a,b;q,h_1,h_2)\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

the required result. □

For appropriate and suitable choice of λ, we obtain several new results for two points midpoint, Trapezoidal rule, three points Trapezoidal rule and Simpson's rule.

4. Conclusions

In this paper, we have established several Hermite–Hadamard type inequalities for (h_1, h_2)-preinvex functions. These results can be viewed as refinement and significant improvements of the previously known and new classes of preinvex functions. The ideas and techniques of this paper may be extended for other classes of convex functions and their variant forms.

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