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Global Optimization for Quasi-Noncyclic Relatively Nonexpansive Mappings with Application to Analytic Complex Functions

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Abstract: The purpose of this article is to resolve a global optimization problem for quasi-noncyclic relatively nonexpansive mappings by giving an algorithm that determines an optimal approximate solution of the following minimization problem,

$$\min_{x\in A} d(x,Tx),\quad \min_{y\in B} d(y,Ty)\ \text{ and } \min_{(x,y)\in A\times B} d(x,y);$$

also, we provide some illustrative examples to support our results. As an application, the existence of a solution of the analytic complex function is discussed.

Keywords: best proximity point; noncyclic mapping; quasi-noncyclic relatively nonexpansive; semi-sharp proximal pair

1. Introduction

The fixed point theorem is one of the most essential branches of functional analysis because it is significantly useful for and capable of solving many problems. This can be seen from popular applications of the fixed point theorem in the fields of science and technology, as well as other disciplines. However, in the case that A and B are nonempty disjoint subsets of metric space X and T is a mapping from A to B or T is a cyclic mapping on $A\cup B$, the equation $Tx = x$ does not necessarily have a solution. For this case, its approximate solution x is the minimum of $d(x,Tx)$. This is the main idea supporting the best approximation theory, which was introduced by Fan [1]. The necessary condition to guarantee the existence of x in A is satisfying $d(Tx,x) = d(Tx,A) := \inf\{d(Tx,y) : y \in A\}$, which is called the best proximity point. After that, many authors studied and developed Fan's theorem by using different assumptions on various kinds of mappings in many directions; one can refer to [2–8].

In 2005, Eldred et al. [9] defined a mapping $T : A\cup B \to A\cup B$ with properties $T(A) \subseteq A$ and $T(B) \subseteq B$, which is called a noncyclic mapping, and studied the existence of the following minimization problem to find $x \in A$ and $y \in B$ satisfying:

$$\min_{x\in A} d(x,Tx),\quad \min_{y\in B} d(y,Ty)\ \text{ and } \min_{(x,y)\in A\times B} d(x,y), \tag{1}$$

and a solution of (1) is an element $(x,y) \in A\times B$ with the property:

$$x = Tx,\ \ y = Ty \text{ and } d(x,y) = dist(A,B).$$

Later, many authors studied the existence of a solution of (1); see [10–13].

The aim of this paper is to establish the existence of a solution of the minimization problem (1) for quasi-noncyclic relatively nonexpansive mappings, which was defined by Gabeleh and Otafudu [12], and to provide some illustrative examples that support our results. Furthermore, we establish the existence of a solution of the analytic complex functions by applying our new results.

2. Preliminaries

Let A and B be nonempty subsets of a metric space (X,d); we recall some basic concepts that will be used in the next sections.

$$F_A(T) := \{x \in A : x = Tx\}, \ F_B(T) := \{y \in B : y = Ty\},$$

$$dist(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x,y) = dist(A,B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x,y) = dist(A,B) \text{ for some } x \in A\},$$

$$Prox_{A\times B}(T) := \{(x,y) \in A \times B : x = Tx, \ y = Ty \text{ and } \ d(x,y) = dist(A,B)\}.$$

Definition 1. *Let $T : A \cup B \to A \cup B$ be a mapping. A point $x \in A$ is called a best proximity point if it satisfies:*

$$d(x,Tx) = dist(A,B)$$

Definition 2. *A noncyclic mapping $T : A \cup B \to A \cup B$ is called relatively nonexpansive if and only if:*

$$d(Tx,Ty) \leq d(x,y) \quad \textit{for all} \quad x \in A \ \textit{ and } y \in B$$

Note that in Definition 2, if $A = B$, then the mapping T becomes to nonexpansive.

In 2016, Gabeleh and Otafudu [12] (see also [11]) defined the concept of quasi-noncyclic relatively nonexpansive mappings, which includes the class of relatively nonexpansive mappings as follows.

Definition 3. *Let A and B be nonempty subsets of metric space (X,d) such that A_0 is nonempty. A mapping $T : A \cup B \to A \cup B$ is called quasi-noncyclic relatively nonexpansive if and only if:*

(i) *T is noncyclic*
(ii) *$(F_{A_0}(T), F_{B_0}(T))$ is nonempty*
(iii) *for each $(a,b) \in (F_{A_0}(T), F_{B_0}(T))$, we have:*

$$\begin{cases} d(Tx,a) \leq d(x,a) & \textit{for all } \ x \in A, \\ d(b,Ty) \leq d(b,y) & \textit{for all } \ y \in B. \end{cases}$$

Note that in Definition 3, if $A = B$, then the mapping T becomes a quasi-nonexpansive mapping (see [14]). For example, the class of quasi-noncyclic relatively nonexpansive mappings is not a subclass of noncyclic relatively nonexpansive mappings, as we can see in [11].

Definition 4. *A subset A of the metric spaces is said to be approximatively compact with respect to B if and only if every sequence $\{x_n\}$ in A satisfying the condition that $d(y,x_n) \to d(y,A)$ for some $y \in B$ has a convergent subsequence.*

Remark 1. *Let A and B be nonempty subsets of a metric space (X,d) and A_0 and B_0 be nonempty sets; we have:*

(a) *A is approximatively compact with respect to A*
(b) *if A is a compact set, then A is approximatively compact with respect to any set,*
(c) *if A is compact, then B is approximatively compact with respect to A.*

Definition 5. *[2] Let A and B be nonempty subsets of metric space (X,d). A pair (A,B) is called sharp proximinal if and only if, for each x in A and y in B, there exist a unique element x' in B and a unique element y' in A such that:*

$$d(x,x') = d(y',y) = dist(A,B).$$

Definition 6. *[2] Let A and B be nonempty subsets of metric space (X,d). A pair (A,B) is called semi-sharp proximinal if and only if, for each x in A and y in B, there exists at most one element x' in B and at most one element y' in A such that:*

$$d(x,x') = d(y',y) = dist(A,B).$$

3. Main Result

In this section, we establish the existence theorems of the minimization problem (1) by using different assumptions. Furthermore, we provide some illustrative examples to support our results.

Theorem 1. *Let (X,d) be a complete metric space and A, B be nonempty subsets of X such that A is closed and $A_0 \neq \emptyset$. Suppose that B is approximatively compact with respect to A and that $T : A \cup B \to A \cup B$ is a noncyclic mapping satisfying the following conditions.*

(i) *$T|_A$ is a contraction in the sense of Banach and $T(A_0) \subseteq A_0$,*
(ii) *T is quasi-noncyclic relatively nonexpansive,*
(iii) *the pair (A,B) is semi-sharp proximal.*

Then, there exists $(x_\star, y_\star) \in A \times B$, which is a solution of (1).

Proof. Let $x_1 \in A_0$. Since $T(A_0) \subseteq A_0$, there exists an element $x_{n+1} \in A$ such that $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. By the Banach contraction of $T|_A$ and A being closed, there exists $x_\star \in A$ such that the sequence $\{x_n\}$ converges to $x_\star$ with $x_\star = Tx_\star$. From the fact that $T(A_0) \subseteq A_0$, we can find a point y_n in B,

$$d(x_n, y_n) = dist(A,B) \quad \text{for each} \quad n \in \mathbb{N}.$$

By the triangle inequality and the definition of $dist(A,B)$, we have:

$$\begin{aligned} d(x_\star, y_n) &\leq d(x_\star, x_n) + d(x_n, y_n) \\ &= d(x_\star, x_n) + dist(A,B) \\ &\leq d(x_\star, x_n) + dist(x_\star, B), \end{aligned}$$

then $d(x_\star, y_n) \to d(x_\star, B)$ as $n \to \infty$. By the hypothesis that B is approximatively compact with respect to A, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y_\star \in B$ such that $y_{n_k} \to y_\star$ as $k \to \infty$. Therefore,

$$d(x_\star, y_\star) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = dist(A,B). \tag{2}$$

By Assumption (ii) , we get:

$$d(x_\star, Ty_\star) \leq d(x_\star, y_\star) = dist(A,B)$$

and hence:

$$d(x_\star, Ty_\star) = dist(A,B). \tag{3}$$

By Assumption (iii), (2) and (3), we have $Ty_\star = y_\star$. Therefore, we get $(x_\star, y_\star) \in Prox_{A\times B}(T)$, and hence, the minimization problem (1) has a solution. $\square$

Next, we give an example to support Theorem 1.

Example 1. *Let $X = \mathbb{R}^2$, and let d be the metric on X defined by:*

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for all $(x_1, x_2), (y_1, y_2) \in X$. Let $A = \{(0, a) : 0 \le a \le 1\}$ and $B = \{(1, b) : 0 \le b \le 1\}$. Then, $A_0 = A$, $B_0 = B$, and $dist(A, B) = 1$. Define $T : A \cup B \to A \cup B$ by:

$$T(0, a) = (0, \frac{a}{2}) \quad and \quad T(1, b) = (1, \frac{b}{1+b}).$$

Then, T is noncyclic mapping with $T(A_0) \subseteq A_0$, and $p := (0, 0) \in A$, $q := (1, 0) \in B$ are the unique fixed points of T in A and B, respectively. Moreover, the pair (A,B) is semi-sharp proximal. Now, let $(0, a_1), (0, a_2) \in A$, we have:

$$d(T(0, a_1), T(0, a_2)) = \frac{1}{2} d((0, a_1), (0, a_2)).$$

This means that T is a contraction on A. Moreover, let $\boldsymbol{a} = (0, a) \in A$ and $\boldsymbol{b} = (1, b) \in B$,

$$d(T\boldsymbol{a}, q) = 1 + \frac{a}{2} \le 1 + a = d(\boldsymbol{a}, q)$$

and:

$$d(p, T\boldsymbol{b}) = 1 + \frac{b}{1+b} \le 1 + b = d(p, \boldsymbol{b}).$$

Then, Tis a quasi-noncyclic relatively nonexpansive mapping. On the other hand, since $A_0 \neq \emptyset$ and A is a closed subset of $\mathbb{R}^2$, then A is compact. By Remark 1, we get that B is approximatively compact with respect to A. Thus, all the conditions of Theorem 1 are satisfied, and $(0, 0) \in A$, $(1, 0) \in B$ is a solution of Problem (1). That is:

$$(0, 0) = T(0, 0), \ (1, 0) = T(1, 0) \ and \ d((0, 0), (1, 0)) = dist(A, B).$$

Next, we will remove the contraction of T on A by replacing the other conditions to prove a new theorem of the minimization problem (1) as follows.

Theorem 2. *Let (X, d) be a complete metric space and A, B be nonempty subsets of X such that B is approximatively compact with respect to A with $A_0 \neq \emptyset$. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic mapping and the following conditions hold.*

- *(i)* $T|_A$ *is continuous and* $T(A_0) \subseteq A_0$*,*
- *(ii)* *T is quasi-noncyclic relatively nonexpansive,*
- *(iii)* *the pair (A, B) is semi-sharp proximal.*
- *(iv)* *for any sequence $\{x_n\}$ in A, if $d(x_n, y_n) = dist(A, B)$ for some $y_n \in B$, then there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in A$ such that $x_{n_k} \to x$ as $k \to \infty$.*

Then, there exists $(x_\star, y_\star) \in A \times B$, which is a solution of (1).

Proof. Let $x_1 \in A_0$; by the same method as Theorem 1, there exists a sequence $\{x_n\}$ in A and a sequence $\{y_n\}$ in B such that:

$$d(x_n, y_n) = dist(A, B) \quad \text{for all} \quad n \in \mathbb{N}.$$

By Assumption (iv), there exists there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_\star \in A$ such that $x_{n_k} \to x_\star$ as $k \to \infty$. Consequently, $x_\star = Tx_\star$ because $T|_A$ is continuous. Using the same argument as the proof of Theorem 1 and the hypothesis that B is approximatively compact with respect to A, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_\star$ for some $y_\star \in B$. Therefore,

$$d(x_\star, y_\star) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = dist(A, B).$$

By Assumption (ii), we get:

$$d(x_\star, Ty_\star) \le d(x_\star, y_\star) = dist(A, B). \tag{4}$$

and thus:

$$d(x_\star, Ty_\star) = dist(A, B). \tag{5}$$

By Assumption (iii), (4), and (5), we have $Ty_\star = y_\star$. This completes the proof. □

Now, we give an example to illustrate Theorem 2.

Example 2. *Let $X = \mathbb{R}$ with the usual metric, and let $A = [-1, 0]$ and $B = [2, 3]$. Obviously, $dist(A, B) = 2$ and $A_0 = \{0\}$ and $B_0 = \{2\}$. Define the noncyclic mapping $T : A \cup B \to A \cup B$ by:*

$$T(x) = \begin{cases} x + \dfrac{x^2}{2} & x \in A \\ \dfrac{x+2}{2} & x \in B. \end{cases}$$

Then, T is continuous on A and $0 \in A$, $2 \in B$ are unique fixed points of T in A and B, respectively. Moreover, for each $x_1, x_2 \in A$ with $x_1 \ne x_2$, if

$$d(Tx_1, Tx_2) \le \lambda d(x_1, x_2)$$

for some $\lambda \in [0, 1)$, then:

$$|(x_1 + \frac{x_1^2}{2}) - (x_2 + \frac{x_2^2}{2})| \le \lambda |x_1 - x_2|.$$

Letting $x_1 < 0$ and $x_2 = 0$, we have:

$$1 = \lim_{x_1\to 0^-} |1 + \frac{x_1}{2}| = \lim_{x_1\to 0^-} \frac{|x_1 + \frac{x_1^2}{2}|}{|x_1|} \le \lambda < 1,$$

which is a contradiction. Therefore, T is not a contraction on A. Now, for $x \in A$ and $y \in B$,

$$d(Tx, 2) = |x + \frac{x^2}{2} - 2| \le |x - 2| = d(x, 2), \quad \textit{because} \;\; x < 0$$

and:

$$d(0, Ty) = \frac{y+2}{2} \le y = d(0, y).$$

Then, Tis a quasi-noncyclic relatively nonexpansive mapping. It is easy to check that the other conditions of Theorem 2 are satisfied and that $(0, 2) \in A \times B$ is a solution of Problem (1). This example is interesting because $T|_A$ is not a contraction. Therefore, Theorem 1 cannot be applied to this example.

4. Application to Analytic Complex Function Theory

In this section, we will apply Theorem 1 to show the existence theorem of (1) in analytic complex functions. First, we give some properties of our consideration as follow.

Recall that a Banach space X is said to be:

(1) uniformly convex if there exists $\delta : (0,2] \to [0,1]$, which is a strictly increasing function such that, for all $x, y, l \in X, L > 0$ and $r \in [0, 2L]$,

$$\left.\begin{array}{l} \|x-l\| \leq L \\ \|y-l\| \leq L \\ \|x-y\| \geq r \end{array}\right\} \implies \left\|\frac{x+y}{2} - l\right\| \leq \Big(1 - \delta(\frac{r}{L})\Big)L;$$

(2) strictly convex if, for all x, y, l in X and $L > 0$,

$$\left.\begin{array}{l} \|x-l\| \leq L \\ \|y-l\| \leq L \\ x \neq y \end{array}\right\} \implies \left\|\frac{x+y}{2} - l\right\| < L.$$

Remark 2. *It is well known that:*

(a) *Every uniformly convex Banach space is strictly convex.*
(b) *Banach space X is strictly convex if and only if* $\|x_1 + x_2\| < 2$ *whenever* x_1 *and* x_2 *are different points such that* $\|x_1\| = \|x_2\| = 1$.

Proposition 1. *Let A and B be nonempty closed subsets of a strictly convex Banach space X. Then, (A, B) is semi-sharp proximal pair.*

Proof. Let $x \in A$ and $y_1, y_2 \in B$ such that:

$$\|y_1 - x\| = dist(A, B) \text{ and } \|y_2 - x\| = dist(A, B).$$

If $y_1 \neq y_2$, then:

$$\begin{array}{rcl} dist(A,B) & \leq & \|\frac{y_1+y_2}{2} - x\| \\ & \leq & \frac{1}{2}(\|y_1 - x\| + \|y_2 - x\|) \\ & = & dist(A,B) \end{array}$$

which is a contradiction, and hence, $y_1 = y_2$. Similarly, if $x_1, x_2 \in A$ and $y \in B$ such that:

$$\|x_1 - y\| = dist(A, B) \text{ and } \|x_2 - y\| = dist(A, B),$$

hence, $x_1 = x_2$. Therefore, (A, B) is semi-sharp proximal pair. □

Proposition 2. *The complex plane $\mathbb{C}$ with the usual norm $\|x + iy\| = \sqrt{x^2 + y^2}$ for all $x, y \in \mathbb{C}$ is strictly convex.*

Proof. Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$ be a point such that $z_1 \neq z_2$ and:

$$\|z_1\| = \|z_2\| = 1.$$

Since $z_1 \neq z_2$, we know that either:

$$(x_1 - x_2)^2 > 0 \text{ or } (y_1 - y_2)^2 > 0,$$

which further yields:

$$2x_1x_2 < x_1^2 + x_2^2 \text{ or } 2y_1y_2 < y_1^2 + y_2^2.$$

Hence,

$$2x_1x_2 + 2y_1y_2 < x_1^2 + x_2^2 + y_1^2 + y_2^2$$

and thus:

$$\begin{aligned} \|z_1 + z_2\| &= \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \\ &= \sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1x_2 + y_1y_2)} \\ &< \sqrt{2(x_1^2 + y_1^2) + 2(x_1^2 + y_2^2)} \\ &= 2. \end{aligned}$$

Therefore, $\mathbb{C}$ is strictly convex. □

Theorem 3. *Let A and B be nonempty, compact, and convex subsets of a domain D of the complex plane with the usual norm. Let f and g be functions in D such that f is an analytic function. Suppose that $F_B(g)$ is nonempty and the following hold.*

(i) $f(A) \subseteq A$ *and* $f(B) \subseteq B$,
(ii) $|f'(z)| < 1$ *for all* $z \in A$,
(iii) *for* $z_1 \in F_A(f)$ *and* $z_2 \in F_B(g)$, $|fx - z_2| \leq |x - z_2|$ *and* $|z_1 - gy| \leq |z_1 - y|$ *for all* $x \in A$, $y \in B$.

Then, the problem (1) has a solution.

Proof. Since f is an analytic function, then f is continuous, and since A is a compact and convex subset of a domain D, by applying Brouwer's fixed point theorem, this implies that $F_A(f)$ is nonempty. Again, since A is a compact set and $|f'|$ is continuous on A, there exists $\hat{z} \in A$ such that maximum point and $f(\hat{z}) = \lambda < 1$. Therefore, $|f'(z)| \leq \lambda < 1$. Let $z, v \in A$; we have:

$$|f(z) - f(v)| = |\int_v^z f'(z)| \leq \lambda|z - v|.$$

This means that f is a contraction mapping on A. Let $T : A \cup B \to A \cup B$ with:

$$T(z) = \begin{cases} f(z) & \text{if } z \in A \\ g(z) & \text{if } z \in B. \end{cases}$$

Therefore, we have that $T|_A$ is a contraction. Further, by (iii), if $z_1 \in F_A(f)$ and $z_2 \in F_B(g)$, we have:

$$|Tx - z_2| = |fx - z_2| \leq |x - z_2|$$

and:

$$|z_1 - Ty| = |z_1 - gy| \leq |z_1 - y|$$

for all $x \in A$ and $y \in B$. Hence, T is a quasi-noncyclic relatively nonexpansive mapping. Since A and B are nonempty, compact subsets of a domain D, then A is closed, $A_0 \neq \emptyset$, and B is approximatively compact with respect to A. On the other hand, by Proposition 1 and Proposition 2, (A, B) have the semi-sharp proximal property. Therefore, all conditions of Theorem 1 are satisfied, and the conclusion of this theorem follows from Theorem 1. □

5. Conclusions

The existence of the minimization problem (1) for a noncyclic mapping was first studied by Eldred et al. [9]. Later, many authors studied the existence of a solution of (1); see [10–13]. This article resolves a minimization problem (1) for quasi-noncyclic relatively nonexpansive mappings by giving necessary and sufficient conditions with an approximate algorithm for finding the existence of the minimization problem (1). Furthermore, we provide some illustrative examples that support our results. Finally, we apply our results to show the existence of the solution of the analytic complex function.

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