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Some Identities on Degenerate Bernstein and Degenerate Euler Polynomials

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Abstract: In recent years, intensive studies on degenerate versions of various special numbers and polynomials have been done by means of generating functions, combinatorial methods, umbral calculus, p-adic analysis and differential equations. The degenerate Bernstein polynomials and operators were recently introduced as degenerate versions of the classical Bernstein polynomials and operators. Herein, we firstly derive some of their basic properties. Secondly, we explore some properties of the degenerate Euler numbers and polynomials and also their relations with the degenerate Bernstein polynomials.

Keywords: degenerate Bernstein polynomials; degenerate Bernstein operators; degenerate Euler polynomials

1. Introduction

Let us denote the space of continuous functions on $[0,1]$ by $C[0,1]$, and the space of polynomials of degree $\leq n$ by $\mathbb{P}_n$. The Bernstein operator $\mathbb{B}_n$ of order n, $(n \geq 1)$, associates to each $f \in C[0,1]$ the polynomial $\mathbb{B}_n(f|x) \in \mathbb{P}_n$, and was introduced by Bernstein as (see [1,2]):

$$\begin{aligned}\mathbb{B}_n(f|x) &= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x),\end{aligned} \tag{1}$$

(see [1–14]) where

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k},\ (n,k \in \mathbb{Z}_{\geq 0}) \tag{2}$$

are called either Bernstein polynomials of degree n or Bernstein basis polynomials of degree n.

The Bernstein polynomials of degree n can be defined in terms of two such polynomials of degree $n-1$. That is, the k-th Bernstein polynomial of degree n can be written as

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x),\ (k,n \in \mathbb{N}). \tag{3}$$

From (2), the first few Bernstein polynomials $B_{k,n}(x)$ are given by

$$\begin{aligned}&B_{0,1}(x) = 1-x,\ B_{1,1}(x) = x,\ B_{0,2}(x) = (1-x)^2,\ B_{1,2}(x) = 2x(1-x), \\ &B_{2,2}(x) = x^2,\ B_{0,3}(x) = (1-x)^3,\ B_{1,3}(x) = 3x(1-x)^2, \\ &B_{2,3}(x) = 3x^2(1-x),\ B_{3,3}(x) = x^3, \cdots.\end{aligned}$$

Thus, we note that

$$\begin{aligned} x^k = x(x^{k-1}) &= x \sum_{i=k-1}^{n} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} B_{i,n}(x) \\ &= \sum_{i=k-1}^{n} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} \frac{i+1}{n+1} B_{i+1,n+1}(x) \\ &= \sum_{i=k-1}^{n} \frac{\binom{i+1}{k}}{\binom{n+1}{k}} B_{i+1,n+1}(x). \end{aligned}$$

For $\lambda \in \mathbb{R}$, L. Carlitz introduced the degenerate Euler poynomials given by the generating function (see [15,16])

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{4}$$

When $x = 0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers. It is easy to show that $\lim_{\lambda \to 0} \mathcal{E}_{n,\lambda}(x) = E_n(x)$, where $E_n(x)$ are the Euler polynomials given by (see [15–18])

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \tag{5}$$

For $n \geq 0$, we define the λ-product as follows (see [8]):

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1), \tag{6}$$

Observe here that $\lim_{\lambda \to 0}(x)_{n,\lambda} = x^n,\ (n \geq 1)$.

Recently, the degenerate Bernstein polynomials of degree n are introduced as (see [8])

$$B_{k,n}(x|\lambda) = \binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda},\ (x \in [0,1],\ n,k \geq 0), \tag{7}$$

From (7), it is not difficult to show that the generating function for $B_{k,n}(x|\lambda)$ is given by (see [8])

$$\frac{1}{k!}(x)_{k,\lambda}t^k(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=k}^{\infty} B_{k,n}(x|\lambda)\frac{t^n}{n!}, \tag{8}$$

By (8), we easily get $\lim_{\lambda \to 0} B_{k,n}(x|\lambda) = B_{k,n}(x)$, $(n,k \geq 0)$.

The Bernstein polynomials are the mathematical basis for Bézier curves which are frequently used in computer graphics and related fields. In this paper, we investigate the degenerate Bernstein polynomials and operators. We study their elementary properties (see also [8]) and then their further properties in association with the degenerate Euler numbers and polynomials.

Finally, we would like to briefly go over some of the recent works related with Bernstein polynomials and operators.

Kim-Kim in Ref. [19] gave identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind. The authors in Ref. [20] introduced a generalization of the Bernstein polynomials associated with Frobenius–Euler polynomials. The paper [21] deals with some identities of q-Euler numbers and polynomials associated with q-Bernstein polynomials. In Ref. [22], the authors studied a space-time fractional diffusion equation with initial boundary conditions and presented a numerical solution for that. Both normalized Bernstein polynomials with collocation and Galerkin methods are applied to turn the problem into an algebraic system. Kim in Ref. [23] introduced some identities on the q-integral representation of the product of the several q-Bernstein type polynomials. Grouped data are commonly encountered in applications. In Ref. [24], Kim-Kim studied some properties on degenerate Eulerian numbers and polynomials. The authors in Ref. [25] give an overview of several results related to partially degenerate poly-Bernoulli polynomials associated with Hermit polynomials.

2. Degenerate Bernstein Polynomials and Operators

The degenerate Bernstein operator of order n is defined, for $f \in C[0,1]$, as

$$\mathbb{B}_{n,\lambda}(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x|\lambda), \tag{9}$$

where $x \in [0,1]$ and $n, k \in \mathbb{Z}_{\geq 0}$.

Theorem 1. *For $n \geq 0$, we have*

$$\mathbb{B}_{n,\lambda}(f|0) = f(0)(1)_{n,\lambda}, \ \mathbb{B}_{n,\lambda}(f|1) = f(1)(1)_{n,\lambda},$$

and

$$\mathbb{B}_{n,\lambda}(1|x) = (1)_{n,\lambda}, \ \mathbb{B}_{n,\lambda}(x|x) = x\sum_{k=0}^{n-1}(-1)^k\lambda^k(n-1)_k(1)_{n-1-k,\lambda}, \ (n \geq 1),$$

where $(x)_k = x(x-1)\cdots(x-k+1)$, $(k \geq 1)$, $(x)_0 = 1$.

Proof. From (9), we clearly have

$$\mathbb{B}_{n,\lambda}(1|x) = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}. \tag{10}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!} &= (1+\lambda t)^{\frac{1}{\lambda}} = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{1-x}{\lambda}} \\ &= \left(\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1-x)_{m,\lambda}\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x)_{l,\lambda}(1-x)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{11}$$

Comparing the coefficients on both sides of (11), we derive

$$(1)_{n,\lambda} = \sum_{l=0}^{n}\binom{n}{l}(x)_{l,\lambda}(1-x)_{n-l,\lambda}. \tag{12}$$

Combining (10) with (12), we have

$$\mathbb{B}_{n,\lambda}(1|x) = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = (1)_{n,\lambda}, \ (n \geq 0). \tag{13}$$

Furthermore, we get from (9) that for $f(x) = x$,

$$\begin{aligned}\mathbb{B}_{n,\lambda}(x|x) &= \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} \\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}(x)_{k,\lambda}(1-x)_{n-k,\lambda} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-1-k,\lambda}(x-k\lambda) \\ &= x(1)_{n-1,\lambda} - (n-1)\lambda\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-1-k,\lambda}\frac{k}{n-1} \\ &= x(1)_{n-1,\lambda} - (n-1)\lambda B_{n-1,\lambda}(x|x).\end{aligned} \tag{14}$$

From (14), we can easily deduce the following Equation (15):

$$\begin{aligned}\mathbb{B}_{n,\lambda}(x|x) &= x(1)_{n-1,\lambda} - (n-1)\lambda\{x(1)_{n-2,\lambda} - (n-2)\lambda B_{n-2,\lambda}(x|x)\} \\ &= x(1)_{n-1,\lambda} - x(n-1)\lambda(1)_{n-2,\lambda} + (-1)^2(n-1)(n-2)\lambda^2 B_{n-2,\lambda}(x|x) \\ &= x(1)_{n-1,\lambda} - x(n-1)\lambda(1)_{n-2,\lambda} \\ &\quad + (-1)^2(n-1)(n-2)\lambda^2\{x(1)_{n-3,\lambda} - (n-3)\lambda B_{n-3,\lambda}(x|x)\} \\ &= x(1)_{n-1,\lambda} - x(n-1)\lambda(1)_{n-2,\lambda} + (-1)^2(n-1)(n-2)\lambda^2 x(1)_{n-3,\lambda} \\ &\quad + (-1)^3(n-1)(n-2)(n-3)\lambda^3 B_{n-3,\lambda}(x|x) \\ &= \cdots \\ &= x\sum_{k=0}^{n-1}(-1)^k\lambda^k(n-1)_k(1)_{n-1-k,\lambda}.\end{aligned} \tag{15}$$

□

Let f, g be continuous functions defined on $[0,1]$. Then, we clearly have

$$\mathbb{B}_{n,\lambda}(\alpha f + \beta g|x) = \alpha\mathbb{B}_{n,\lambda}(f|x) + \beta\mathbb{B}_{n,\lambda}(g|x), \ (n \geq 0), \tag{16}$$

where α, β are constants.

So, the degenerate Bernstein operator is linear. From (7), we note that

$$\begin{aligned}&\mathbb{B}_{0,1}(x|\lambda) = 1-x, \ \mathbb{B}_{1,1}(x|\lambda) = x, \ \mathbb{B}_{0,2}(x|\lambda) = (1-x)^2 - \lambda(1-x), \\ &\mathbb{B}_{1,2}(x|\lambda) = 2x(1-x), \ \mathbb{B}_{2,2}(x|\lambda) = x^2 - \lambda x.\end{aligned}$$

It is not hard to see that

$$\begin{aligned}\sum_{n=0}^{\infty}(1-x)_{n,\lambda}\frac{t^n}{n!} = (1+\lambda t)^{\frac{1-x}{\lambda}} &= (1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t)^{-\frac{x}{\lambda}} \\ &= \left(\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-x)_{m,\lambda}\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}(-1)^l(x)_{l,-\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

This shows that we have

$$(1-x)_{n,\lambda} = \sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}(-1)^l(x)_{l,-\lambda}, \ (n \geq 0). \tag{17}$$

Theorem 2. *For $f \in C[0,1]$ and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathbb{B}_{n,\lambda}(f|x) = \sum_{m=0}^{n} \binom{n}{m} (x)_{m,-\lambda} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (1)_{n-m,\lambda} \frac{(x)_{k,\lambda}}{(x+(m-1)\lambda)_{k,\lambda}} f(\frac{k}{n}).$$

Proof. From (9), it is immediate to see that

$$\begin{aligned}\mathbb{B}_{n,\lambda}(f|x) &= \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x|\lambda) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} (x)_{k,\lambda} (1-x)_{n-k,\lambda} \\ &= \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} (x)_{k,\lambda} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j (1)_{n-k-j,\lambda} (x)_{j,-\lambda}.\end{aligned} \tag{18}$$

We need to note the following:

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{k+j}\binom{k+j}{k}, \ (n,k \geq 0), \tag{19}$$

and

$$(x)_{j,-\lambda} = \frac{(x)_{k+j,-\lambda}}{(x+(j+k-1)\lambda)_{k,\lambda}}. \tag{20}$$

Let $k+j=m$. Then, by (19), we obviously have

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{m}\binom{m}{k}. \tag{21}$$

Combining (18) with (19)–(21) gives the following result:

$$\mathbb{B}_{n,\lambda}(f|x) = \sum_{m=0}^{n} \binom{n}{m} (x)_{m,-\lambda} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (1)_{n-m,\lambda} \frac{(x)_{k,\lambda}}{(x+(m-1)\lambda)_{k,\lambda}} f(\frac{k}{n}).$$

□

Theorem 3. *For $n,k \in \mathbb{Z}_{\geq 0}$ and $x \in [0,1]$, we have*

$$B_{k,n}(x|\lambda) = \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i}\binom{i}{k} (x)_{i,-\lambda} \frac{(x)_{k,\lambda}}{(x+(i-1)\lambda)_{k,\lambda}} (1)_{n-i,\lambda}.$$

Proof. From (7), (17), and (20), we observe that

$$\begin{aligned}B_{k,n}(x|\lambda) &= \binom{n}{k} (x)_{k,\lambda} (1-x)_{n-k,\lambda} \\ &= \binom{n}{k} (x)_{k,\lambda} \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i (x)_{i,-\lambda} (1)_{n-k-i,\lambda} \\ &= \sum_{i=0}^{n-k} (-1)^i \binom{n}{k}\binom{n-k}{i} (x)_{k+i,-\lambda} \frac{(x)_{k,\lambda}}{(x+(k+i-1)\lambda)_{k,\lambda}} (1)_{n-k-i,\lambda} \\ &= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{k}\binom{n-k}{i-k} (x)_{i,-\lambda} \frac{(x)_{k,\lambda}}{(x+(i-1)\lambda)_{k,\lambda}} (1)_{n-i,\lambda} \\ &= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i}\binom{i}{k} (x)_{i,-\lambda} \frac{(x)_{k,\lambda}}{(x+(i-1)\lambda)_{k,\lambda}} (1)_{n-i,\lambda}.\end{aligned} \tag{22}$$

□

3. Degenerate Euler Polynomials Associated with Degenerate Bernstein Polynomials

Theorem 4. *For $n \geq 0$, the following holds true:*

$$\sum_{l=0}^{n} \binom{n}{l} (1)_{n-l,\lambda} \mathcal{E}_{l,\lambda} + \mathcal{E}_{n,\lambda} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

Proof. From (4), we remark that

$$\begin{aligned} 2 &= \left(\sum_{l=0}^{\infty} \mathcal{E}_{l,\lambda} \frac{t^l}{l!} \right) \left((1+\lambda t)^{\frac{1}{\lambda}} + 1 \right) = \left(\sum_{l=0}^{\infty} \mathcal{E}_{l,\lambda} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (1)_{m,\lambda} \frac{t^m}{m!} + 1 \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,\lambda} (1)_{n-l,\lambda} \right) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} (1)_{n-l,\lambda} \mathcal{E}_{l,\lambda} + \mathcal{E}_{n,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{23}$$

The result follows by comparing the coefficients on both sides of (23).
□

From Theorem 4, we note that

$$\mathcal{E}_{0,\lambda} = 1, \ \mathcal{E}_{n,\lambda} = -\sum_{l=0}^{n} \binom{n}{l} (1)_{n-l,\lambda} \mathcal{E}_{l,\lambda}, \ (n > 0).$$

From these recurrence relations, we note that the first few degenerate Euler numbers are given by

$$\begin{aligned} \mathcal{E}_{0,\lambda} = 1, \ \mathcal{E}_{1,\lambda} = -\frac{1}{2}, \ \mathcal{E}_{2,\lambda} = \frac{1}{2}\lambda, \ \mathcal{E}_{3,\lambda} = \frac{1}{4} - \lambda^2, \ \mathcal{E}_{4,\lambda} = -\frac{3}{2}\lambda + 3\lambda^3, \\ \mathcal{E}_{5,\lambda} = -\frac{1}{2} + \frac{35}{4}\lambda^2 - 12\lambda^4, \ \mathcal{E}_{6,\lambda} = \frac{15}{2}\lambda - \frac{225}{4}\lambda^3 + 60\lambda^5. \end{aligned}$$

Theorem 5. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}(1-x) = (-1)^n \mathcal{E}_{n,-\lambda}(x).$$

Especially, we have

$$\mathcal{E}_{n,\lambda}(2) = (-1)^n \mathcal{E}_{n,-\lambda}(-1), \ (n \geq 0).$$

Proof. By (4), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(1-x) \frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{1-x}{\lambda}} = \frac{2}{(1+\lambda t)^{-\frac{1}{\lambda}} + 1} (1+\lambda t)^{-\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \mathcal{E}_{n,-\lambda}(x) (-1)^n \frac{t^n}{n!}. \end{aligned} \tag{24}$$

Comparing the coefficients on both sides of (24), we have

$$\mathcal{E}_{n,\lambda}(1-x) = (-1)^n \mathcal{E}_{n,-\lambda}(x), \ (n \geq 0). \tag{25}$$

In particular, if we take $x = -1$, we get

$$\mathcal{E}_{n,\lambda}(2) = (-1)^n \mathcal{E}_{n,-\lambda}(-1), \ (n \geq 0). \tag{26}$$

□

Corollary 1. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}(2) = 2(1)_{n,\lambda} + \mathcal{E}_{n,\lambda}.$$

Proof. From (4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(2)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{2}{\lambda}} \\ &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1-1) \\ &= 2(1+\lambda t)^{\frac{1}{\lambda}} - \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(2(1)_{n,\lambda} - \sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{27}$$

Thus, by (27), we get

$$\mathcal{E}_{n,\lambda}(2) = 2(1)_{n,\lambda} - \sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda} = 2(1)_{n,\lambda} + \mathcal{E}_{n,\lambda}, \ (n \geq 0).$$

□

Theorem 6. *For $n \geq 0, k \geq 1$, we have*

$$\mathcal{E}_{n,\lambda}(x) = 2\sum_{i=1}^{k}(-1)^{i-1}(x-i)_{n,\lambda} + (-1)^k \mathcal{E}_{n,\lambda}(x-k).$$

Proof. From (4), we easily see that

$$\mathcal{E}_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,\lambda}(x)_{n-l,\lambda}, \ (n \geq 0).$$

Now, we observe that

$$\begin{aligned}
&\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}((1+\lambda t)^{\frac{1}{\lambda}}+1-1)(1+\lambda t)^{\frac{x-1}{\lambda}} \\
&= 2(1+\lambda t)^{\frac{x-1}{\lambda}} - \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-1}{\lambda}} \\
&= 2(1+\lambda t)^{\frac{x-1}{\lambda}} - 2(1+\lambda t)^{\frac{x-2}{\lambda}} + \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-2}{\lambda}} \\
&= 2(1+\lambda t)^{\frac{x-1}{\lambda}} - 2(1+\lambda t)^{\frac{x-2}{\lambda}} + 2(1+\lambda t)^{\frac{x-3}{\lambda}} - \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-3}{\lambda}} \\
&= 2(1+\lambda t)^{\frac{x-1}{\lambda}} - 2(1+\lambda t)^{\frac{x-2}{\lambda}} + 2(1+\lambda t)^{\frac{x-3}{\lambda}} - 2(1+\lambda t)^{\frac{x-4}{\lambda}} \\
&\quad + \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-4}{\lambda}}.
\end{aligned} \tag{28}$$

Continuing the process in (28) gives the following result:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!} = 2\sum_{i=1}^{k}(-1)^{i-1}(1+\lambda t)^{\frac{x-i}{\lambda}} + (-1)^k\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-k}{\lambda}} \\
&= \sum_{n=0}^{\infty}\left(2\sum_{i=1}^{k}(-1)^{i-1}(x-i)_{n,\lambda}\right)\frac{t^n}{n!} + (-1)^k\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-k}{\lambda}}.
\end{aligned} \tag{29}$$

The desired result now follows from (4) and (29).
□

Theorem 7. *For $n, k \geq 0$, we have*

$$B_{k,n+k}(x|\lambda) = \frac{1}{2}(x)_{k,\lambda}\binom{n+k}{k}(\mathcal{E}_{n,\lambda}(2-x)+\mathcal{E}_{n,\lambda}(1-x)).$$

Proof. In view of (8), we have

$$(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} = \frac{k!}{t^k}\sum_{n=k}^{\infty}B_{k,n}(x|\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}B_{k,n+k}(x|\lambda)\frac{1}{\binom{n+k}{n}}\frac{t^n}{n!}. \tag{30}$$

On the other hand, (30) is also given by

$$(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{k,\lambda}(1-x)_{n,\lambda}\frac{t^n}{n!}. \tag{31}$$

From (30) and (31), we have

$$(x)_{k,\lambda}(1-x)_{n,\lambda} = \frac{1}{\binom{n+k}{n}}B_{k,n+k}(x|\lambda), \;\; (n,k \geq 0). \tag{32}$$

Now, we observe that

$$
\begin{aligned}
(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} &= \frac{(x)_{k,\lambda}}{2}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1-x}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1) \\
&= \frac{(x)_{k,\lambda}}{2}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{2-x}{\lambda}}+\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1-x}{\lambda}}\right) \\
&= \frac{(x)_{k,\lambda}}{2}\left(\sum_{n=0}^{\infty}(\mathcal{E}_{n,\lambda}(2-x)+\mathcal{E}_{n,\lambda}(1-x))\frac{t^n}{n!}\right).
\end{aligned}
\tag{33}
$$

By (31) and (33), we get

$$
(x)_{k,\lambda}(1-x)_{n,\lambda} = \frac{(x)_{k,\lambda}}{2}(\mathcal{E}_{n,\lambda}(2-x)+\mathcal{E}_{n,\lambda}(1-x)),\ (n \geq 0). \tag{34}
$$

Therefore, from (32) and (34), we have the result. □

4. Conclusions

In 1912, Bernstein first used Bernstein polynomials to give a constructive proof for the Stone–Weierstrass approximation theorem. The convergence of the Bernstein approximation of a function f to f is of order $1/n$, even for smooth functions, and hence the related approximation process is not used for computational purposes. However, by combining Bernstein approximations and the use of ad hoc extrapolation algorithms, fast techniques were designed (see the recent review [14] and paper [13]). Furthermore, about half a century later, they were used to design automobile bodies at Renault by Pierre Bézier. The Bernstein polynomials are the mathematical basis for Bézier curves, which are frequently used in computer graphics and related fields such as animation, modeling, CAD, and CAGD.

The study of degenerate versions of special numbers and polynomials began with the papers by Carlitz in Refs. [15,16]. Kim and his colleagues have been studying various degenerate numbers and polynomials by means of generating functions, combinatorial methods, umbral calculus, p-adic analysis, and differential equations. This line of study led even to the introduction to degenerate gamma functions and degenerate Laplace transforms (see [26]). These already demonstrate that studying degenerate versions of known special numbers and polynomials can be very promising and rewarding. Furthermore, we can hope that many applications will be found not only in mathematics but also in sciences and engineering. As we mentioned in the above, it was not until about fifty years later that Bernstein polynomials found their applications in real-world problems.

With this hope in mind, here we investigated the degenerate Bernstein polynomials and operators which were recently introduced as degenerate versions of the classical Bernstein polynomials and operators. We derived some of their basic properties. In addition, we studied some further properties of the degenerate Bernstein polynomials related to the degenerate Euler numbers and polynomials.

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