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System of Variational Inclusions and Fixed Points of Pseudocontractive Mappings in Banach Spaces

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Abstract: The purpose of this paper is to solve the general system of variational inclusions (GSVI) with hierarchical variational inequality (HVI) constraint, for an infinite family of continuous pseudocontractive mappings in Banach spaces. By utilizing the equivalence between the GSVI and the fixed point problem, we construct an implicit multiple-viscosity approximation method for solving the GSVI. Under very mild conditions, we prove the strong convergence of the proposed method to a solution of the GSVI with the HVI constraint, for infinitely many pseudocontractions.

Keywords: implicit multiple viscosity approximation method; system of variational inclusions; pseudocontractive mapping; nonexpansive mapping

MSC: 47H05; 47H10; 47J25

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $S : C \rightarrow C$ be nonexpansive, with its fixed point set $F(S) \neq \emptyset$. Let $A, B : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone, respectively. In 2008, Ceng et al. [1] introduced and considered the following general system of variational inequality problems of finding $(x^*, y^*) \in C \times C$, such that

$$\begin{cases} \langle \rho Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C, \\ \langle \eta Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in C. \end{cases} \quad (1)$$

They first transformed problem (1) into a fixed point problem for the mapping $G = P_C(I - \rho A)P_C(I - \eta B)$, and then proved strong convergence of the following relaxed extragradient method for solving the problem (1), and the fixed point problem of S :

$$\begin{cases} y_n = P_C(x_n - \eta Bx_n), n \geq 0, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \rho A y_n), \end{cases} \tag{2}$$

where $\rho \in (0, 2\alpha)$, $\eta \in (0, 2\beta)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$.

Let E be a real Banach space with the dual E^* and C a nonempty closed convex subset of E . A self-mapping $f : C \rightarrow C$ is said to be k -Lipschitz on C if $k \in \mathbf{R}_+ = [0, +\infty)$ and $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in C$. If f is k -Lipschitz with $k < 1$, then f is called a k -contraction mapping (or a contraction mapping with coefficient k). A self-mapping $f : C \rightarrow C$ is said to be nonexpansive if it is Lipschitz with $k = 1$. Also, recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is called pseudocontractive if the following inequality holds

$$\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|, \forall x, y \in D(T), r > 0,$$

which is equivalent to the inequality (see [2]) that for each $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2. \tag{3}$$

It is known that the class of pseudocontractive mappings is an important and significant generator of nonexpansive mappings [3]. Moreover, interest in pseudocontractive mappings stems mainly from their firm connection with the class of accretive mappings.

Let $A_1, A_2 : C \rightarrow E$ and $M_1, M_2 : C \rightarrow 2^E$ be nonlinear mappings. The general system of variational inclusions (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} 0 \in x^* - y^* + \rho_1(A_1 y^* + M_1 x^*), \\ 0 \in y^* - x^* + \rho_2(A_2 x^* + M_2 y^*), \end{cases} \tag{4}$$

where ρ_1 and ρ_2 are two positive constants.

In 2010, Qin et al. [4] introduced a relaxed extragradient-type method for solving GSVI (4), and proved a strong convergence theorem for the proposed method (for its related results in the literature, see, e.g., [1,5–18]). Furthermore, Aoyama et al. [19] considered the following variational inequality: Find $x^* \in C$, such that

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{5}$$

They proved that the problem (5) is equivalent to a fixed point problem; that is, the element $x^* \in C$ is a solution of problem (5) if and only if $x^* \in C$ satisfies the following equation:

$$x^* = \Pi_C(x^* - \eta Ax^*), \tag{6}$$

where $\eta > 0$ is a constant and Π_C is a sunny nonexpansive retraction from E onto C . In particular, if $E = H$ a Hilbert space, then Π_C coincides with the metric projection P_C from H onto C . Recently, many authors have studied the problem of finding a common element of the set of fixed points of nonlinear mappings and the set of solutions to variational inequalities by iterative methods (see, e.g., [1–3,5,6,8–10,12,14–16,18–24]).

In particular, Ceng et al. [22] introduced an implicit viscosity approximation method for computing approximate fixed points of a pseudocontractive mapping T , and derived strong convergence of the proposed implicit method to a point in $F(T)$, which solves a certain variational inequality.

The purpose of this paper is to solve the GSVI (4) with the hierarchical variational inequality (HVI) constraint, for an infinite family of continuous pseudocontractive mappings $\{T_n\}_{n=1}^\infty$ in a uniformly convex and two-uniformly smooth Banach space E . By utilizing the equivalence between the GSVI (4) and the fixed point problem, we construct an implicit multiple-viscosity approximation method for solving the GSVI (4) with the HVI constraint, for infinitely many pseudocontractions $\{T_n\}_{n=1}^\infty$. Under very mild conditions, we prove the strong convergence of the proposed method to a solution of the GSVI (4) with the HVI constraint, for infinitely many pseudocontractions $\{T_n\}_{n=1}^\infty$. Our results improve and extend the corresponding results announced by some others; for example, Yao et al. [13] and Ceng et al. [22].

2. Preliminaries

Let E be a real Banach space with dual E^* . Throughout this paper, we write $x_n \rightharpoonup x$ (respectively, $x_n \rightarrow x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x . Let C be a nonempty closed convex subset of E . Recall that a mapping $T : C \rightarrow E$ is said to be

- (a) accretive if, for each $x, y \in C$, $\exists j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y) \rangle \geq 0$, where J is the normalized duality mapping;
- (b) α -strongly accretive if, for each $x, y \in C$, $\exists j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y) \rangle \geq \alpha \|x - y\|^2$ for some $\alpha \in (0, 1)$;
- (c) β -inverse-strongly accretive if, for each $x, y \in C$, $\exists j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y) \rangle \geq \beta \|Tx - Ty\|^2$ for some $\beta > 0$.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . Then E is said to be strictly convex if for any $x, y \in U, x \neq y \Rightarrow \|\frac{x+y}{2}\| < 1$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U, \|x - y\| \geq \epsilon \Rightarrow \|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space E is reflexive, then E is strictly convex if and only if E^* is smooth, as well as that E is smooth if and only if E^* is strictly convex. A Banach space E is said to have a Gateaux differentiable norm if the limit $\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$ exists for each $x, y \in U$ and, in this case, we call E smooth. E is said to have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U$ and, in this case, we call E uniformly smooth. E is also said to have a Fréchet differentiable norm if for each $x \in U$, the limit is attained uniformly for $y \in U$ and, in this case, we call E strongly smooth. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau \right\},$$

where $\rho : [0, \infty) \rightarrow [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau \rightarrow 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q -uniformly smooth if there exists a constant $\kappa > 0$, such that $\rho(\tau) \leq \kappa \tau^q$ for all $\tau > 0$.

Let q be a real number with $1 < q \leq 2$ and let E be a Banach space. Then E is q -uniformly smooth if and only if there exists a constant $c > 0$ such that

$$\|x + y\|^q + \|x - y\|^q \leq 2(\|x\|^q + \|cy\|^q), \quad \forall x, y \in E.$$

The best constant c in the above inequality is called the q -uniformly smooth constant of E ; see [11] for more details. Note that no Banach space is q -uniformly smooth for $q > 2$; see [25].

For $q > 1$, the generalized duality mapping $J_q : E \rightarrow 2^{E^*}$ is defined by

$$J_q(x) = \{\varphi \in E^* : \langle x, \varphi \rangle = \|x\|^q, \|\varphi\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$, $\forall x \in E$. If E is a Hilbert space, then $J = I$ the identity mapping. Recall that

- (1) if E is smooth, then J is single-valued and norm-to-weak* continuous on E ;
- (2) if E is uniformly smooth, then J is single-valued and norm-to-norm uniformly continuous on bounded subsets of E ;
- (3) all Hilbert spaces, L^p (or l^p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$), are two-uniformly smooth, while L^p (or l^p) and W_m^p spaces ($1 < p \leq 2$) are p -uniformly smooth;
- (4) typical examples of both uniformly convex and uniformly smooth Banach space are L^p , where $p > 1$. More precisely, L^p is $\min\{p, 2\}$ -uniformly smooth for any $p > 1$.

Proposition 1 ([26]). *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow \mathbf{R}$, $g(0) = 0$ such that $g(\|x - y\|) \leq \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2$, $\forall x, y \in B_r$, where $B_r = \{x \in E : \|x\| \leq r\}$.*

Proposition 2 ([27]). *If E is a two-uniformly smooth Banach space, then $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + 2\|cy\|^2$, $\forall x, y \in E$, where c is the two-uniformly smooth constant of E . In particular, if E is a Hilbert space, then the duality pairing $\langle \cdot, \cdot \rangle$ reduces to the inner product, $j = I$ the identity mapping, and $c = 1/\sqrt{2}$.*

Let D be a subset of C and let Π be a mapping of C into D . Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π . A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D .

Proposition 3 ([28]). *Let C be a nonempty closed convex subset of a smooth Banach space E , D be a nonempty subset of C and Π be a retraction of C onto D . Then the following are equivalent:*

- (i) Π is sunny and nonexpansive;
- (ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, j(\Pi(x) - \Pi(y)) \rangle, \forall x, y \in C$;
- (iii) $\langle x - \Pi(x), j(y - \Pi(x)) \rangle \leq 0, \forall x \in C, y \in D$.

Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with the fixed point set $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C ; see [29].

Proposition 4 ([30]). *Let C be a nonempty closed convex subset of a Banach space E and $T : C \rightarrow C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C .*

Recall that a possibly multivalued operator $M \subset E \times E$ with domain $D(M)$ and range $R(M)$ in E is accretive if, for each $x_i \in D(M)$ and $y_i \in Mx_i$ ($i = 1, 2$), there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2) \rangle \geq 0$. An accretive operator M is said to satisfy the range condition if $\overline{D(M)} \subset R(I + rM)$ for all $r > 0$. An accretive operator M is m -accretive if $R(I + rM) = E$ for each $r > 0$. If M is

an accretive operator which satisfies the range condition, then we can define, for each $r > 0$, a mapping $J_r^M : R(I + rM) \rightarrow D(M)$ by $J_r^M = (I + rM)^{-1}$, which is called the resolvent of M . It is well known that J_r^M is nonexpansive and $F(J_r^M) = M^{-1}0, \forall r > 0$; see [31]. Hence, $F(J_r^M) = M^{-1}0 = \{x \in D(M) : 0 \in Mx\}$. If $M^{-1}0 \neq \emptyset$, then the inclusion $0 \in Mx$ is solvable. We below present some lemmas which will be used in the sequel. Some of them are known, and others are not hard to prove.

Lemma 1 ([24]). Let C be a nonempty closed convex subset of a smooth Banach space E and $M : C \rightarrow 2^E$ be an m -accretive mapping. Then for any given $r > 0$, the inequality holds: $\|J_r^M x - J_r^M y\|^2 \leq \langle x - y, j(J_r^M x - J_r^M y) \rangle, \forall x, y \in E$. This means that $J_r^M : E \rightarrow C$ is nonexpansive.

Lemma 2 ([24]). Let $M : C \rightarrow 2^E$ be an m -accretive mapping and $A : C \rightarrow E$ is a mapping. Then $x^* \in C$ is a solution of the variational inclusion $0 \in Ax + Mx$ if and only if $x^* = J_\rho^M(x^* - \rho Ax^*)$, for all $\rho > 0$, that is, $VI(C, A, M) = F(J_\rho^M(I - \rho A)), \forall \rho > 0$, where $VI(C, A, M)$ denotes the set of solutions to this variational inclusion.

Lemma 3 ([24]). Let $M_1, M_2 : C \rightarrow 2^E$ be two m -accretive mappings and $A_1, A_2 : C \rightarrow E$ be two mappings. For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of the GSVI (1.4) if and only if x^* is a fixed point of the mapping $Q := J_{\rho_1}^{M_1}(I - \rho_1 A_1)J_{\rho_2}^{M_2}(I - \rho_2 A_2)$, where $y^* = J_{\rho_2}^{M_2}(I - \rho_2 A_2)x^*$.

Lemma 4 ([32]). Let C be a nonempty closed convex subset of a strictly convex Banach space E . Let $T_1, T_2 : C \rightarrow E$ be nonexpansive mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping $S : C \rightarrow E$ by $Sx = \nu T_1 x + (1 - \nu)T_2 x, \forall x \in C$, where ν is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 5 ([24]). Let C be a nonempty closed convex subset of a two-uniformly smooth Banach space E . Let the mapping $A : C \rightarrow E$ be α -inverse-strongly accretive. Then,

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(c^2\lambda - \alpha)\|Ax - Ay\|^2.$$

In particular, if $0 \leq \lambda \leq \frac{\alpha}{c^2}$, then $I - \lambda A$ is nonexpansive.

Lemma 6 ([24]). Let C be a nonempty closed convex subset of a two-uniformly smooth Banach space E . Let $M_1, M_2 : C \rightarrow 2^E$ be two m -accretive mappings and $A_i : C \rightarrow E$ be ζ_i -inverse-strongly accretive for $i = 1, 2$. Let the mapping $Q : C \rightarrow C$ be defined as $Q := J_{\rho_1}^{M_1}(I - \rho_1 A_1)J_{\rho_2}^{M_2}(I - \rho_2 A_2)$. If $0 \leq \rho_i \leq \frac{\zeta_i}{c^2}$ for $i = 1, 2$, then $Q : C \rightarrow C$ is nonexpansive.

Lemma 7 ([33]). Let J be the normalized duality mapping on a real Banach space E . Then for all $x, y \in E$, the inequality holds: $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \forall j(x + y) \in J(x + y)$.

Lemma 8 ([33]). Let C be a nonempty closed convex subset of a uniformly smooth Banach space $E, A : C \rightarrow C$ be a nonexpansive mapping with $F(A) \neq \emptyset$, and $f : C \rightarrow C$ be a fixed contraction mapping. For each $t \in (0, 1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1 - t)Az$ on C , that is, $z_t = tf(z_t) + (1 - t)Az_t$. Then $\{z_t\}$ converges strongly to a point $x^* \in F(A)$, which solves the variational inequality: $\langle (I - f)x^*, j(x^* - x) \rangle \leq 0, \forall x \in F(A)$.

Lemma 9 ([33]). Let $\{a_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \sigma_n, \forall n \geq 0$, where $\{\lambda_n\}_{n=0}^\infty$ and $\{\sigma_n\}_{n=0}^\infty$ are real sequences such that (i) $\{\lambda_n\}_{n=0}^\infty \subset (0, 1), \sum_{n=0}^\infty \lambda_n = \infty$, and (ii) either $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ or $\sum_{n=0}^\infty |\lambda_n \sigma_n| < \infty$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 10 ([34]). *Let C be a nonempty closed convex subset of a Banach space E . Let T_0, T_1, \dots be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_n x - T_{n-1} x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C . Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n \rightarrow \infty} T_n y$ for all $y \in C$. Then $\lim_{n \rightarrow \infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.*

3. Main Results

Now, we are in a position to state and prove our main result.

Theorem 1. *Let C be a nonempty closed convex subset of a uniformly convex and two-uniformly smooth Banach space E . Let $M_1, M_2 : C \rightarrow 2^E$ be two m -accretive mappings and $A_i : C \rightarrow E$ be ζ_i -inverse-strongly accretive for $i = 1, 2$. Let the mapping $Q : C \rightarrow C$ be defined as $Q := J_{\rho_1}^{M_1}(I - \rho_1 A_1)J_{\rho_2}^{M_2}(I - \rho_2 A_2)$, where $0 < \rho_i < \frac{\zeta_i}{c^2}$, $i = 1, 2$, for c the 2-uniformly smooth constant of E . Let $f : C \rightarrow C$ be a fixed contraction mapping with coefficient $k \in [0, 1)$, $S : C \rightarrow C$ be a nonexpansive mapping, and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of continuous pseudocontractive mappings of C into itself, such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0, 1)$ satisfying the following conditions:*

- (i) $\alpha_n + \beta_n + \gamma_n \leq 1, \forall n \geq 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$;
- (iii) $\lim_{n \rightarrow \infty} \gamma_n = 1$;
- (iv) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For arbitrary initial value $x_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} y_n = J_{\rho_2}^{M_2}(x_n - \rho_2 A_2 x_n), \\ x_n = (1 - \alpha_n - \beta_n - \gamma_n)x_{n-1} + \alpha_n f(x_{n-1}) + \beta_n Sx_{n-1} \\ \quad + \gamma_n [\mu T_n x_n + (1 - \mu)J_{\rho_1}^{M_1}(y_n - \rho_1 A_1 y_n)], \quad \forall n \geq 1, \end{cases} \tag{7}$$

where $\mu \in (0, 1)$, and $J_{\rho_i}^{M_i}$ is the resolvent of M_i for $i = 1, 2$. Assume that $\sum_{n=1}^{\infty} \sup_{x \in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C , let T be a mapping of C into itself defined by $Tx = \lim_{n \rightarrow \infty} T_n x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* (\in \Omega)$ and y^* , respectively, where

- (a) (x^*, y^*) solves the GSVI (4);
- (b) x^* solves the variational inequality: $\langle (I - f)x^*, j(x^* - x) \rangle \leq 0, \forall x \in \Omega$ (i.e., $x^* = \Pi_{\Omega} f(x^*)$ where Π_{Ω} is a sunny nonexpansive retraction from C onto Ω).

Proof. Note that the mapping $Q : C \rightarrow C$ is defined as $Q := J_{\rho_1}^{M_1}(I - \rho_1 A_1)J_{\rho_2}^{M_2}(I - \rho_2 A_2)$, where $0 < \rho_i < \frac{\zeta_i}{c^2}$, $i = 1, 2$, for c the two-uniformly smooth constant of E . So, by Lemma 6, we know that Q is nonexpansive. It is easy to see that the implicit iterative scheme (7) can be rewritten as

$$x_n = (1 - \alpha_n - \beta_n - \gamma_n)x_{n-1} + \alpha_n f(x_{n-1}) + \beta_n Sx_{n-1} + \gamma_n (\mu T_n x_n + (1 - \mu)Qx_n), \quad \forall n \geq 1. \tag{8}$$

Consider the mapping

$$F_n x = (1 - \alpha_n - \beta_n - \gamma_n)x_{n-1} + \alpha_n f(x_{n-1}) + \beta_n Sx_{n-1} + \gamma_n (\mu T_n x + (1 - \mu)Qx), \quad \forall x \in C.$$

Since $Q : C \rightarrow C$ is a nonexpansive mapping and $T_n : C \rightarrow C$ is a continuous pseudocontraction mapping, we deduce that all $x, y \in C$,

$$\begin{aligned} \langle F_n x - F_n y, j(x - y) \rangle &= \gamma_n [\mu \langle T_n x - T_n y, j(x - y) \rangle + (1 - \mu) \langle Qx - Qy, j(x - y) \rangle] \\ &\leq \gamma_n [\mu \|x - y\|^2 + (1 - \mu) \|Qx - Qy\| \|x - y\|] \leq \gamma_n \|x - y\|^2. \end{aligned}$$

Hence F_n is a continuous and strong pseudocontraction mapping of C into itself (due to $\gamma_n \in (0, 1)$). By Proposition 4, we know that for each $n \geq 1$ there exists a unique element $x_n \in C$, satisfying (8).

Next, we divide the rest of the proof into several steps.

Step 1. We claim that $\{x_n\}, \{y_n\}, \{f(x_n)\}, \{Sx_n\}, \{T_n x_n\}$ and $\{Qx_n\}$ are bounded. Indeed, take an arbitrarily given $p \in \Omega$. Then we have $T_n p = p$ and $Qp = p$. Putting $W_n := \mu T_n + (1 - \mu)Q$, we know that W_n is a continuous pseudocontraction mapping of C into itself. Then it follows that $W_n p = p$ and

$$\begin{aligned} \|x_n - p\|^2 &= \langle x_n - p, j(x_n - p) \rangle \\ &\leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_{n-1} - p\| \|x_n - p\| + \alpha_n \|f(x_{n-1}) - p\| \|x_n - p\| \\ &\quad + \beta_n \|Sx_{n-1} - p\| \|x_n - p\| + \gamma_n \|x_n - p\|^2, \end{aligned}$$

which hence implies that

$$\begin{aligned} \|x_n - p\| &\leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_{n-1} - p\| + \alpha_n (\|f(x_{n-1}) - f(p)\| + \|f(p) - p\|) \\ &\quad + \beta_n (\|Sx_{n-1} - Sp\| + \|Sp - p\|) + \gamma_n \|x_n - p\| \\ &\leq (1 - (1 - k)\alpha_n - \gamma_n) \|x_{n-1} - p\| + \alpha_n \|f(p) - p\| + \beta_n \|Sp - p\| + \gamma_n \|x_n - p\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$, we may assume, without loss of generality, that $\beta_n \leq \alpha_n$ for all $n \geq 1$. This implies that

$$\begin{aligned} \|x_n - p\| &\leq [1 - (1 - k) \frac{\alpha_n}{1 - \gamma_n}] \|x_{n-1} - p\| + \frac{\alpha_n}{1 - \gamma_n} \|f(p) - p\| + \frac{\beta_n}{1 - \gamma_n} \|Sp - p\| \\ &\leq [1 - (1 - k) \frac{\alpha_n}{1 - \gamma_n}] \|x_{n-1} - p\| + \frac{\alpha_n}{1 - \gamma_n} (\|f(p) - p\| + \|Sp - p\|) \\ &\leq \max\{\|x_{n-1} - p\|, \frac{1}{1 - k} (\|f(p) - p\| + \|Sp - p\|)\}. \end{aligned} \tag{9}$$

By induction, we derive $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - k} (\|f(p) - p\| + \|Sp - p\|)\}, \forall n \geq 0$. So, $\{x_n\}$ is bounded. Observe that $\|f(x_n)\| \leq \|f(x_n) - f(p)\| + \|f(p)\| \leq k \|x_n - p\| + \|f(p)\|$ and $\|Sx_n\| \leq \|Sx_n - Sp\| + \|Sp\| \leq \|x_n - p\| + \|Sp\|$. This implies that $\{f(x_n)\}$ and $\{Sx_n\}$ are bounded. Similarly, by the nonexpansivity of Q we know that $\{Qx_n\}$ is bounded. Note that $\lim_{n \rightarrow \infty} \gamma_n = 1$. Hence there exist $n_0 \geq 1$ and $\varepsilon_0 \in (0, 1)$, such that $\gamma_n \geq \varepsilon_0$ for all $n \geq n_0$. Consequently, we have

$$\begin{aligned} \varepsilon_0 \|W_n x_n\| &\leq \gamma_n \|W_n x_n\| = \|x_n - (1 - \alpha_n - \beta_n - \gamma_n)x_{n-1} - \alpha_n f(x_{n-1}) - \beta_n Sx_{n-1}\| \\ &\leq \|x_n\| + \|x_{n-1}\| + \|f(x_{n-1})\| + \|Sx_{n-1}\|. \end{aligned}$$

This means that $\{W_n x_n\}$ is bounded. Since $W_n x_n = \mu T_n x_n + (1 - \mu)Qx_n$, we get

$$\mu \|T_n x_n\| = \|W_n x_n - (1 - \mu)Qx_n\| \leq \|W_n x_n\| + (1 - \mu) \|Qx_n\| \leq \|W_n x_n\| + \|Qx_n\|.$$

Hence $\{T_n x_n\}$ is bounded. In addition, from Lemma 3 and $p \in \Omega \subset F(Q)$, it follows that (p, q) is a solution of GSVI (4) where $q = J_{\rho_2}^{M_2}(I - \rho_2 A_2)p$. So, by Lemmas 1 and 5 we get $\|y_n\| \leq \|J_{\rho_2}^{M_2}(I - \rho_2 A_2)x_n - q\| + \|q\| \leq \|x_n - p\| + \|q\|$. That is, $\{y_n\}$ is bounded.

Step 2. We show that $\|x_n - Qx_n\| \rightarrow 0$ and $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, where $T : C \rightarrow C$ is defined as $Tx = \lim_{n \rightarrow \infty} T_n x, \forall x \in C$. For simplicity, put $q = J_{\rho_2}^{M_2}(p - \rho_2 A_2 p)$ and $z_n = J_{\rho_1}^{M_1}(y_n - \rho_1 A_1 y_n)$. Then $z_n = Qx_n, \forall n \geq 1$. From Lemmas 1 and 5, we have

$$\begin{aligned} \|y_n - q\|^2 &= \|J_{\rho_2}^{M_2}(x_n - \rho_2 A_2 x_n) - J_{\rho_2}^{M_2}(p - \rho_2 A_2 p)\|^2 \\ &\leq \|x_n - p\|^2 - 2\rho_2(\zeta_2 - c^2\rho_2)\|A_2 x_n - A_2 p\|^2. \end{aligned} \tag{10}$$

Similarly, we get

$$\|z_n - p\|^2 \leq \|y_n - q\|^2 - 2\rho_1(\zeta_1 - c^2\rho_1)\|A_1 y_n - A_1 q\|^2. \tag{11}$$

Substituting (10) into (11), we obtain

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\rho_2(\zeta_2 - c^2\rho_2)\|A_2 x_n - A_2 p\|^2 - 2\rho_1(\zeta_1 - c^2\rho_1)\|A_1 y_n - A_1 q\|^2. \tag{12}$$

From (8) and (12), we conclude

$$\begin{aligned} \|x_n - p\|^2 &\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + \alpha_n[\langle f(x_{n-1}) - f(p), j(x_n - p) \rangle + \langle f(p) - p, j(x_n - p) \rangle] \\ &\quad + \beta_n(\|Sx_{n-1} - Sp\| + \|Sp - p\|)\|x_n - p\| \\ &\quad + \gamma_n(\mu\|x_n - p\| + (1 - \mu)\|z_n - p\|)\|x_n - p\| \\ &\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + \alpha_n[k\|x_{n-1} - p\|\|x_n - p\| + \langle f(p) - p, j(x_n - p) \rangle] + \beta_n(\|x_{n-1} - p\| \\ &\quad + \|Sp - p\|)\|x_n - p\| + \gamma_n \frac{(\mu\|x_n - p\| + (1 - \mu)\|z_n - p\|)^2 + \|x_n - p\|^2}{2} \\ &\leq [1 - (1 - k)\alpha_n - \gamma_n] \frac{\|x_{n-1} - p\|^2 + \|x_n - p\|^2}{2} + \alpha_n \langle f(p) - p, j(x_n - p) \rangle \\ &\quad + \beta_n\|Sp - p\|\|x_n - p\| + \gamma_n\|x_n - p\|^2 - \gamma_n(1 - \mu)[\rho_2(\zeta_2 - c^2\rho_2)\|A_2 x_n - A_2 p\|^2 \\ &\quad + \rho_1(\zeta_1 - c^2\rho_1)\|A_1 y_n - A_1 q\|^2], \end{aligned} \tag{13}$$

which together with $\alpha_n + \beta_n + \gamma_n \leq 1$, immediately yields

$$\begin{aligned} &\gamma_n(1 - \mu)[\rho_2(\zeta_2 - c^2\rho_2)\|A_2 x_n - A_2 p\|^2 + \rho_1(\zeta_1 - c^2\rho_1)\|A_1 y_n - A_1 q\|^2] \\ &\leq [1 - (1 - k)\alpha_n - \gamma_n] \frac{\|x_{n-1} - p\|^2 + \|x_n - p\|^2}{2} + \alpha_n\|f(p) - p\|\|x_n - p\| + \beta_n\|Sp - p\|\|x_n - p\|. \end{aligned}$$

From $\alpha_n \rightarrow 0, \frac{\beta_n}{\alpha_n} \rightarrow 0, \gamma_n \rightarrow 1, \rho_i \in (0, \frac{\zeta_i}{c^2})$ and the boundedness of $\{x_n\}$, we deduce that

$$\lim_{n \rightarrow \infty} \|A_2 x_n - A_2 p\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|A_1 y_n - A_1 q\| = 0. \tag{14}$$

Also, utilizing Lemma 1 and Proposition 1, we have

$$\begin{aligned} \|y_n - q\|^2 &\leq \langle x_n - p, j(y_n - q) \rangle + \rho_2 \langle A_2 p - A_2 x_n, j(y_n - q) \rangle \\ &\leq \frac{1}{2} [\|x_n - p\|^2 + \|y_n - p\|^2 - g_1(\|x_n - y_n - (p - q)\|)] + \rho_2 \|A_2 p - A_2 x_n\| \|y_n - q\|, \end{aligned}$$

which implies that

$$\|y_n - q\|^2 \leq \|x_n - p\|^2 - g_1(\|x_n - y_n - (p - q)\|) + 2\rho_2 \|A_2 p - A_2 x_n\| \|y_n - q\|. \tag{15}$$

Similarly, we get

$$\|z_n - p\|^2 \leq \|y_n - q\|^2 - g_2(\|y_n - z_n + (p - q)\|) + 2\rho_1 \|A_1 q - A_1 y_n\| \|z_n - p\|. \tag{16}$$

Substituting (15) into (16), we get

$$\begin{aligned} \|z_n - p\|^2 &\leq \|x_n - p\|^2 - g_1(\|x_n - y_n - (p - q)\|) - g_2(\|y_n - z_n + (p - q)\|) \\ &\quad + 2\rho_2 \|A_2 p - A_2 x_n\| \|y_n - q\| + 2\rho_1 \|A_1 q - A_1 y_n\| \|z_n - p\|. \end{aligned} \tag{17}$$

From (13) and (17), we have

$$\begin{aligned} \|x_n - p\|^2 &\leq [1 - (1 - k)\alpha_n - \gamma_n] \frac{\|x_{n-1} - p\|^2 + \|x_n - p\|^2}{2} + \alpha_n \langle f(p) - p, j(x_n - p) \rangle \\ &\quad + \beta_n \|Sp - p\| \|x_n - p\| + \frac{\gamma_n}{2} \{ \|x_n - p\|^2 + \mu \|x_n - p\|^2 + (1 - \mu) [\|x_n - p\|^2 \\ &\quad - g_1(\|x_n - y_n - (p - q)\|) - g_2(\|y_n - z_n + (p - q)\|) + 2\rho_2 \|A_2 p - A_2 x_n\| \|y_n - q\| \\ &\quad + 2\rho_1 \|A_1 q - A_1 y_n\| \|z_n - p\|] \} \\ &= [1 - (1 - k)\alpha_n - \gamma_n] \frac{\|x_{n-1} - p\|^2 + \|x_n - p\|^2}{2} + \alpha_n \langle f(p) - p, j(x_n - p) \rangle \\ &\quad + \beta_n \|Sp - p\| \|x_n - p\| + \gamma_n \|x_n - p\|^2 - \frac{\gamma_n(1 - \mu)}{2} [g_1(\|x_n - y_n - (p - q)\|) \\ &\quad + g_2(\|y_n - z_n + (p - q)\|)] + \gamma_n(1 - \mu)(\rho_2 \|A_2 p - A_2 x_n\| \|y_n - q\| \\ &\quad + \rho_1 \|A_1 q - A_1 y_n\| \|z_n - p\|), \end{aligned}$$

which together with $\alpha_n + \beta_n + \gamma_n \leq 1$, leads to

$$\begin{aligned} &\frac{\gamma_n(1 - \mu)}{2} [g_1(\|x_n - y_n - (p - q)\|) + g_2(\|y_n - z_n + (p - q)\|)] \\ &\leq [1 - (1 - k)\alpha_n - \gamma_n] \frac{\|x_{n-1} - p\|^2 + \|x_n - p\|^2}{2} + \alpha_n \|f(p) - p\| \|x_n - p\| + \beta_n \|Sp - p\| \|x_n - p\| \\ &\quad + \rho_2 \|A_2 p - A_2 x_n\| \|y_n - q\| + \rho_1 \|A_1 q - A_1 y_n\| \|z_n - p\|. \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\frac{\beta_n}{\alpha_n} \rightarrow 0$ and $\gamma_n \rightarrow 1$ as $n \rightarrow \infty$ (from (14)), and by the boundedness of $\{y_n\}$ and $\{z_n\}$, we deduce that $\lim_{n \rightarrow \infty} g_1(\|x_n - y_n - (p - q)\|) = 0$ and $\lim_{n \rightarrow \infty} g_2(\|y_n - z_n + (p - q)\|) = 0$. Utilizing the properties of g_1 and g_2 , we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - y_n - (p - q)\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|y_n - z_n + (p - q)\| = 0. \tag{18}$$

From (18), we get $\|x_n - z_n\| \leq \|x_n - y_n - (p - q)\| + \|y_n - z_n + (p - q)\| \rightarrow 0$ as $n \rightarrow \infty$. That is,

$$\lim_{n \rightarrow \infty} \|x_n - Qx_n\| = 0. \tag{19}$$

Note that

$$\gamma_n \|x_n - W_n x_n\| \leq (1 - \alpha_n - \beta_n - \gamma_n) \|x_{n-1} - x_n\| + \alpha_n \|f(x_{n-1}) - x_n\| + \beta_n \|Sx_{n-1} - x_n\|.$$

From $\alpha_n \rightarrow 0, \frac{\beta_n}{\alpha_n} \rightarrow 0, \gamma_n \rightarrow 1$, and the boundedness of $\{x_n\}, \{f(x_n)\}$, and $\{Sx_n\}$, we know that

$$\lim_{n \rightarrow \infty} \|x_n - W_n x_n\| = 0. \tag{20}$$

Observe that $\mu \|T_n x_n - x_n\| = \|W_n x_n - x_n - (1 - \mu)(Qx_n - x_n)\| \leq \|W_n x_n - x_n\| + \|Qx_n - x_n\|$. In terms of (19) and (20), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0. \tag{21}$$

It is easy to see that $\overline{\text{conv}}\{x_n\}$ is a nonempty bounded closed convex subset of C , where $\overline{\text{conv}}\{x_n\}$ is the closed convex hull of the set $\{x_n\}$. By assumption, we get $\sum_{n=1}^{\infty} \sup_{x \in \overline{\text{conv}}\{x_n\}} \|T_{n+1}x - T_n x\| < \infty$. By Lemma 10, we have $\lim_{n \rightarrow \infty} \sup_{x \in \overline{\text{conv}}\{x_n\}} \|T_n x - Tx\| = 0$. Therefore, by (21), we conclude that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x_n - Tx_n\| &\leq \limsup_{n \rightarrow \infty} (\|x_n - T_n x_n\| + \|T_n x_n - Tx_n\|) \\ &\leq \limsup_{n \rightarrow \infty} \|x_n - T_n x_n\| + \limsup_{n \rightarrow \infty} \sup_{x \in \overline{\text{conv}}\{x_n\}} \|T_n x - Tx\| = 0. \end{aligned}$$

That is, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Step 3. We claim that

$$\limsup_{n \rightarrow \infty} \langle x^* - f(x^*), j(x^* - x_n) \rangle \leq 0, \quad x^* \in \Omega, \tag{22}$$

where z_t is the fixed point of the mapping $z \mapsto tf(z) + (1 - t)(\mu A + (1 - \mu)Q)z$ with $A := (2I - T)^{-1}$, $x^* = \lim_{t \rightarrow 0^+} z_t$ and x^* solves the VI: $\langle (I - f)x^*, j(x^* - x) \rangle \leq 0, \forall x \in \Omega$.

Indeed, note that the mapping $T : C \rightarrow C$ is defined as $Tx := \lim_{n \rightarrow \infty} T_n x$ for all $x \in C$. By assumption, we have that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Let us show that $T : C \rightarrow C$ is pseudocontractive and continuous. As a matter of fact, observe that for all $x, y \in C, \lim_{n \rightarrow \infty} \|T_n x - Tx\| = 0$ and $\lim_{n \rightarrow \infty} \|T_n y - Ty\| = 0$. Since each T_n is pseudocontractive, we get

$$\langle Tx - Ty, j(x - y) \rangle = \lim_{n \rightarrow \infty} \langle T_n x - T_n y, j(x - y) \rangle \leq \|x - y\|^2.$$

This means that T is pseudocontractive. In order to show the continuity of T on C , we suppose that $u_n \rightarrow u$ as $n \rightarrow \infty$ with $u \in C$ and $\{u_n\} \subset C$. Let $D := \overline{\text{conv}}(\{u_n\} \cup \{u\})$, where $\overline{\text{conv}}(\{u_n\} \cup \{u\})$ is the closed convex hull of the set $\{u_n\} \cup \{u\}$. Then D is a nonempty bounded closed convex subset of C . By assumption, we obtain $\sum_{n=1}^{\infty} \sup_{x \in D} \|T_{n+1}x - T_n x\| < \infty$ for such a subset D of C . So, by Lemma 10 we deduce that $\lim_{m \rightarrow \infty} \sup_{x \in D} \|T_m x - Tx\| = 0$.

We now observe that, for each given $m, n \geq 1$,

$$\begin{aligned} \|Tu_n - Tu\| &\leq \|Tu_n - T_m u_n\| + \|T_m u_n - T_m u\| + \|T_m u - Tu\| \\ &\leq \sup_{x \in D} \|Tx - T_m x\| + \|T_m u_n - T_m u\| + \|T_m u - Tu\|. \end{aligned}$$

Since each T_m is continuous and $u_n \rightarrow u$ as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \|T_m u_n - T_m u\| = 0$, which together with the last inequality, implies that for each given $m \geq 1$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Tu_n - Tu\| &\leq \limsup_{n \rightarrow \infty} (\sup_{x \in D} \|Tx - T_m x\| + \|T_m u_n - T_m u\| + \|T_m u - Tu\|) \\ &\leq \sup_{x \in D} \|Tx - T_m x\| + \limsup_{n \rightarrow \infty} \|T_m u_n - T_m u\| + \|T_m u - Tu\| \\ &= \sup_{x \in D} \|Tx - T_m x\| + \|T_m u - Tu\|. \end{aligned}$$

Since $\lim_{m \rightarrow \infty} \sup_{x \in D} \|Tx - T_m x\| = 0$ and $\lim_{m \rightarrow \infty} \|T_m u - Tu\| = 0$, we obtain

$$\limsup_{n \rightarrow \infty} \|Tu_n - Tu\| \leq \lim_{m \rightarrow \infty} \sup_{x \in D} \|Tx - T_m x\| + \lim_{m \rightarrow \infty} \|T_m u - Tu\| = 0,$$

that is, $\lim_{n \rightarrow \infty} \|Tu_n - Tu\| = 0$. This means that T is continuous on C .

Suppose $A := (2I - T)^{-1}$. Then A is nonexpansive and $F(A) = F(T)$ as a consequence of Theorem 6 of [35]. So it follows that $F(A) \cap F(Q) = F(T) \cap F(Q) = \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) (=:\Omega) \neq \emptyset$. Also, we observe that

$$\|x_n - Ax_n\| = \|AA^{-1}x_n - Ax_n\| \leq \|A^{-1}x_n - x_n\| = \|(2I - T)x_n - x_n\| = \|x_n - Tx_n\|.$$

Since $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, we have $\lim_{n \rightarrow \infty} \|x_n - Ax_n\| = 0$. Meanwhile, from Lemma 4 it is easy to see that $\mu A + (1 - \mu)Q$ is nonexpansive and $F(\mu A + (1 - \mu)Q) = F(A) \cap F(Q) = \bigcap_{n=1}^{\infty} F(T_n) \cap F(Q) (=:\Omega) \neq \emptyset$. Obviously, the mapping $z \mapsto tf(z) + (1 - t)(\mu A + (1 - \mu)Q)z$ is a contraction of C into itself for each $t \in (0, 1)$. So, z_t solves the fixed point equation $z_t = tf(z_t) + (1 - t)(\mu A + (1 - \mu)Q)z_t$. Then, we have

$$z_t - x_n = (1 - t)[\mu(Az_t - x_n) + (1 - \mu)(Qz_t - x_n)] + t(f(z_t) - x_n). \tag{23}$$

Thus, from Lemma 7 and (23), we obtain

$$\begin{aligned} \|z_t - x_n\|^2 &\leq (1 - t)^2 \|\mu(Az_t - x_n) + (1 - \mu)(Qz_t - x_n)\|^2 + 2t \langle f(z_t) - x_n, j(z_t - x_n) \rangle \\ &\leq (1 - t)^2 \{ \mu(\|z_t - x_n\|^2 + 2\|z_t - x_n\| \|Ax_n - x_n\| + \|Ax_n - x_n\|^2) \\ &\quad + (1 - \mu)(\|z_t - x_n\|^2 + 2\|z_t - x_n\| \|Qx_n - x_n\| + \|Qx_n - x_n\|^2) \} \\ &\quad + 2t \langle f(z_t) - x_n, j(z_t - x_n) \rangle \\ &\leq (1 - t)^2 \{ \|z_t - x_n\|^2 + \|Ax_n - x_n\|(2\|z_t - x_n\| + \|Ax_n - x_n\|) \\ &\quad + \|Qx_n - x_n\|(2\|z_t - x_n\| + \|Qx_n - x_n\|) \} + 2t \langle f(z_t) - x_n, j(z_t - x_n) \rangle, \end{aligned}$$

that is,

$$\begin{aligned} \|z_t - x_n\|^2 &\leq (1 + t^2)\|z_t - x_n\|^2 + \|Ax_n - x_n\|(2\|z_t - x_n\| + \|Ax_n - x_n\|) \\ &\quad + \|Qx_n - x_n\|(2\|z_t - x_n\| + \|Qx_n - x_n\|) + 2t \langle f(z_t) - z_t, j(z_t - x_n) \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \langle z_t - f(z_t), j(z_t - x_n) \rangle &\leq \frac{t}{2} \|z_t - x_n\|^2 + \frac{1}{2t} [\|Ax_n - x_n\| (2\|z_t - x_n\| + \|Ax_n - x_n\|) \\ &\quad + \|Qx_n - x_n\| (2\|z_t - x_n\| + \|Qx_n - x_n\|)]. \end{aligned} \tag{24}$$

Letting $n \rightarrow \infty$ in (24), from $\|x_n - Ax_n\| \rightarrow 0$ and $\|x_n - Qx_n\| \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \langle z_t - f(z_t), j(z_t - x_n) \rangle \leq \frac{t}{2} \ell \tag{25}$$

where ℓ is a constant such that $\|z_t - x_n\|^2 \leq \ell$ for all $n \geq 0$ and $t \in (0, 1)$. Utilizing Lemma 8 we deduce that $\{z_t\}$ converges strongly to a fixed point $x^* \in F(\mu A + (1 - \mu)Q) = F(A) \cap F(Q) (= \Omega)$, which solves the variational inequality: $\langle (I - f)x^*, j(x^* - x) \rangle \leq 0, \forall x \in \Omega$. Since j is norm-to-norm uniformly continuous on bounded sets of E , as $t \rightarrow 0^+$ in (25), we get (22).

Step 4. We claim that $x_n \rightarrow x^*$ and $y_n \rightarrow y^*$ as $n \rightarrow \infty$, where (x^*, y^*) solves the GSVI (4). Indeed, putting $p = x^*$ in (13), we obtain

$$\begin{aligned} \|x_n - x^*\|^2 &\leq [1 - (1 - k)\alpha_n - \gamma_n] \frac{\|x_{n-1} - x^*\|^2 + \|x_n - x^*\|^2}{2} + \alpha_n \langle f(x^*) - x^*, j(x_n - x^*) \rangle \\ &\quad + \beta_n \|Sx^* - x^*\| \|x_n - x^*\| + \gamma_n \|x_n - x^*\|^2, \end{aligned}$$

which hence implies that

$$\begin{aligned} \|x_n - x^*\|^2 &\leq \frac{1 - (1 - k)\alpha_n - \gamma_n}{1 + (1 - k)\alpha_n - \gamma_n} \|x_{n-1} - x^*\|^2 + \frac{2\alpha_n}{1 + (1 - k)\alpha_n - \gamma_n} \langle f(x^*) - x^*, j(x_n - x^*) \rangle \\ &\quad + \frac{2\beta_n}{1 + (1 - k)\alpha_n - \gamma_n} \|Sx^* - x^*\| \|x_n - x^*\| \\ &= (1 - \lambda_n) \|x_{n-1} - x^*\|^2 + \lambda_n \sigma_n, \end{aligned} \tag{26}$$

where $\lambda_n = \frac{2(1 - k)\alpha_n}{1 + (1 - k)\alpha_n - \gamma_n}$ and $\sigma_n = \frac{1}{1 - k} \langle f(x^*) - x^*, j(x_n - x^*) \rangle + \frac{\beta_n}{(1 - k)\alpha_n} \|Sx^* - x^*\| \|x_n - x^*\|$.

Now, observe that $(1 - k)\alpha_n = \frac{2(1 - k)\alpha_n}{2} \leq \frac{2(1 - k)\alpha_n}{1 - \gamma_n + (1 - k)\alpha_n} = \lambda_n$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we infer that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Note that $\lim_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n} = 0$ and $\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, j(x_n - x^*) \rangle \leq 0$, due to (22). Thus, in terms of the boundedness of $\{\|x_n - x^*\|\}$, we have $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$. Therefore, applying Lemma 9 to (26) implies that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Moreover, putting $p = x^*$ and $q = y^* = J_{\rho_2}^{M_2} (I - \rho_2 A_2)x^*$ in (18), we have $\lim_{n \rightarrow \infty} \|x_n - y_n - (x^* - y^*)\| = 0$. Also, since $\|y_n - y^*\| \leq \|x_n - y_n - (x^* - y^*)\| + \|x^* - x_n\|$, we know that $y_n \rightarrow y^*$ as $n \rightarrow \infty$. In addition, in terms of Lemma 3 and as $x^* \in \Omega \subset F(Q)$, (x^*, y^*) solves the GSVI (4). \square

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