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Fixed Point Results for the Family of Multivalued *F*-Contractive Mappings on Closed Ball in Complete Dislocated *b*-Metric Spaces

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Abstract: The purpose of this paper is to find out fixed point results for the family of multivalued mappings fulfilling a generalized rational type *F*-contractive conditions on a closed ball in complete dislocated *b*-metric space. An application to the system of integral equations is presented to show the novelty of our results. Our results extend several comparable results in the existing literature.

Keywords: fixed point; closed ball; family of multivalued mapping; dislocated *b*-metric space; application to the system of integral equations

MSC: 46S40; 47H10; 54H25.

1. Introduction and Preliminaries

Fixed point theory plays a fundamental role in functional analysis. Nadler [1] initiated the study of fixed point theorems for the multivalued mappings. Due to its significance, a large number of authors have proved many interesting multiplications of his result (see [2–8]).

Rasham et al. [9] proved the multivalued fixed point results for new generalized *F*-contractive mappings on dislocated metric spaces with application to the system of integral equations. Nazir et al. [10] showed common fixed point results for a family of generalized multivalued *F*-contraction mappings in ordered metric spaces (see also [11–21]). Recently Shoaib et al. [7] discussed the results for the family of multivalued mappings satisfying contranction on a sequence in a closed ball in Hausdorff fuzzy metric space. For further results on closed ball, see [7,8,22–27].

In this paper, we have obtained common fixed point for the family of multivalued mappings satisfying conditions only on a sequence contained in a closed ball. We have used a weaker class of strictly increasing mappings F rather than the class of mappings F used by different authors. An example which supports the proved results is also given. Moreover, we investigate our results in a better framework of dislocated *b*-metric space (see [28–30]). New results in ordered spaces, partial *b*-metric space, dislocated metric space, partial metric space, *b*-metric space, and metric space can be obtained as corollaries of our results. We give the following definitions and results which will be needed in the sequel.

Definition 1 ([28]). *Let X be a nonempty set and let $d_b : X \times X \to [0, \infty)$ be a function, called a dislocated b-metric (or simply d_b-metric). If there exists $b \geq 1$ such that for any $x, y, z \in X$, the following conditions holds:*

(i) *If $d_b(x, y) = 0$, then $x = y$;*
(ii) $d_b(x, y) = d_b(y, x)$;

(iii) $d_b(x,y) \leq b[d_b(x,z) + d_b(z,y)]$.

The pair (X, d_b) is called a dislocated b-metric space. It should be noted that every dislocated metric is a dislocated b-metric with $b = 1$.

It is clear that if $d_b(x,y) = 0$, then from (i), $x = y$. But if $x = y$, $d_b(x,y)$ may not be 0. For $x \in X$ and $\varepsilon > 0$, $\overline{B(x,\varepsilon)} = \{y \in X : d_b(x,y) \leq \varepsilon\}$ is a closed ball in (X, d_b). We will use D.B.M space instead of dislocated b-metric space.

Definition 2 ([28]). *Let (X, d_b) be a D.B.M space .*

(i) A sequence $\{x_n\}$ in (X, d_b) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $n, m \geq n_0$ we have $d_b(x_m, x_n) < \varepsilon$ or $\lim_{n,m\to\infty} d_b(x_n, x_m) = 0$.

(ii) A sequence $\{x_n\}$ dislocated b-converges (for short d_b -converges) to x if $\lim_{n\to\infty} d_b(x_n, x) = 0$. In this case x is called a d_b-limit of $\{x_n\}$.

(iii) (X, d_b) is called complete if every Cauchy sequence in X converges to a point $x \in X$ such that $d_b(x,x) = 0$.

Definition 3. *Let K be a nonempty subset of D.B.M space of X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if*

$$d_b(x,K) = d_b(x,y_0), \text{ where } d_b(x,K) = \inf_{y\in K} d_b(x,y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.
We denote $P(X)$ be the set of all closed proximinal subsets of X.

Definition 4 ([8]). *The function $H_{d_b} : P(X) \times P(X) \to R^+$, defined by*

$$H_{d_b}(N,M) = \max\{\sup_{n\in N} d_b(n,M), \sup_{m\in M} d_b(N,m)\}$$

is called dislocated Hausdorff $b-$metric on $P(X)$.

Definition 5 ([21]). *Let (X,d) be a metric space. A mapping $H : X \to X$ is said to be an $F-$contraction if there exists $\tau > 0$ such that*

$$\forall j,k \in X,\ d(Hj,Hk) > 0 \Rightarrow \tau + F\left(d(Hj,Hk)\right) \leq F\left(d(j,k)\right)$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $j,k \in \mathbb{R}_+$ such that $j < k$, $F(j) < F(k)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if

$\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

Lemma 1. *Let (X, d_b) be a D.B.M space. Let $(P(X), H_{d_b})$ be a dislocated Hausdorff $b-$metric space on $P(X)$. Then, for all $G, H \in P(X)$ and for each $g \in G$ such that $d_b(g,H) = d_b(g,h_g)$, where $h_g \in H$. Then the following holds:*

$$H_{d_b}(G,H) \geq d_b(g,h_g).$$

2. Main Result

Let (Z, d_b) be a *D.B.M* space, $c_0 \in Z$ and let $\{S_\beta : \beta \in \Omega\}$ be a family of multivalued mappings from Z to $P(Z)$. Then there exist $c_1 \in S_\alpha c_0$ for some $\alpha \in \Omega$, such that $d_b(c_0, S_\alpha c_0) = d_b(c_0, c_1)$. Let $c_2 \in S_\beta c_1$ be such that $d_b(c_1, S_\beta c_1) = d_b(c_1, c_2)$. Continuing this method, we get a sequence c_n of

points in Z such that $c_{n+1} \in S_\gamma c_n$, $d_b(c_n, S_\gamma c_n) = d_b(c_n, c_{n+1})$ for some $\gamma \in \Omega$. We denote this iterative sequence by $\{ZS_\beta(c_n) : \beta \in \Omega\}$. We say that $\{ZS_\beta(c_n) : \beta \in \Omega\}$ is a sequence in Z generated by c_0.

Theorem 1. *Let (Z, d_b) be a complete D.B.M space with constant $b \geq 1$ and $\{S_\beta : \beta \in \Omega\}$ be a family of multivalued mappings from Z to $P(Z)$ and $\{ZS_\beta(c_n) : \beta \in \Omega\}$ be a sequence in Z generated by c_0. Assume that the following hold:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + \eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_b}(S_i e, S_j y)) \leq F\left(\begin{array}{c} \eta_1 d_b(e, y) + \eta_2 d_b(e, S_i e) \\ +\eta_3 d_b(y, S_j y) + \eta_4 \dfrac{d_b^2(e, S_i e).d_b(y, S_j y)}{1 + d_b^2(e, y)} \end{array}\right) \tag{1}$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{ZS_\beta(c_n) : \beta \in \Omega\}$ with $e \neq y$, $i, j \in \Omega$ with $i \neq j$ and $H_{d_b}(S_i e, S_j y) > 0$.

(ii) If $\lambda = \dfrac{\eta_1 + \eta_2}{1 - \eta_3 - \eta_4}$, then

$$d_b(c_0, S_\alpha c_0) \leq \lambda(1 - b\lambda)r. \tag{2}$$

Then $\{ZS_\beta(c_n) : \beta \in \Omega\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$ and $\{ZS_\beta(c_n) : \beta \in \Omega\} \to u \in \overline{B_{d_b}(c_0, r)}$. Also, if the inequality (1) holds for u, then there exist a common fixed point for the family of multivalued mappings $\{S_\beta : \beta \in \Omega\}$ in $\overline{B_{d_b}(c_0, r)}$ and $d_b(u, u) = 0$.

Proof. Let $\{ZS_\beta(c_n) : \beta \in \Omega\}$ be a sequence in Z generated by c_0. If $c_0 = c_1$, then c_0 is a common fixed point of S_α for all $\alpha \in \Omega$. Let $c_0 \neq c_1$. From (2), we get

$$d_b(c_0, c_1) = d_b(c_0, S_\alpha c_0) \leq \lambda(1 - b\lambda)r < r.$$

It follows that,

$$c_1 \in \overline{B_{d_b}(c_0, r)}.$$

Let $c_2, \cdots, c_j \in \overline{B_{d_b}(c_0, r)}$ for some $j \in \mathbb{N}$. Now by using Lemma 1, we have

$$\begin{aligned} \tau + F(d_b(c_j, c_{j+1})) &\leq \tau + F(H_{d_b}(S_\delta c_{j-1}, S_\eta c_j)) \\ &\leq F[\eta_1 d_b\left(c_{j-1}, c_j\right) + \eta_2 d_b\left(c_{j-1}, S_\delta c_{j-1}\right) + \eta_3 d_b\left(c_j, S_\eta c_j\right) \\ &\quad + \eta_4 \frac{d_b\left(c_{j-1}, S_\delta c_{j-1}\right).d_b(c_j, S_\eta c_j)}{1 + d_b\left(c_{j-1}, c_j\right)} \\ &\leq F[\eta_1 d_b\left(c_{j-1}, c_j\right) + \eta_2 d_b\left(c_{j-1}, c_j\right) + \eta_3 d_b\left(c_j, c_{j+1}\right) \\ &\quad + \eta_4 \frac{d_b^2\left(c_{j-1}, c_j\right).d_b(c_j, c_{j+1})}{1 + d_b^2\left(c_{j-1}, c_j\right)} \\ &\leq F((\eta_1 + \eta_2) d_b\left(c_{j-1}, c_j\right) + (\eta_3 + \eta_4) d_b\left(c_j, c_{j+1}\right)). \end{aligned}$$

This implies

$$F(d_b(c_j, c_{j+1})) < F((\eta_1 + \eta_2) d_b\left(c_{j-1}, c_j\right) + (\eta_3 + \eta_4) d_b\left(c_j, c_{j+1}\right)).$$

As F is strictly increasing. So, we have

$$d_b(c_j, c_{j+1}) < (\eta_1 + \eta_2) d_b\left(c_{j-1}, c_j\right) + (\eta_3 + \eta_4) d_b\left(c_j, c_{j+1}\right).$$

Which implies

$$\begin{aligned}(1-\eta_3-\eta_4)d_b(c_j,c_{j+1}) &< (\eta_1+\eta_2)d_b\left(c_{j-1},c_j\right)\\ d_b(c_j,c_{j+1}) &< \left(\frac{\eta_1+\eta_2}{1-\eta_3-\eta_4}\right)d_b\left(c_{j-1},c_j\right).\end{aligned}$$

As $\lambda=\dfrac{\eta_1+\eta_2}{1-\eta_3-\eta_4}<1$. Hence

$$d_b(c_j,c_{j+1})<\lambda d_b\left(c_{j-1},c_j\right)<\lambda^2 d_b\left(c_{j-2},c_j\right)<\cdots<\lambda^j d_b\left(c_0,c_1\right).$$

Now, we have

$$d_b(c_j,c_{j+1})<\lambda^j d_b\left(c_0,c_1\right) \text{ for some } j\in\mathbb{N}. \tag{3}$$

Now,

$$\begin{aligned}d_b(x_0,c_{j+1}) &\leq bd_b(c_0,c_1)+b^2d_b(c_1,c_2)+\cdots+b^{j+1}d_b(c_j,c_{j+1})\\ &\leq bd_b(c_0,c_1)+b^2\lambda(d_b(c_0,c_1))+\cdots\\ &\quad +b^{j+1}\lambda^{j+1}(d_b(c_0,c_1)), \qquad \text{(by (3))}\\ d_b(c_0,c_{j+1}) &\leq \frac{b(1-(b\lambda)^{j+1})}{1-b\lambda}\lambda(1-b\lambda)r<r,\end{aligned}$$

which implies $c_{j+1}\in\overline{B_{d_b}(c_0,r)}$. Hence, by induction $c_n\in\overline{B_{d_b}(c_0,r)}$ for all $n\in N$. Now,

$$d_b(c_n,c_{n+1})<\lambda^n d_b\left(c_0,c_1\right) \text{ for all } n\in\mathbb{N}. \tag{4}$$

Now, for any positive integers m,n $(n>m)$, we have

$$\begin{aligned}d_b(c_m,c_n) &\leq b(d_b(c_m,c_{m+1}))+b^2(d_b(c_{m+1},c_{m+2}))+\cdots\\ &\quad +b^{n-m}(d_b(c_{n-1},c_n)),\\ &< b\lambda^m d_b(c_0,c_1)+b^2\lambda^{m+1}d_b(c_0,c_1)+\cdots\\ &\quad +b^{n-m}\lambda^{n-1}d_b(c_0,c_1), \qquad \text{(by (4))}\\ &< b\lambda^m(1+b\lambda+\cdots)d_b(c_0,c_1)\end{aligned}$$

As $\eta_1,\eta_2,\eta_3,\eta_4>0$, $b\geq 1$ and $b\eta_1+b\eta_2+\eta_3+\eta_4<1$, so $|b\lambda|<1$. Then, we have

$$d_b(c_m,c_n)<\frac{b\lambda^m}{1-b\lambda}d_b(c_0,c_1)\to 0 \text{ as } m\to\infty.$$

Hence $\{ZS_\beta(c_n)\}$ is a Cauchy sequence in $\overline{B_{d_b}(c_0,r)}$. Since $(\overline{B_{d_b}(c_0,r)},d_b)$ is a complete metric space, so there exist $u\in\overline{B_{d_b}(c_0,r)}$ such that $\{ZS_\beta(c_n)\}\to u$ as $n\to\infty$, then

$$\lim_{n\to\infty}d_b(c_n,u)=0. \tag{5}$$

Suppose that $d_b(u,S_qu)>0$, then there exist a positive integer k such that $d_b(c_n,S_qu)>0$ for all $n\geq k$. For $n\geq k$, we have

$$\begin{aligned} d_b(u, S_q u) &\leq d_b(u, c_n) + d_b(c_n, S_q u) \\ &\leq d_b(u, c_n) + H_{d_b}(S_\gamma c_{n-1}, S_q u) \\ &< d_b(u, c_n) + \eta_1 d_b(c_{n-1}, u) + \eta_2 d_b(c_{n-1}, S_\gamma c_{n-1}) \\ &\quad + \eta_3 d_b(u, S_q u) + \eta_4 \frac{d_b^2(c_{n-1}, S_\gamma c_{n-1}).d_b(u, S_q u)}{1 + d_b^2(c_{n-1}, u)}. \end{aligned}$$

Letting $n \to \infty$, and by using (5) we get

$$d_b(u, S_q u) < \eta_3 d_b(u, S_q u) < d_b(u, S_q u),$$

which is a contradiction. So our supposition is wrong. Hence $d_b(u, S_q u) = 0$ or $u \in S_q u$. Similarly, by using Lemma 1, inequality (1), we can show that $d_b(u, S_i u) = 0$ or $u \in S_i u$ for all $i \in \Omega$. Now, for some $i \in \Omega$

$$d_b(u, u) \leq b d_b(u, S_i u) + b d_b(S_i u, u) \leq 0.$$

This implies that $d_b(u, u) = 0$. This completes the proof. □

Example 1. *Let $Z = [0, \infty)$ and $d_b : Z \times Z \to \mathbb{R}$ be a complete D.B.M space defined by*

$$d_b(x, y) = (x + y)^2 \text{ for all } x, y \in Z.$$

Consider the family of multivalued mappings $S_\beta : Z \to P(Z)$ where $\beta \in \Omega = \alpha, 1, 2, 3, \cdots$, defined as

$$S_n(x) = \begin{cases} [\frac{x}{3n}, \frac{x}{2n}] \text{ if } x \in \overline{B_{d_b}(x_0, r)}, \\ [2nx, 3nx] \text{ if } x \in (4, \infty) \cap Z, \end{cases} \quad \text{where } n = 1, 2, 3, \cdots,$$

and

$$S_\alpha(x) = \begin{cases} [\frac{x}{3}, \frac{5x}{12}] \text{ if } x \in [0, 4] \cap Z \\ [2x, 3x] \text{ if } x \in (4, \infty) \cap Z. \end{cases}$$

Suppose that, $x_0 = 1$, $r = 25$, then $\overline{B_{d_b}(x_0, r)} = [0, 4] \cap Z$. Now, $d_b(x_0, S_\alpha x_0) = d_b(1, S_\alpha 1) = d_b(1, \frac{1}{3}) = \frac{16}{9}$. So $x_1 = \frac{1}{3}$. Now, $d_b(x_1, S_1 x_1) = d_b(\frac{1}{3}, S_1 \frac{1}{3}) = d_b(\frac{1}{3}, \frac{1}{9})$. So $x_2 = \frac{1}{9}$. Now, $d_b(x_2, S_2 x_2) = d_b(\frac{1}{9}, S_2 \frac{1}{9}) = d_b(\frac{1}{9}, \frac{1}{54})$. So $x_3 = \frac{1}{54}$. Continuing in this way, we have $\{ZS_\beta(x_n)\} = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{54} ....\}$. Take $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$, $\eta_3 = \frac{1}{60}$, $\eta_4 = \frac{1}{30}$, then $b\eta_1 + b\eta_2 + \eta_3 + \eta_4 < 1$ and $\lambda = \frac{3}{19}$. Now

$$d_b(x_0, S_\alpha x_0) = \frac{16}{9} < \frac{3}{19}(1 - \frac{6}{19})25 = \lambda(1 - b\lambda)r.$$

Now, take S_α, S_n where $n = 1, 2, 3, \cdots$. Now, if $x, y \in \overline{B_{d_b}(x_0, r)} \cap \{ZS_\beta(x_n)\}$, then, we have

$$\begin{aligned} H_{d_b}(S_n x, S_\alpha y) &= \max\{\sup_{a \in S_n x} d_b(a, S_\alpha y), \sup_{y \in S_\alpha y} d_b(S_n x, b)\} \\ &= \max\{\sup_{a \in S_n x} d_b(a, [\frac{y}{3}, \frac{5y}{12}]), \sup_{b \in S_\alpha y} d_b([\frac{x}{3n}, \frac{x}{2n}], b)\} \\ &= \max\{d_b(\frac{x}{2n}, \frac{y}{3}), d_b(\frac{x}{3n}, \frac{5y}{12})\} \\ &= \max\left\{\left(\frac{x}{2n} + \frac{y}{3}\right)^2, \left(\frac{x}{3n} + \frac{5y}{12}\right)^2\right\} \\ &\leq \frac{1}{10} d_b(x, y) + \frac{1}{20} d_b(x, [\frac{x}{3n}, \frac{x}{2n}]) + \frac{1}{60} d_b(y, [\frac{y}{3}, \frac{5y}{12}]) \\ &\quad + \frac{1}{30} \frac{d_b^2(x, [\frac{x}{3n}, \frac{x}{2n}]).d_b(y, [\frac{y}{3}, \frac{5y}{12}])}{1 + d_b^2(x, y)}. \end{aligned}$$

Thus,

$$H_{d_b}(S_n x, S_\alpha y) < \eta_1 d_b(x,y) + \eta_2 d_b(x, S_n x) + \eta_3 d_b(y, S_\alpha y) + \eta_4 \frac{d_b^2(x, S_n x).d_b(y, S_\alpha y)}{1 + d_b^2(x,y)},$$

which implies that, for any $\tau \in (0, \frac{12}{95}]$ *and for a strictly increasing mapping* $F(s) = \ln s$*, we have*

$$\tau + F(H_{d_b}(S_n x, S_\alpha y)) \leq F \left(\begin{array}{c} \eta_1 d_b(x,y) + \eta_2 d_b(x, S_n x) + \eta_3 d_b(y, S_\alpha y) \\ + \eta_4 \dfrac{d_b^2(x, S_n x).d_b(y, S_\alpha y)}{1 + d_b^2(x,y)} \end{array} \right).$$

Similarly, for some $i, j \in \Omega$ *and* $\tau > 0$*, we can prove*

$$\tau + F(H_{d_b}(S_i x, S_j y)) \leq F \left(\begin{array}{c} \eta_1 d_b(x,y) + \eta_2 d_b(x, S_i x) + \eta_3 d_b(y, S_j y) \\ + \eta_4 \dfrac{d_b^2(x, S_i x).d_b(y, S_j y)}{1 + d_b^2(x,y)} \end{array} \right).$$

Note that, for $x = 6 \in Z$, $y = 7 \in Z$*, then, we have*

$$\tau + F(H_{d_b}(S_1 6, S_\alpha 7)) > F \left(\begin{array}{c} \eta_1 d_b(6,7) + \eta_2 d_b(6, S7) + \eta_3 d_b(6, S7) \\ + \eta_4 \dfrac{d_b^2(6, S7).(7, S7)}{1 + d_b^2(6,7)} \end{array} \right).$$

So condition (1) does not hold on Z*. Thus the mappings* S_β *satisfying all the conditions of Theorem 1 only for* $x, y \in \overline{B_{d_b}(x_0, r)} \cap \{ZS_\beta(x_n)\}$*. Hence there exist a common fixed point for the family of multivalued mappings* $\{S_\beta : \beta \in \Omega\}$ *in* $\overline{B_{d_b}(c_0, r)}$*.*

If we take $\eta_2 = 0$ in Theorem 1 then we are left with the following result.

Corollary 1. *Let* (Z, d_b) *be a complete D.B.M space with constant* $b \geq 1$ *and* $\{S_\beta : \beta \in \Omega\}$ *be a family of multivalued mappings from* Z *to* $P(Z)$ *and* $\{ZS_\beta(c_n) : \beta \in \Omega\}$ *be a sequence in* Z *generated by* c_0*. Assume that the following hold:*

(i) There exist $\tau, \eta_1, \eta_3, \eta_4 > 0$ *satisfying* $b\eta_1 + \eta_3 + \eta_4 < 1$ *and a strictly increasing mapping* F *such that*

$$\tau + F(H_{d_b}(S_i e, S_j y)) \leq F \left(\eta_1 d_b(e,y) + \eta_3 d_b(y, S_j y) + \eta_4 \frac{d_b^2(e, S_i e).d_b(y, S_j y)}{1 + d_b^2(e,y)} \right) \tag{6}$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{ZS_\beta(c_n) : \beta \in \Omega\}$ *with* $e \neq y$*,* $i, j \in \Omega$ *with* $i \neq j$ *and* $H_{d_b}(S_i e, S_j y) > 0$*.*

(ii) If $\lambda = \frac{\eta_1}{1 - \eta_3 - \eta_4}$*, then*

$$d_b(c_0, S_\alpha c_0) \leq \lambda(1 - b\lambda) r.$$

Then $\{ZS_\beta(c_n) : \beta \in \Omega\}$ *is a sequence in* $\overline{B_{d_b}(c_0, r)}$ *and* $\{ZS_\beta(c_n) : \beta \in \Omega\} \to u \in \overline{B_{d_b}(c_0, r)}$*. Also, if the inequality (6) holds for* u*, then there exist a common fixed point for the family of multivalued mappings* $\{S_\beta : \beta \in \Omega\}$ *in* $\overline{B_{d_b}(c_0, r)}$ *and* $d_b(u, u) = 0$*.*

If we take $\eta_3 = 0$ *in Theorem 1 then we are left with the following result.*

Corollary 2. *Let* (Z, d_b) *be a complete D.B.M space with constant* $b \geq 1$ *and* $\{S_\beta : \beta \in \Omega\}$ *be a family of multivalued mappings from* Z *to* $P(Z)$ *and* $\{ZS_\beta(c_n) : \beta \in \Omega\}$ *be a sequence in* Z *generated by* c_0*. Assume that the following hold:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ *satisfying* $b\eta_1 + b\eta_2 + \eta_4 < 1$ *and a strictly increasing mapping F such that*

$$\tau + F(H_{d_b}(S_i e, S_j y)) \leq F\left(\eta_1 d_b(e,y) + \eta_2 d_b(e, S_i e) + \eta_4 \frac{d_b^2(e, S_i e).d_b(y, S_j y)}{1 + d_b^2(e,y)}\right) \tag{7}$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{ZS_\beta(c_n) : \beta \in \Omega\}$ *with* $e \neq y$, $i, j \in \Omega$ *with* $i \neq j$ *and* $H_{d_b}(S_i e, S_j y) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4}$, *then*

$$d_b(c_0, S_\alpha c_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{ZS_\beta(c_n) : \beta \in \Omega\}$ *is a sequence in* $\overline{B_{d_b}(c_0, r)}$ *and* $\{ZS_\beta(c_n) : \beta \in \Omega\} \to u \in \overline{B_{d_b}(c_0, r)}$. *Also, if the inequality (7) holds for u, then there exist a common fixed point for the family of multivalued mappings* $\{S_\beta : \beta \in \Omega\}$ *in* $\overline{B_{d_b}(c_0, r)}$ *and* $d_b(u, u) = 0$.

If we take $\eta_4 = 0$ *in Theorem 1 then we are left with the following result.*

Corollary 3. *Let* (Z, d_b) *be a complete D.B.M space with constant* $b \geq 1$ *and* $\{S_\beta : \beta \in \Omega\}$ *be a family of multivalued mappings from Z to* $P(Z)$ *and* $\{ZS_\beta(c_n) : \beta \in \Omega\}$ *be a sequence in Z generated by* c_0. *Assume that the following hold:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3 > 0$ *satisfying* $b\eta_1 + b\eta_2 + \eta_3 < 1$ *and a strictly increasing mapping F such that*

$$\tau + F(H_{d_b}(S_i e, S_j y)) \leq F\left(\eta_1 d_b(e,y) + \eta_2 d_b(e, S_i e) + \eta_3 d_b(y, S_j y)\right) \tag{8}$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{ZS_\beta(c_n) : \beta \in \Omega\}$ *with* $e \neq y$, $i, j \in \Omega$ *with* $i \neq j$ *and* $H_{d_b}(S_i e, S_j y) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_3}$, *then*

$$d_b(c_0, S_\alpha c_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{ZS_\beta(c_n) : \beta \in \Omega\}$ *is a sequence in* $\overline{B_{d_b}(c_0, r)}$ *and* $\{ZS_\beta(c_n) : \beta \in \Omega\} \to u \in \overline{B_{d_b}(c_0, r)}$. *Also, if the inequality (8) holds for u, then there exist a common fixed point for the family of multivalued mappings* $\{S_\beta : \beta \in \Omega\}$ *in* $\overline{B_{d_b}(c_0, r)}$ *and* $d_b(u, u) = 0$.

3. Application to the Systems of Integral Equations

Theorem 2. *Let* (Z, d_b) *be a complete D.B.M space with constant* $b \geq 1$. *Let* $c_0 \in Z$ *and* $\{S_\beta : \beta \in \Omega\}$ *be a family of mappings from Z to Z. Assume that, there exist* $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ *satisfying* $b\eta_1 + b\eta_2 + \eta_3 + \eta_4 < 1$ *and a strictly increasing mapping F such that the following holds:*

$$\tau + F(d_b(S_\alpha e, S_\beta y)) \leq F\left(\begin{array}{c} \eta_1 d_b(e,y) + \eta_2 d_b(e, S_\alpha e) \\ +\eta_3 d_b(y, S_\beta y) + \eta_4 \dfrac{d_b^2(e, S_\alpha e).d_b(y, S_\beta y)}{1 + d_b^2(e,y)} \end{array}\right), \tag{9}$$

for all $e, y \in Z$ *and* $d(S_\alpha e, S_\beta y) > 0$ *where* $\alpha, \beta \in \Omega$ *with* $\alpha \neq \beta$. *Also if the inequality (9) holds for u, then the family* $\{S_\beta : \beta \in \Omega\}$ *has a unique common fixed point u in Z.*

Proof. The proof of this theorem is similar as Theorem 1. We have to prove the uniqueness only. Let v be another common fixed point of S. Suppose $d_b(S_\alpha u, S_\beta v) > 0$. Then, we have

$$\tau + F(d_b(S_\alpha u, S_\beta v)) \leq F\left(\begin{array}{c} \eta_1 d_b(u,v) + \eta_2 d_b(u, S_\alpha u) \\ +\eta_3 d_b(v, S_\beta v) + \eta_4 \dfrac{d_b^2(u, S_\alpha u).d_b(v, S_\beta v)}{1 + d_b^2(u,v)} \end{array}\right).$$

This implies that

$$d_b(u,v) < \eta_1 d_b(u,v) < d_b(u,v),$$

which is a contradiction. So $d_b(S_\alpha u, S_\beta v) = 0$. Hence $u = v$. □

In this section, we discuss the application of fixed point Theorem 2 in form of Volterra type integral equation.

$$u(k) = \int_0^k H_\alpha(k, h, u(h))dh, \tag{10}$$

for all $k, h \in [0,1]$ and $\alpha \in \Omega$. We find the solution of (10). Let $Z = C([0,1], \mathbb{R})$ be the set of all real valued continuous functions on $[0,1]$, endowed with the complete dislocated b-metric. For $u \in C([0,1], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{k\in[0,1]} \{|u(k)|\, e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$d_\tau(u, c) = \left[\sup_{k\in[0,1]} \{|u(k) + c(k)|\, e^{-\tau k}\}\right]^2 = \|u + c\|_\tau^2$$

for all $u, c \in C([0,1], \mathbb{R})$, with these settings, $(C([0,1], \mathbb{R}), d_\tau)$ becomes a complete *D.B.M.S*.

Now we prove the following theorem to ensure the existence of solution of integral equation.

Theorem 3. *Assume the following conditions are satisfied:*

(i) $H_\alpha : [0,1] \times [0,1] \times C([0,1], \mathbb{R}) \to \mathbb{R}$;
(ii) *Define* $S_\alpha : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$, *where* $\alpha \in \Omega$

$$S_\alpha u(k) = \int_0^k H_\alpha(k, h, u(h))dh.$$

Suppose there exist $\tau > 0$, *such that*

$$\left|H_\alpha(k, h, u(h)) + H_\beta(k, h, c(h))\right| \leq \frac{\tau N(u(h), c(h))}{\tau \|N(u, c)\|_\tau + 1}$$

for all $k, h \in [0,1]$ *and* $u, c \in C([0,1], \mathbb{R})$, *where*

$$\begin{aligned} N(u(h), c(h)) &= \eta_1[|u(h) + c(h)|]^2 + \eta_2[|u(h) + S_\alpha u(h)|]^2 + \eta_3[|c(h) + S_\beta c(h)|]^2 \\ &\quad + \eta_4 \frac{[|u(h) + S_\alpha u(h)|]^4 . [|c(h) + S_\beta c(h)|]^2}{1 + [|u(h) + c(h)|]^4}, \end{aligned}$$

where $\eta_1, \eta_2, \eta_3, \eta_4 \geq 0$, *and* $2\eta_1 + 2\eta_2 + \eta_3 + \eta_4 < 1$. *Then integral Equation* (10) *has a solution.*

Proof. By assumption (ii)

$$\begin{aligned} \left|S_\alpha u(k) + S_\beta c(k)\right| &= \int_0^k \left|H_\alpha(k, h, u(h) + H_\beta(k, h, c(h)))\right| dh, \\ &\leq \int_0^k \frac{\tau}{\tau \|N(u, c)\|_\tau + 1} ([N(u(h), c(h))] e^{-\tau h}) e^{\tau h} dh \\ &\leq \int_0^k \frac{\tau}{\tau \|N(u, c)\|_\tau + 1} \|N(u, c)\|_\tau e^{\tau h} dh \end{aligned}$$

$$\begin{aligned} &\leq \frac{\tau\|N(u,c)\|_\tau}{\tau\|N(u,c)\|_\tau+1}\int_0^k e^{\tau h}dh, \\ &\leq \frac{\|N(u,c)\|_\tau}{\tau\|N(u,c)\|_\tau+1}e^{\tau k}. \end{aligned}$$

This implies

$$\left|S_\alpha u(k)+S_\beta c(k)\right|e^{-\tau k} \leq \frac{\|N(u,c)\|_\tau}{\tau\|N(u,c)\|_\tau+1}.$$

$$\|S_\alpha u(k)+S_\beta c(k)\|_\tau \leq \frac{\|N(u,c)\|_\tau}{\tau\|N(u,c)\|_\tau+1}.$$

$$\frac{\tau\|N(u,c)\|_\tau+1}{\|N(u,c)\|_\tau} \leq \frac{1}{\|S_\alpha u(k)+S_\beta c(k)\|_\tau}.$$

$$\tau+\frac{1}{\|N(u,c)\|_\tau} \leq \frac{1}{\|S_\alpha u(k)+S_\beta c(k)\|_\tau}.$$

which further implies

$$\tau-\frac{1}{\|S_\alpha u(k)+S_\beta c(k)\|_\tau} \leq \frac{-1}{\|N(u,c)\|_\tau}.$$

So all the conditions of Theorem 3 are satisfied for $F(c)=\frac{-1}{\sqrt{c}}; c>0$ and $d_\tau(u,c)=\|u+c\|_\tau^2$, $b=2$. Hence integral equations given in (10) have a unique common solution. □

Example 2. *Consider the system of integral equations*

$$g(k)=\frac{1}{\alpha}\int_0^k g(h)dh, \text{ where } k\in[0,1] \text{ and } \alpha\in\Omega=\mathbb{N}.$$

Define $H_\alpha:[0,1]\times[0,1]\times C([0,1],\mathbb{R}_+)\to\mathbb{R}$ *by* $H_\alpha=\frac{1}{\alpha}g(h), \alpha\in\Omega=\mathbb{N}$. *Now,*

$$S_\alpha g(k)=\frac{1}{\alpha}\int_0^k g(h)dh.$$

Take $\eta_1=\frac{1}{10}$, $\eta_2=\frac{1}{20}$, $\eta_3=\frac{1}{60}$, $\eta_4=\frac{1}{30}$, $\tau=\frac{12}{95}$, *then* $2\eta_1+2\eta_2+\eta_3+\eta_4<1$. *Moreover, all conditions of Theorem 3 are satisfied and* $g(k)=0$ *for all k, is a unique common solution to the above equations.*

4. Conclusions

In the present paper, we have achieved common fixed point of a family of multivalued mappings satisfying conditions only on a sequence contained in a closed ball. We have used a weaker class of strictly increasing mappings F rather than the class of mappings F used by many potential authors. Examples and an application are given to demonstrate the variety of our results. New results for families of multivalued mappings and singlevalued contractive mappings in ordered spaces, partial b-metric space, dislocated metric space, partial metric space, b-metric space, and metric space can be obtained as corollaries of our results.

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