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The Structure of Moduloid on a Nexus

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Abstract: In this paper, the notion of moduloid on a nexus is introduced. Moreover, the concept of submoduloid is defined and some different submoduloids are introdused. Also, several interesting facts about submoduloids are proved. Finally, a homomorphism between two moduloids is defined and some related results are investigated.

Keywords: $\mathbb{N}^{\infty}$-moduloid; $\mathbb{N}^{\infty}$-submoduloid; $\mathbb{N}^{\infty}$-homomorphism of $\mathbb{N}^{\infty}$-moduloid

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1. Introduction and Preliminaries

The roots of the concepts of 'formex' (plural: formices) and 'plenix' (plural: plenices) go back to the nineteen seventies. At that time, an extensive programme of research was led by H. Nooshin in the Space Structures Research Centre of the University of Surrey. The aim of the research was to find convenient ways of generating data for analysis and design of space structures consisting of many thousands of elements. The geometry of such a structural system often involves many types of symmetries that can be used to simplify the generation of information. However, in addition to the geometric information, it is necessary to produce information about the properties of material(s) of the elements, positions and particulars of the supports and the nature and magnitudes of the external loads. Also, the information about the external loads should include the details of dead weights, snow loads, wind effects, earthquake forces, temperature changes and so on.

The starting point was the introduction of the concepts of 'formex algebra' [1,2]. These concepts are used for the algebraic representation and processing of all types of geometric forms and, in particular, structural configurations [3–5]. Subsequently, a software called 'Formian' was introduced that provides a convenient medium for the use of formex algebra [6]. However, The concepts of formex algebra are general and can be used in many different fields.

Several years later, in order to be able to conveniently handle the vast amount of varied data that defines a space structure, a sophisticated form of data base was evolved which was called a 'plenix' [7,8]. A plenix is capable of containing any type of information either in explicit constant form, or in a 'generic form', that is, as a 'parametric formulation'. The term 'plenix' comes from the Latin word 'plenus' meaning 'full'. This choice was a reflection of the intention of a plenix being capable of representing the 'full spectrum' of mathematical objects. However, these pioneering works were mainly concerned with plenices as data structures. The generic nature of a plenix as a data base places the concept in a class of its own with capabilities which are far beyond any normal data base.

Fundamantaly, a plenix is a mathematical object consisting of an arrangement of a mathematical object. A plenix is like a tree structure in which every branch is a mathematical object. For instance, Figure 1 shows a graphic representation of a plenix, consisting of a sequence of elements, each of which consists of a sequence of elements and so on. The graphical representation of plenix Q in Figure 1 is referred to as the dendrogram of Q.

The following construct is another way to represent a plenix Q.

$$Q =< [7,1,0], <>, < 5,3, < 2, FALSE >, 0 >>$$

In this plenix, the first principal panel is a vector and the second principal is an empty plenix, that is, a plenix that without any principal panel. A dendrogram of Q is shown in Figure 1.

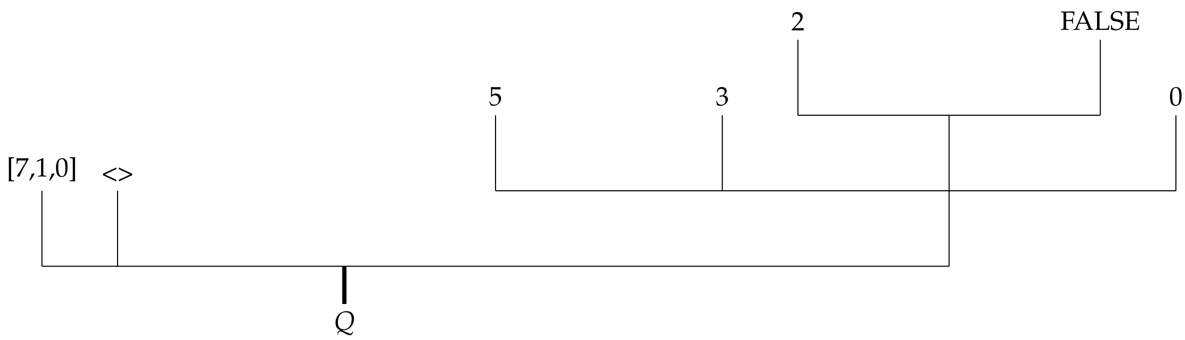

Figure 1. Dendrogram of Q.

In the early two thousands, the basic idea of a plenix was further developed as a mathematical object by M. Bolourian [9,10]. The aim of the research was to create an algebra based on plenices. That is, to define meaningful relations, operations and functions for plenices and to investigate the properties of the resulting algebra. This work turned the concept of a plenix into a proper mathematical system with the potential for applications in all branches of human knowledge. Cosider the plenix Q, every panel of a plenix may be associated with a sequence of positive integers that indicates the position of a panel in the plenix. This sequence of positive integers is referred to as the address of the panel. For instance, referring to plenix Q, Figure 1, the panels addresses are given in the following Table 1.

Table 1. Addresses of plenix of Q.

Panel	**Address**
[7, 1, 0]	(1)
$<>$	(2)
$< 5,3, < 2, FALSE >, 0 >$	(3)
5	(3, 1)
3	(3, 2)
$< 2, FALSE >$	(3, 3)
0	(3, 4)
2	(3, 3, 1)
$FALSE$	(3, 3, 2)

An address (i, j, k) refers to the kth principal panel of the jth principal panel of the ith principal panel of the plenix. For example, the address of 0 is $(3, 4)$, indicating the 4th principal panel of the 3rd principal panel of the plenix.

The set of addresses for all panels of a plenix is called the 'address set' of that plenix. For instance, the set

$$A = \{(1), (2), (3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 3, 1), (3, 3, 2)\}$$

is the address set of plenix Q. therefore, the address set represents the constitution of a plenix.

The constitution of a plenix plays an important role in the theory of plenices. As a result, one of the interesting domains for research in plenix theory is the constitution of a plenix, irrespective of the values of its primion panel. The mathematical object that represents the constitution of a plenix is called a nexus. The notion of a nexus is introduced in [9] and a nexus is defined axiomatically, by using the concept of the address set. Also, an interesting fact about a relationship between plenix and nexus is shown, that is, the concept of plenix is defined via of the concept of nexus.

The idea of 'nexus algebra' is another important mathematical structure that has come out of Bolourian's work [9].The concept of the nexus, as an abstract algebraic structure, is certainly worthy of attention. In [9], the properties of nexuses are investigated from the view point of pure mathematics. Many familiar concepts in an abstract algebra such as substructures, cyclic substructures, generators of an algebra, homomorphism of an algebra, direct product and direct sum of an algebra, metric space, prime and maximal substructures, decomposition theorem and so on, are studied deeply in the context of nexus algebra [9,11–17]. This means that nexus algebra has great potential as an algebraic structure.

The main aim of this paper is to create a structure of moduloid on a nexus. This paper is subdivided into four sections. Section 1 includes an introduction and some basic definitions of the structure of nexus, such as address set, level of the element of the nexus, order of the nexus as well as definitions of subnexuses of the nexus and the cyclic subnexuses. In Section 2, the main notion of this paper, that is, a moduloid on a nexus, is defined. In Section 3, the concept of submoduloid is defined and some interesting facts about submoduloid are proven such as every cyclic subnexus of the nexus N is a submoduloid of N. In Section 4, a homomorphism between two moduloids is defined and some of these results are investigated such as Theorem 9, that is, let f be a moduloid homomorphim and $f(a) = c$ then $f(b) = c$ for every b in panel of a.

Definition 1. *A groupoid is a set that is closed under a binary operation. A semigroup G is a groupoid with a binary operation* $\circ$ *, which satisfies the associative property,*

$$(a \circ b) \circ c = a \circ (b \circ c)$$

for all $a, b, c \in G$*.*

A monoid G is a semigroup containing an identity element. A semiring is a set R with two operations $+$ *and* $\circ$*, such that* $(R, +)$ *is a commutative monoid and* $(R, \circ)$ *is a semigroup. The operation* $\circ$ *is distributive with respect to* $+$*, that is,*

$$a \circ (b + c) = a \circ b + a \circ c$$

$$(b + c) \circ a = b \circ a + c \circ a$$

for all $a, b, c \in R$*. Also, for any* $a \in R, a \circ 0 = 0 \circ a = 0$*, where* 0 *is the identity element of the monoid* $(R, +)$*.*

Definition 2. *A moduloid N over the semiring R consists of a commutative groupoid* $(M, +)$ *with the identity element and the scalar multiplication:* $R \times M \to M$*, which maps* $(r, a) \to ra$*. Also, for all r and s in R, and a in M, the following equations are valid,*

(i) $(r + s)a = ra + sa$
(ii) $r(a + b) = ra + rb$
(iii) $(rs)a = r(sa)$
(iv) $0a = r0 = 0$.

Definition 3 ([9])**.**

(i) *Let* $\mathbb{N}^*$ *be the set of non-negative integers. Then, an address is a sequence whose elements belong to* $\mathbb{N}^*$*. Also,* $a_k = 0$ *implies that* $a_i = 0$*, for all* $i \geq k$*. The sequence of zero is called the empty address and is denoted by* $()$*. In other words, every nonempty address is of the form* $(a_1, a_2, ..., a_n, 0, 0, ...)$*, where* a_i *and* n *belong to* $\mathbb{N}$*. Hereafter, this address will be denoted by* $(a_1, a_2, ..., a_n)$*.*
(ii) *A nexus N is a nonempty set of address with the following properties:*

$$(a_1, a_2, ..., a_{n-1}, a_n) \in N \Rightarrow (a_1, a_2, ..., a_{n-1}, t) \in N, \forall\, 0 \leq t \leq a_n, \tag{1}$$

and for an infinite nexus

$$\{a_i\}_{i=1}^{\infty} \in N, a_i \in \mathbb{N} \Rightarrow \forall n \in \mathbb{N}, \forall 0 \leq t \leq a_n, (a_1, a_2, ..., a_n - t) \in N. \tag{2}$$

Note that condition (2) does imply condition (1).

Definition 4 ([9])**.** *Let N be a nexus. A subset S of N is called a subnexus of N provided that S itself is a nexus.*

Definition 5 ([9])**.** *Let N be a nexus and $\emptyset \neq A \subseteq N$. Then the smallest subnexus of N containing A is called the subnexus of N generated by A and is denoted by $< A >$. If $A = \{a_1, a_2, ..., a_n\}$, then instead of $< A >$ one may write $< a_1, a_2, ..., a_n >$. If A has only one element a, then the subnexus $< a >$ is called a cyclic subnexus of N. Clearly, $()$ and N are trivial subnexuses of the nexus N.*

Definition 6 ([9])**.** *Let N be a nexus and $a \in N$. The level of a is said to be:*

(i) *n, if $a = (a_1, a_2, ..., a_n)$, for some $a_n \in N$.*
(ii) *∞, if a is an infinite sequence of N.*
(iii) *0, if $a = ()$.*

The level of a is denoted by $l(a)$.

Definition 7 ([9])**.** *The highest level of the elements of M is referred to as the rise of M and is denoted by $rise(M)$. In particular, the highest level of the elements of a nexus N is referred to as the rise of a nexus N and denoted by $rise(N)$.*

Example 1. *The set $N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (1,2,1)\}$ is a nexus and $S = \{(), (1), (2), (2,1), (2,2), (2,3)\}$ is a subnexus of N. Also, $< (2,3) >= S$, $l(2,1) = 2$ and $rise(N) = 3$.*

Definition 8 ([9])**.** *Every address of a nexus N containing only one term is called a 'principal address' of N. In other words, a principal address is of the form (a) where a is a positive integer.*

Definition 9 ([9])**.** *The number of principal addresses of a nexus N is called the order of the nexus and is denoted by $Ord(N)$.*

Example 2. *Consider the nexus $N = \{(), (1), (2), (1,1), (1,2)\}$. The addresses (1) and (2) are the principal addresses of N.*

Definition 10 ([9])**.** *Let N be a nexus and let $a = (a_1, a_2, ..., a_n)$ be an address of N. The first term a_1 is said to be the stem of a and is denoted by $stem(a)$.*

Example 3. *Consider the nexus $M = \{(), (1), (2), (3), (1,1), (1,2), (3,1), (3,2), (3,2,1)\}$. The number of principal addresses of M is three, and hence, $Ord(N) = 3$. The stem of the address $()$ is zero. Moreover, the stem of the addresses$(3), (3,1), (3,2)$ and $(3,2,1)$ is 3.*

Definition 11 ([9])**.** *Let $a = \{a_i\}$ and $b = \{b_i\}$, $i \in \mathbb{N}$, be two addresses. Then, $a \leq b$ if $l(a) = 0$ or if one of the following cases is satisfied:*

Case 1. *If $l(a) = 1$, that is $a = (a_1)$, for some $a_1 \in \mathbb{N}$ and $a_1 \leq b_1$.*
Case 2. *If $1 < l(a) < \infty$, then $l(a) \leq l(b)$ and $a_{l(a)} \leq b_{l(b)}$ and for any $1 \leq i < l(a), a_i = b_i$.*
Case 3. *If $l(a) = \infty$, then $a = b$.*

Definition 12 ([9])**.** *Let N be a nexus and let $a = (a_1, a_2, ..., a_k)$ be an address of N. The set*

$$\{(a_1, a_2, ..., a_k, a_{k+1}, ..., a_n) \in N : a_{k+i} \in \mathbb{N} \quad for\, i = 1, 2, ..., n-k\}$$

is called the 'panel' of a and is denoted by q_a. In other words, if $a = (a_1, a_2, ..., a_k)$, then every address b of N is an address in q_a provided that the first k terms of b are the same as the corresponding terms of a. Note that, the 'panel' of a does not include a. Also, $q_{()}$ includes all the addresses of N except for the empty address itself.

Definition 13 ([9])**.** *Let N be a nexus and let a be an address of N. The set $\{b \in N : a \leq b\}$ is called the quasi panel of a and is denoted by Q_a.*

Example 4. *Consider the nexus*

$$N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (2,2,1), (2,2,2), (2,3,1), (2,3,2)\}.$$

Now, consider the address $a = (2,2)$, of N. Then $q_a = \{(2,2,1), (2,2,2)\}$ and

$$Q_a = \{(2,2), (2,2,1), (2,2,2), (2,3), (2,3,1), (2,3,2)\}.$$

2. A Nexus as a Moduloid

We denoted the $\max\{m,n\}$ and the $\min\{m,n\}$ by $m \vee n$ and $m \wedge n$, respectively, where $m, n \in \mathbb{N}^*$. Now, consider $\mathbb{N}^*$ with the two operations $\vee$ and $\wedge$, as mentioned above. Then $(\mathbb{N}^*, \vee)$ is a commutative monoid with number 0, as the identity element and $(\mathbb{N}^*, \wedge)$ is a semigroup. Furthermore, for all $a, b, c \in \mathbb{N}^*$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Also, $x \wedge 0 = 0$, for all $x \in \mathbb{N}^*$. Therefore, $(\mathbb{N}^*, \vee, \wedge, 0)$ is a commutative semiring.

Definition 14. *Let N be a nexus and let $a = (a_1, a_2, ...)$ and $b = (b_1, b_2, ...)$ be two addresses of N. Now, the operation $+$ is defined on N as follows:*

Suppose that there exists a k such that

$$(a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k) \in N$$

and

$$(a_1 \vee b_1, a_2 \vee b_2, ..., a_{k+1} \vee b_{k+1}) \notin N,$$

then

$$a + b := (a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k).$$

In this case one may write $index_N(a,b) = k$. On the other hand, if there is no such a k, then

$$a + b := (a_1 \vee b_1, a_2 \vee b_2, ...),$$

and we write $index_N(a,b) = \infty$. Note that always $(a_1 \vee b_1) \in N$.

Example 5. *Consider the nexus*

$$N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (1,2,1), (1,2,2), (1,2,3), (1,2,4), (2,3,1), (2,3,2)\}.$$

The dendrogram of N is shown in Figure 2.

Suppose that $a = (1,2,4)$ and $b = (2,3,1)$ then $a + b = (1,2,4) + (2,3,1) = (1 \vee 2, 2 \vee 3, 4 \vee 1) = (2,3,4)$. As one can see, $(2,3,4) \notin N$, so, by definition of summation of two addresses of a nexus N, the last component, that is 4, must be eliminated. Since, $(2,3) \in N$, then one may consider $(2,3)$ as the summation of the addresses a and b. In this case, $index_N(a,b) = 2$. As another example, suppose that $a = (1)$ and $b = (2,3,1)$. Since $l(a) = 1$ and $l(b) = 3$, one may write $a = (1,0,0)$. Therefore, $a + b = (1,0,0) + (2,3,1) = (1 \vee 2, 0 \vee 3, 0 \vee 1) = (2,3,1) = b$.

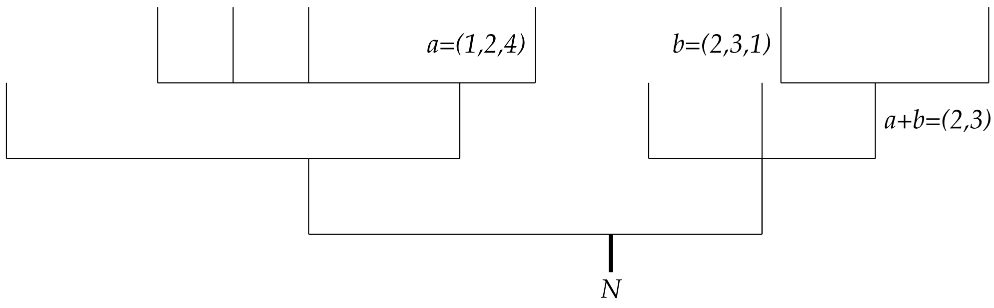

Figure 2. Dendrogram of N.

Remark 1. *For any non-empty address $a \in N$, we have:*

(i) $a + b = b + a$,
(ii) $a + 0 = a$,
(iii) $a + (1) = a$.

Lemma 1.

(i) *Suppose that N is a nexus, and a and b are two addresses in N. If $a \leq b$ then $a + b = b$.*
(ii) *In a cyclic nexus N, since every two addresses are comparable, then the sumation of two addresses is equal to the greater summand.*

Proof.

(i) Suppose that $a = (a_1, a_2, ..., a_n)$. Since $a \leq b$, so b can be considered as $b = (a_1, a_2, ..., a_n + k, b_1, b_2, ..., b_m)$ for some $k \in \mathbb{N}^*$. Therefore $a + b = b$.
(ii) Since every cyclic nexus is a chain, then all the addresses are comparable. Now, by using part one, the proof is complete.

□

Remark 2.

(i) *Generally, $+$ is not associative.*
(ii) *It is possible that $a \leq b$ but $a + c \geq b + c$, for some a, b and c in a nexus.*

Example 6. *Suppose that,*

$$N = \{(), (1), (2), (3), (1,1)(1,2), (2,1), (3,1)\}.$$

Now, consider the addresses

$$a = (1,2),\ b = (2,1),\ c = (3,1)$$

then

$$(a + b) + c = ((1,2) + (2,1)) + (3,1) = (2,2) + (3,1),$$

since $(2,2) \notin N$, *so* $a + b = (2)$. *Therefore,*

$$(a + b) + c = (2) + (3,1) = (3,1)$$

but

$$a + (b + c) = (1,2) + (3,1) = (3,2),$$

since $(3,2) \notin N$,

$$a + (b + c) = (3).$$

Example 7. *Consider the nexus N whose generators are the addresses* $(1,2,3,9)$ *and* $(2,3,4,8)$*, namely:*

$$N =< (1,2,3,9),(2,3,4,8) > .$$

Suppose that,

$$a = (1,2,3,8)\; b = (1,2,3,9)\; c = (2,3,4,7)$$

As one can see $a \leq b$ *but,*

$$a + c = (2,3,4,8) \qquad \text{and} \qquad b + c = (2,3,4).$$

Therefore,

$$a + c \geq b + c.$$

Remark 3. $(N,+,0)$ *is a commutative groupoid with the identity.*

Definition 15. *Let* $\mathbb{N}^\infty = \mathbb{N} \cup \{0,\infty\}$*, N be a nexus and the scalar multiplication*

$$\circ : \mathbb{N}^\infty \times N \to N$$

is defined on N as follows: $r \circ w = \begin{cases} (a_1,a_2,...,a_r), & if\;\; l(w) > r,\; r > 0 \\ (a_1,a_2,...,a_n), & if\;\; l(w) \leq r,\; r > 0 \\ 0 & if\;\; r = 0 \\ w & if\;\; r = \infty \end{cases}$

for all, $r \in \mathbb{N}^\infty$ *and* $w = (a_1,a_2,...,a_n) \in N$*. In other words,* $r \circ w = \{a_i\}_{i=1}^{l(w)\wedge r}$*. From now on, operation* $\circ$ *is called dot product.*

Theorem 1. *Let N be a nexus. Then* $(N,+,0)$ *is a moduloid over* $(\mathbb{N}^\infty,\vee,\wedge,0)$ *together with scalar multiplication* $\circ$*. For simplicity N is called an* $\mathbb{N}^\infty$*-moduloid.*

Proof. We show that, the following properties are valid.

(i) $(r \wedge s) \circ a = r \circ (s \circ a)$,
(ii) $r \circ (a + b) = r \circ a + r \circ b$,
(iii) $(r \vee s) \circ a = r \circ a + s \circ a$,
(iv) $0 \circ a = r \circ 0 = 0$,

for all $r,s \in \mathbb{N}^\infty$ and $a,b \in N$.
(i) Let $r,s \in \mathbb{N}^\infty$ and $a = \{a_i\}_{i\in\mathbb{N}^*}$. Then

$$r \circ (s \circ a) = \{a_i\}_{i=1}^{l(a)\wedge r\wedge s} = (r \wedge s) \circ \{a_i\}_{i\in\mathbb{N}^*} = (r \wedge s) \circ a.$$

(ii) Let $a = \{a_i\}_{i\in\mathbb{N}^*}$ and $b = \{b_i\}_{i\in\mathbb{N}^*}$ be two elements of N. Without loss of generality, suppose that $l(a) \leq l(b)$. If $index_N(a,b) = k < \infty$ then

$$r \circ (a + b) = (a_1 \vee b_1, a_2 \vee b_2, ..., a_{r\wedge k} \vee b_{r\wedge k}). \tag{3}$$

On the other hand

$$r \circ a = \{a_i\}_{i=1}^{l(a)\wedge r}$$

and

$$r \circ b = \{a_i\}_{i=1}^{l(b)\wedge r}.$$

Now, consider the two following cases:

Case 1: Let $r \leq l(a) \leq l(b)$. If $r \leq k$, then there exists $s \in \mathbb{N}^*$ such that $r+s=k$. In this case,

$$c = (a_1 \vee b_1, a_2 \vee b_2, ..., a_r \vee b_r) \leq (a_1 \vee b_1, a_2 \vee b_2, ..., a_{r+s} \vee b_{r+s}),$$

since $index_N(a,b) = k$, so,

$$(a_1 \vee b_1, a_2 \vee b_2, ..., a_{r+s} \vee b_{r+s}) \in N.$$

However, N is a nexus therefore, $c \in N$. So,

$$\begin{aligned} r \circ a + r \circ b &= (a_1, a_2, ..., a_r) + (b_1, b_2, ..., b_r) \\ &= (a_1 \vee b_1, a_2 \vee b_2, ..., a_r \vee b_r) \\ &= (a_1 \vee b_1, a_2 \vee b_2, ..., a_{r \wedge k} \vee b_{r \wedge k}). \end{aligned} \tag{4}$$

The last equality is valid, because $r \leq k$. By (3) and (4)

$$r \circ (a+b) = r \circ a + r \circ b.$$

Now, if $r > k$, then

$$r \circ a + r \circ b = (a_1, a_2, ..., a_k, a_{k+1}, ..., a_r) + (b_1, b_2, ..., b_k, b_{k+1}, ..., b_r)$$

$$= (a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k),$$

since $index_N(a,b) = k$, the last equality is valid. Also, since $r > k$ then,

$$(a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k) = (a_1 \vee b_1, a_2 \vee b_2, ..., a_{r \wedge k} \vee b_{r \wedge k}).$$

Therefore, $r \circ a + r \circ b = r \circ (a+b)$.

Case 2: Let $l(a) < r \leq l(b)$, then $r \circ a = (a_1, a_2, ..., a_n)$ and $r \circ b = (b_1, b_2, ..., b_r)$. where $l(a) = n \in \mathbb{N}$. If $r \leq k$, that is, $n < r \leq k$ and since $index_N(a,b) = k$, then

$$r \circ a + r \circ b = (a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_r)$$

$$= (a_1 \vee b_1, a_2 \vee b_2, ..., a_r \vee b_r)$$

$$= (a_1 \vee b_1, a_2 \vee b_2, ..., a_n \vee b_n, b_{n+1}, ..., b_r).$$

On the other hand

$$a + b = (a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_{l(b)})$$

$$= (a_1 \vee b_1, a_2 \vee b_2, ..., a_n \vee b_n, b_{n+1}, ..., b_{l(b)})$$

and

$$r \circ (a+b) = (a_1 \vee b_1, a_2 \vee b_2, ..., a_n \vee b_n, b_{n+1}, ..., b_r).$$

Therefore $r \circ a + r \circ b = r \circ (a+b)$.

Now, if $r > k$, then $r \circ a = (a_1, a_2, ..., a_n)$ and since $r \leq l(b)$, then $r \circ b = (b_1, b_2, ..., b_n, ..., b_k, ..., b_r)$ or $r \circ b = (b_1, b_2, ..., b_k, ..., b_n, ..., b_r)$. Therefore,

$$r \circ a + r \circ b = (a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k)$$

$$= (a_1 \vee b_1, a_2 \vee b_2, ..., a_{r \wedge k} \vee b_{r \wedge k}) = r \circ (a+b)$$

if $index_N(a,b) = \infty$, that is, $a + b = \{a_i \vee b_i\}_{i=1}^{\infty}$, then,

$$r \circ (a+b) = \{a_i \vee b_i\}_{i=1}^{r}.$$

On the other hand, $r \circ a = \{a_i\}_{i=1}^{r}$, $r \circ b = \{a_i\}_{i=1}^{r}$ and hence

$$r \circ a + r \circ b = \{a_i \vee b_i\}_{i=1}^{r} = r \circ (a+b).$$

(iii) Let $r, s \in \mathbb{N}^*$, $s < r$, $a = \{a_i\}_{i=1}^{\infty} \in N$, then,

$$(r \vee s) \circ a = r \circ a = \{a_i\}_{i=1}^{l(a) \wedge r}$$

on the other hand

$$r \circ a = \{a_i\}_{i=1}^{l(a) \wedge r}, \qquad s \circ a = \{a_i\}_{i=1}^{l(a) \wedge s},$$

since, $l(a) \wedge s \leq l(a) \wedge r$. Therefore $r \circ a + s \circ a = \{a_i\}_{i=1}^{l(a) \wedge r} = (r \vee s) \circ a$.

(iv) The proof is straight forward. □

Definition 16. *Let $(N, +, 0)$ be an $\mathbb{N}^{\infty}$-moduloid. If $rise(N) = m < \infty$, then $m \circ a = a$ for all $a \in N$. In this case, $(N, +, 0)$ is called a unitary $\mathbb{N}^{\infty}$-moduloid.*

3. $\mathbb{N}^{\infty}$-Submoduloid

Definition 17. *Let N be an $\mathbb{N}^{\infty}$-moduloid, S be a non-empty subset of N and $0 \in S$. Then S is called a submoduloid of N, if $(S, +, 0)$ is a moduloid over $(\mathbb{N}^{\infty}, \vee, \wedge, 0)$. The set of all $\mathbb{N}^{\infty}$-submoduloid of N is denoted by $SUB_M(N)$.*

Theorem 2. *Let S be a nonempty subset of a nexus N. Then,*

(i) $S \in SUB_M(N)$ *if and only if*

  (a) $r \circ a \in S, \forall r \in \mathbb{N}^{\infty}, \forall a \in S$
  (b) $a + b \in S, \forall a, b \in S.$

(ii) *If N is a unitary moduloid over $(\mathbb{N}^{\infty}, \vee, \wedge, 0)$ and $S \in SUB_M(N)$, then S is a unitary moduloid over $(\mathbb{N}^{\infty}, \vee, \wedge, 0)$.*

Proof.

(i) If $S \in SUB_M(N)$, then, by the definition of $\mathbb{N}^{\infty}$-submoduloid, (a) and (b) are valid. Conversely, since S is closed under operation $+$ and dot product $\circ$, the conditions (i) to (iii) of Theorem 1 hold. By (a) assume that $r = 0$, so, $0 \circ a = 0 \in S$. Thus $(S, +, 0)$ is a commutative groupoid with identity.
(ii) Since, $rise(S) \leq rise(N) = m < \infty$, we have $m \circ a = a$ for all $a \in S$. This implies that $(S, +, 0)$ is a unitary $\mathbb{N}^{\infty}$-moduloid.

□

Remark 4.

(i) *In general, a subnexus of a nexuse is not an $\mathbb{N}^{\infty}$-submoduloid.*
(ii) *In general, an $\mathbb{N}^{\infty}$-submoduloid of a nexus is not a subnexus.*

Example 8. *Consider*

$$N = \{(), (1), (2), (1,1), (1,2), (1,3), (2,1), (2,2)\}, \qquad S = \{(), (1), (2), (1,1), (1,2)\}$$

S is a subnexus of N. But if one considers N as an $\mathbb{N}^\infty$ moduloid, then S is not an $\mathbb{N}^\infty$-submoduloid of N, because $(1,2)$ and (2) belong to S but

$$(1,2) + (2) = (2,2) \notin S.$$

Clearly, each subnexus S of N is closed under dot product, that is,

$$rw \in S, \forall r \in \mathbb{N}^\infty, \forall s \in S.$$

Example 9. *Consider the nexus*

$$N =< (3,2), (2,2) >$$

and the subset

$$S = \{(), (1), (2), (3), (3,2)\}$$

of N. It is easy to check that S is an N^∞-submoduloid of N (closed under addition and dot product), but it is not a subnexus of N because S does not contain the address $(3,1)$.

Corollary 1. *Let N be a nexus and $a \in N$. Then the cyclic subnexus $< a >$ is an submoduloid of N. In particular, if N is a cyclic nexus then every subnexus of N is a submoduloid.*

Proof. Each subnexus of a nexus is closed under dot product. Also, by Lemma 1, if $a \leq b$ then $a + b = b$. Now, the rest of proof follows from Theorem 2. □

Remark 5. *In general, an $\mathbb{N}^\infty$-submoduloid of a cyclic nexus N is not a subnexus of N.*

Example 10. *Consider the cyclic nexus*

$$< (2,3,2) >= \{(), (1), (2), (2,1), (2,2), (2,3), (2,3,1), (2,3,2)\}$$

and the subset

$$\{(2), (2,3), (2,3,2)\}$$

is an $\mathbb{N}^\infty$-submoduloid of $< (2,3,2) >$ but it is not a subnexus.

Theorem 3. *Let N be an $\mathbb{N}^\infty$-moduloid and $n \in \mathbb{N}$. Consider the subset*

$$L(n) = \{a \in N : l(a) \leq n\}$$

of N. Then $L(n)$ is an $\mathbb{N}^\infty$-moduloid of N.

Proof. Suppose that $a, b \in L(n)$. So $l(a)$ and $l(b)$ are less then or equal to n. By definition of $+$ operation, $l(a + b) \leq max\{l(a), l(b)\} \leq n$. Therefore $a + b \in L(n)$. Thus $L(n)$ is closed under $+$ operation. Now, suppose that, $a \in L(n)$ and $k \in \mathbb{N}^\infty$. Then $l(k \circ a) \leq n$. This means that $L(n)$ is closed under dot product. Therefore, $L(n)$ is an $\mathbb{N}^\infty$-submoduloid of N. □

Remark 6. *If N is a nexus and k is a non-negative integer, then by definition of dot product,*

$$k \circ N = \{k \circ a : a \in N\} = \{a \in N : l(a) \leq k\} = L(k).$$

Therefore, by Theorem 3, one can prove that $k \circ N$ is an $\mathbb{N}^\infty$-submoduloid of N for every $k \in \mathbb{N}^\infty$, and it is called k-cut of N.

Example 11. *Suppose that the nexus*

$$N = \{(), (1), (2), (3), (1,1), (1,2), (1,3), (3,1), (3,2), (1,2,1), (1,2,2),$$

$$(3,1,1), (3,1,2), (3,1,3), (3,1,3,1), (3,1,3,2)\},$$

and $k = 2$*, then*

$$2 \circ N = \{(), (1), (2), (3), (1,1), (1,2), (1,3), (3,1), (3,2)\}.$$

The dendrograms of N and $2 \circ N$ *are shown in Figure 3. By Theorem 3,* $2 \circ N$ *is an* $\mathbb{N}^\infty$*-submoduloid of N.*

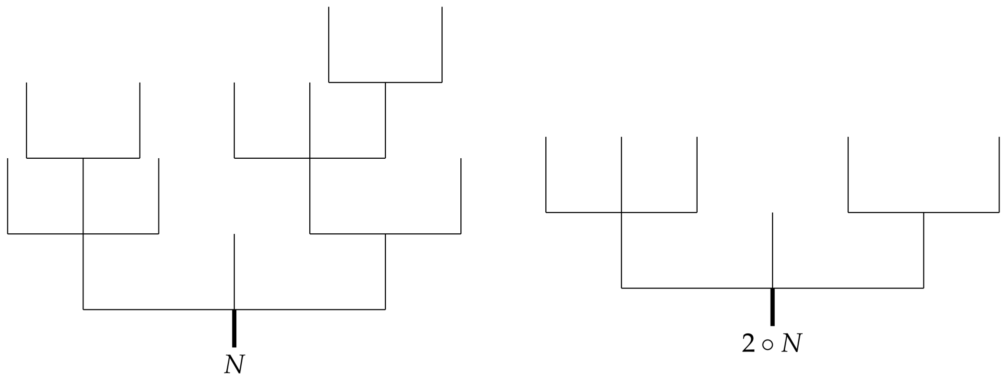

Figure 3. Dendrograms of N and $2 \circ N$.

Theorem 4. *Let N be an* $\mathbb{N}^\infty$*-moduloid, and let M be an* $\mathbb{N}^\infty$*-submoduloid of N. Then for every* $k \in \mathbb{N}^\infty$*,* $k \circ M$ *is an* $\mathbb{N}^\infty$*-submoduloid of N.*

Proof. Suppose that a and b are two addresses in $k \circ M$. So, there exist m_1 and m_2 in M such that $a = k \circ m_1$ and $b = k \circ m_2$. Now consider the summation of a and b, that is,

$$a + b = k \circ m_1 + k \circ m_2 = k \circ (m_1 + m_2)$$

by Theorem 1 (ii), the last equation is held. Since M is an $\mathbb{N}^\infty$-submoduloid of N, $m_1 + m_2 \in M$. Therefore, $k \circ (m_1 + m_2) \in k \circ M$. This means that $k \circ M$ is closed under $+$ operation. Now suppose that $a \in k \circ M$ and $k \in \mathbb{N}^\infty$ then $a = k \circ m$ and

$$r \circ (k \circ M) = (r \wedge k) \circ m = (k \wedge r) \circ m = k \circ (r \circ m).$$

By Theorem 1 (i), the first and the last equations are held. Since M is an $\mathbb{N}^\infty$-submoduloid of N, $r \circ m \in M$. Therefore, $k \circ (r \circ m) \in M$ then $k \circ m$ is closed under dot product. Consequently $k \circ M$ is an N^∞-submoduloid of N. $\square$

Theorem 5. *Let N be an* $\mathbb{N}^\infty$*-moduloid. Consider the subset*

$$N_k = \{a \in N : stem(a) = k\} \cup \{()\} = \{(a_1, a_2, ..., a_n) \in N : a_1 = k\} \cup \{()\}.$$

Then the subset N_k *is an* $\mathbb{N}^\infty$*-submoduloid of N and it is called k-stem.*

Proof. Let $a, b \in N_k$. So that $a = (k, a_2, ..., a_n)$ and $b = (k, b_2, ..., b_n)$. By the definition of $\mathbb{N}^\infty$-moduloid summation, the first term of $a + b$ is $k \vee k = k$. Therefore $a + b \in N_k$. Now suppose that $a \in N_k$ and

$r \in \mathbb{N}^{\infty}$ then by definition of dot product, $r \circ a = K$ for $r \neq 0$ and $r \circ a = 0 = () \in N_k$ for $r = 0$. Thus $r \circ a \in N_k$. Therefore N_k is an $\mathbb{N}^{\infty}$-moduloid of N. □

Example 12. *Suppose that,*

$$N = \{(), (1), (2), (3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2),$$

$$(1,3,1), (1,3,2), (2,2,1), (2,2,2), (3,2,1), (3,2,2), (2,2,1,1), (2,2,1,2)\}$$

and $k = 2$. *Then,*

$$N_2 = \{(), (2), (2,1), (2,2), (2,3), (2,2,1), (2,2,2), (2,2,1,1), (2,2,1,2)\}.$$

The dendrogram of N is shown in Figure 4. By Theorem 5, N_2 is an $\mathbb{N}^{\infty}$-moduloid of N.

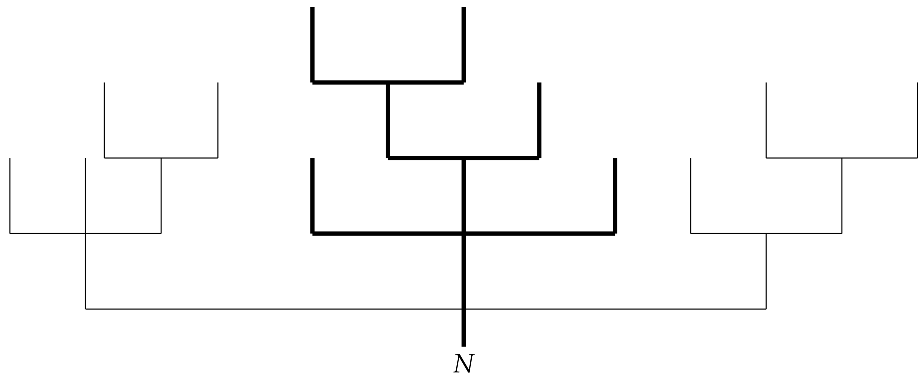

Figure 4. Dendrograms of N. The $\mathbb{N}^{\infty}$-submoduloid N_2 of N is shown by thick lines.

Theorem 6. *Let N be an $\mathbb{N}^{\infty}$-moduloid and let $ord(N) = n$. Now, consider the set $S = \{N_k : for\, k = 0, 1, ..., n\}$, where N_k is an N-submoduloid of N. Then any union of the elements of S is an N-submoduloid of N.*

Proof. Suppose that $\bigcup N_k$ denotes the set of arbitrary union of the elements of S. Now, let $a, b \in \bigcup N_k$. So there exist $0 \leq p, q \leq n$, such that $a \in N_p$ and $b \in N_q$. This means that $a = (p, a_2, ..., a_n)$ and $b = (q, b_2, ..., b_m)$. Without loss of generality, suppose that $p \leq q$ then the first term of $a + b$ is $p \vee q = q$. Therefore, $a + b \in N_k \subseteq (\bigcup N_k)$. Thus $\bigcup N_k$ is closed under $+$ operation. Now, suppose that $a \in (\bigcup N_k)$ and $n \in \mathbb{N}^*$. Therefore there exists $0 \leq p \leq n$, such that $a \in N_p$. By Theorem 5, N_p is an $\mathbb{N}^{\infty}$-submoduloid. Therefore, $n \circ a \in N_p \subseteq (\bigcup N_k)$. Consequently, $\bigcup N_k$ is closed under dot product. So, by Therorem 2 $\bigcup N_k$ is an $\mathbb{N}^{\infty}$-submoduloid of N. □

4. $\mathbb{N}^{\infty}$-Moduloid Homomorphism

In this section a nexus considerd as the $\mathbb{N}^{\infty}$-moduloid and a homomorphism between two $\mathbb{N}^{\infty}$-moduloids are investigated. For simplicity we show that $r \circ a$ by ra for all $r \in \mathbb{N}^{\infty}, a \in N$.

Definition 18. *Let N and M be two $\mathbb{N}^{\infty}$-moduloids, and let $f : N \to M$ be a function. Then f is called an $\mathbb{N}^{\infty}$-moduloid homomorphism if*

(i) $f(a + b) = f(a) + f(b) \qquad \forall a, b \in N,$
(ii) $f(ra) = rf(a) \qquad \forall a \in N, \forall r \in \mathbb{N}^{\infty}.$

The kernel of f is defined by $f^{-1}(\{0\})$ and denoted by Kerf.

Example 13. *Consider two $\mathbb{N}^\infty$-moduloids*

$$N = \{0, (1), (2), (2,1), (2,2)\}, \quad M = \{0, (1), (2), (1,1), (1,2)\}.$$

Suppose that, $f : N \to M$ is defined by $f(0) = 0, f(1) = f(2) = (1), f(2,1) = f(2,2) = (1,1)$. We see that f is an $\mathbb{N}^\infty$-moduloid homomorphism and $Kerf = \{0\}$, but f is not injective. Moreover, the domain of f is N and the rang of f is $\{0, (1), (1,1)\}$. So, f is not surjective. The dendrograms of N and M are shown in Figure 5. In this figure, the $\mathbb{N}^\infty$-moduloid homomorphism between domain and rang of the function f are shown by doted lines.

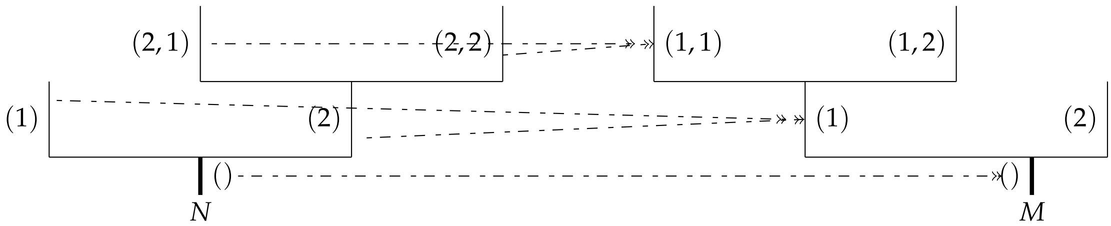

Figure 5. The relationships between domain and range of f are shown by dotted lines.

Theorem 7. *Let $f : N \to M$ be an $\mathbb{N}^\infty$-moduloid homomorphism. Then*

(i) $f(0) = 0$,
(ii) *if $B \in SUB_M(M)$, then $f^{-1}(B) \in SUB_M(N)$, in particular, $Kerf \in SUB_M(N)$,*
(iii) *if $A \in SUB_M(N)$, then $f(A) \in SUB_M(M)$, in particular, $Imf \in SUB_M(M)$.*

Proof.

(i) $f(0) = f(0a) = 0f(a) = 0, \forall a \in N$.
(ii) Let $a, b \in f^{-1}(B), r \in \mathbb{N}^\infty$. Then $f(a), f(b) \in B$, and $f(a+b) = f(a) + f(b) \in B$. Hence, $a + b \in f^{-1}(B)$. Moreover, $f(ra) = rf(a) \in B$, which implies that $ra \in f^{-1}(B)$.
(iii) Let $c, d \in f(A)$ and $r \in \mathbb{N}^\infty$. Hence, there exist $a, b \in A$ such that $f(a) = c, f(b) = d$. Now, $c + d = f(a) + f(b) = f(a+b) \in f(A)$. Also, $rc = rf(a) = f(ra) \in f(A)$.

□

Theorem 8. *Let $f : N \to M$ be an $\mathbb{N}^\infty$-moduloid homomorphism. Then*

(i) *if $a, b \in N, a \leq b$, then $f(a) + f(b) = f(b)$,*
(ii) *if $a \in N, l(a) = n < \infty$, then $l(f(a)) \leq n$. In particular, every principal element is going to 0 or a principal element by f,*
(iii) *if $l(f(a)) \leq l(a) = n$ and $a = (a_1, a_2, ..., a_n)$, then*

$$f((a_1, a_2, ..., a_k)) = f(a), \forall l(f(a)) \leq k \leq n,$$

(iv) *if $l(f(a)) = l(a) \geq 1$, and $a = \{a_i\}_{i \in \mathbb{N}}, f(a) = \{b_i\}_{i \in \mathbb{N}}$, then*

$$f((a_1, a_2, ..., a_k)) = (b_1, b_2, ..., b_k)), \quad \forall 1 \leq k \leq l(a).$$

Proof.

(i) By Lemma 1, if $a \leq b$, then $a + b = b$. Therefore, $f(a+b) = f(a) + f(b) = f(b)$.
(ii) If $f(a) = 0$, we have the result. Let $a = ((a_1, a_2, ..., a_n)$ and let $b = f(a) = \{b_i\}_{i \in \mathbb{N}} \neq 0$. Since $f(na) = nf(a)$, therefore,

$$nb = nf(a) = nf((a_1, a_2, ..., a_n)) = f(n(a_1, a_2, ..., a_n)) = f((a_1, a_2, ..., a_n)) = b = nb = ((b_1, b_2, ..., b_n),$$

it implies that $0 = b_{n+1} = b_{n+2} = \cdots$. Hence, $l(f(a)) \leq l(b)$.

(iii) Let $f(a) = f((a_1, a_2, ..., a_n)) = b = (b_1, b_2, ..., b_m)$, where, $m \leq n$.
Then, for all $m \leq k \leq n$,

$$f(ka) = f(k(a_1, a_2, ..., a_n)) = f((a_1, a_2, ..., a_k)). \tag{5}$$

On the other hand

$$kf(a) = k(b_1, b_2, ..., b_m) = (b_1, b_2, ..., b_m). \tag{6}$$

Since f is an $\mathbb{N}^\infty$-moduloid homomorphism, then for every k the Equations (5) and (6) are equal. Consequently,

$$f((a_1, a_2, ..., a_n)) = (b_1, b_2, ..., b_m) = f(a), \qquad \forall m \leq k < n.$$

(iv) Let $a = (a_1, a_2, ..., a_n)$ be an address in N and let$f((a_1, a_2, ..., a_n)) = (b_1, b_2, ..., b_n)$. Now, consider the equation $f(ka) = kf(a)$. The left side of the equation is equal to

$$f(ka) = f(k(a_1, a_2, ..., a_n)) = f((a_1, a_2, ..., a_k)) \qquad \forall 1 \leq k \leq l(a),$$

and the right side of the equation is equal to

$$kf(a) = kf((a_1, a_2, ..., a_n)) = k(b_1, b_2, ..., b_n) = (b_1, b_2, ..., b_k), \qquad \forall 1 \leq k \leq l(a).$$

Therefore, $f((a_1, a_2, ..., a_k)) = (b_1, b_2, ..., b_k) \qquad \forall 1 \leq k \leq l(a)$. $\square$

Theorem 9. *Let N and M be two $\mathbb{N}^\infty$-moduloids, and let $f : N \to M$ be an $\mathbb{N}^\infty$-moduloid homomorphism. Suppose that $a = (a_1, a_2, ..., a_n)$ is an address in N and $f(a) = c = (c_1, c_2, ..., c_m)$, where $n > m$. Then $f(b) = c$, for all b in q_a.*

Proof. Let b be an address in q_a. Then b is of the form

$$b = (a_1, a_2, ..., a_m, ..., a_n, b_1, b_2, ..., b_p),$$

where t is a non-negative integer. Now, suppose that

$$f((a_1, a_2, ..., a_m, ..., a_n, b_1, b_2, ..., b_p)) = (c_1, c_2, ..., c_m, d_1, d_2, d_q).$$

Since f is an $\mathbb{N}^\infty$-moduloid homomorphism, so,

$$f((m+1)b) = (m+1)f(b) \Rightarrow f((a_1, a_2, ..., a_m, a_{m+1})) = (c_1, c_2, ..., c_m, d_1). \tag{7}$$

But by hypotheses,

$$f((a_1, a_2, ..., a_n)) = (c_1, c_2, ..., c_m),$$

where $n > m$ by Theorem 8 (iii),

$$f((a_1, a_2, ..., a_n)) = (c_1, c_2, ..., c_m), \qquad \forall m \leq k \leq n.$$

Thus,

$$(a_1, a_2, ..., a_m, a_{m+1}) = (c_1, c_2, ..., c_m). \tag{8}$$

Since f is a function, then (7) and (8) implies that

$$(c_1, c_2, ..., c_m, d_1) = (c_1, c_2, ..., c_m),$$

so, $d_1 = 0$. Now, consider the address $(c_1, c_2, ..., c_m, d_1, d_2, ..., d_q)$, by definition of an address, since $d_1 = 0$, then $d_i = 0$ for all $i = 1, 2, ..., q$. Consequently, $f(b) = c$ for all b in q_a. □

Remark 7. *Note that, in the above theorem if q_a replaces by Q_a the theorem is not true.*

Example 14. *Consider, the nexuses*

$$N = \{0, (1), (2), (2,1), (2,2)\}, \quad M = \{0.(1), (2), (2,1)\}.$$

Put $a = (2,1)$, $b = (2,2)$, $c = (2)$ and we defined, $f : N \to M$, where
$f(0) = 0, f(1) = f(2) = (1), f(2,1) = (2), f(2,2) = (2,1)$. It is easy to show that f is 3-homomorphism and $b \in Q_a$, but $f(b) \neq c$.

Theorem 10. *Let N and M be two $\mathbb{N}^\infty$-moduloids, and let $f : N \to M$ be an $\mathbb{N}^\infty$-moduloid homomorphism. Then f is monotone map, that is, $a \leq b$ implies that $f(a) \leq f(b)$.*

Proof. Let a and b be two addresses in N, and let $a \leq b$. Suppose that, $a = (a_1, a_2, ..., a_n)$, then b is of the form

$$b = (a_1, a_2, ..., a_n + k, b_1, b_2, ..., b_p),$$

where k is a nonnegative integer. Suppose that,

$$f((a_1, a_2, ..., a_n)) = c = (c_1, c_2, ..., c_m),$$

where $n \geq m$. Now, consider the two following cases:
Case one: $n > m$, by Theorem 9 , $f(b) = c$. Therefore, $a \leq b$ implies that $f(a) = f(b)$.
Case two: $n = m$. That is, $f((a_1, a_2, ..., a_n)) = c = (c_1, c_2, ..., c_n)$ and $f((a_1, a_2, ..., a_{n-1}, a_n + k, b_1, b_2, ..., b_p)) = (c_1, c_2, ..., c_{n-1}, d_1, d_2, ..., d_q)$, where $p \geq q$. Since f is an $\mathbb{N}^\infty$-moduloid homomorphism, $f(rb) = rf(b)$. Thus,

$$f((a_1, a_2, ..., a_{n-1}, a_n + k)) = (c_1, c_2, ..., c_{n-1}, d_1).$$

Also, $f((a_1, a_2, ..., a_{n-1}, a_n)) = (c_1, c_2, ..., c_{n-1}, c_n)$. Since f is an $\mathbb{N}^\infty$-moduloid homomorphism,

$$f((a_1, a_2, ..., a_{n-1}, a_n + k) + (a_1, a_2, ..., a_{n-1}, a_n)) = f((a_1, a_2, ..., a_{n-1}, a_n + k)) + f((a_1, a_2, ..., a_{n-1}, a_n)).$$

Therefore,

$$f((a_1, a_2, ..., a_{n-1}, a_n + k)) = (c_1, c_2, ..., c_{n-1}, d_1 \vee c_n).$$

Also,

$$f((a_1, a_2, ..., a_{n-1}, a_n + k)) = (c_1, c_2, ..., c_{n-1}, d_1).$$

Since f is a function,

$$(c_1, c_2, ..., c_{n-1}, d_1) = (c_1, c_2, ..., c_{n-1}, d_1 \vee c_n),$$

this implies that $d_1 \vee c_n = d_1$. So, $c_n \leq d_1$. This means that

$$(c_1, c_2, ..., c_{n-1}, c_n) \leq (c_1, c_2, ..., c_{n-1}, d_1) \leq (c_1, c_2, ..., c_{n-1}, d_1, d_2, ..., d_q).$$

Thus,

$$f((a_1, a_2, ..., a_n)) \leq f((a_1, a_2, ..., a_{n-1}, a_n + k)) \leq f((a_1, a_2, ..., a_{n-1}, a_n + k, b_1, b_2, ..., b_q)).$$

Consequently, $f(a) \leq f(b)$. $\square$

5. Conclusions

In this paper, the notions of $\mathbb{N}^\infty$-moduloid nexuses, $\mathbb{N}^\infty$-submoduloid, $\mathbb{N}^\infty$-moduloid homomorphisms, are defined and the relationship between them are investigated. In particular, it was shown that: (i) Every cyclic subnexus $< a >$ of N is an $\mathbb{N}^\infty$-submoduloid of N. Moreover, if N is a cyclic nexus, then every subnexus of N is an $\mathbb{N}^\infty$-submoduloid; (ii) Every k-cut and k-stem of any $\mathbb{N}^\infty$-moduloid is $\mathbb{N}^\infty$-submoduloid; (iii) If f is an $\mathbb{N}^\infty$-moduloid homomorphim and $f(a) = c$ then, $f(b) = c$ for every b in panel of a (Theorem 9); (iv) Every $\mathbb{N}^\infty$-moduloid homomorphism is monotone (Theorem 10).

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