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Coefficient Inequalities of Functions Associated with Hyperbolic Domains

Sarfraz Nawaz Malik [1], Shahid Mahmood [2], Mohsan Raza [3,*], Sumbal Farman [1], Saira Zainab [4] and Nazeer Muhammad [1]

1 Department of Mathematics, COMSATS University Islamabad, Wah Campus 47040, Pakistan; snmalik110@yahoo.com (S.N.M.); sumbalfarman678@gmail.com (S.F.); nazeermuhammad@ciitwah.edu.pk (N.M.)
2 Department of Mechanical Engineering, Sarhad University of Science and I.T, Ring Road, Peshawar 25000, Pakistan; shahidmahmood757@gmail.com
3 Department of Mathematics, Government College University Faisalabad, Faisalabad 38000, Pakistan
4 Department of Mathematical Sciences, Fatima Jinnah Women University, Rawalpindi 46000, Pakistan; sairazainab07@yahoo.com
* Correspondence: mohsan976@yahoo.com

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Abstract: In this work, our focus is to study the Fekete-Szegö functional in a different and innovative manner, and to do this we find its upper bound for certain analytic functions which give hyperbolic regions as image domain. The upper bounds obtained in this paper give refinement of already known results. Moreover, we extend our work by calculating similar problems for the inverse functions of these certain analytic functions for the sake of completeness.

Keywords: analytic functions; starlike functions; convex functions; Fekete-Szegö inequality

MSC: 30C45, 33C10; Secondary: 30C20, 30C75

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1. Introduction and Preliminaries

We consider the class of analytic functions f in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$, defined as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We also consider $\mathcal{S}$, the class of those functions from $\mathcal{A}$ which are univalent in $\mathcal{U}$. Fekete-Szegö problem may be considered as one of the most important results about univalent functions, which is related to coefficients a_n of a function's taylor series and was introduced by Fekete and Szegö [1]. We state it as:

If $f \in \mathcal{S}$ and is of the form (1), then

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} 3 - 4\lambda, & \text{if} \quad \lambda \leq 0, \\ 1 + 2\exp\left(\frac{2\lambda}{\lambda - 1}\right), & \text{if} \quad 0 \leq \lambda \leq 1, \\ 4\lambda - 3, & \text{if} \quad \lambda \geq 1. \end{cases}$$

The problem of maximizing the absolute value of the functional $a_3 - \lambda a_2^2$ is called Fekete-Szegö problem. This result is sharp and is studied thoroughly by many researchers. The equality holds true for Koebe function. The case $0 < \lambda < 1$ provides an example of an extremal problem over $\mathcal{S}$ in which Koebe fails to be extremal. In this regard, one can find a number of results related to the maximization of the non-linear functional $\left|a_3 - \lambda a_2^2\right|$ for various classes and subclasses of univalent functions. Moreover, this functional has also been studied for λ as real as well as complex number. To maximize Fekete-Szegö functional $\left|a_3 - \lambda a_2^2\right|$ for different types of functions, showing interesting geometric characteristics of image domains, several authors used certain classified techniques. For in-depth understanding and more details, we refer the interested readers to study [1–11].

Subordination of two functions f and g is written symbolically as $f \prec g$, and is defined with respect to a schwarz function w such that $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathcal{U}$, as

$$f(z) = g(w(z)), \quad z \in \mathcal{U}. \tag{2}$$

We now include P, the class of analytic functions p such that $p(0) = 1$ and $p \prec \frac{1+z}{1-z}$, $z \in \mathcal{U}$. For details, see [12].

Goodman [13] opened an altogether new area of research with the initiation of the concept of conic domain. He did it in 1991, by introducing parabolic region as image domain of analytic functions. Related to the same, he introduced the class UCV of uniformly convex functions and defined it as follows:

$$UCV = \left\{ f \in \mathcal{A} : \Re\left(1 + (z - \zeta)\frac{f''(z)}{f'(z)}\right) > 0,\ z, \zeta \in \mathcal{U} \right\}.$$

The most suitable one variable characterization of the above defined class UCV of Goodman was independently given by Rønning [14], and Ma and Minda [6]. They defined it as follows:

$$UCV = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U} \right\}.$$

It proved its importance by giving birth to a domain, ever first of its kind, that is, conic (parabolic) domain, given as $\Omega = \{w : \Re w > |w - 1|\}$. Later on, $\beta-$uniformly convex functions were introduced by Kanas and Wiśniowska [15], which are defined as:

$$\beta - UCV = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U} \right\}.$$

This proved to be a remarkable innovation in this area since it gave the most general conic domain Ω_β, given as under, which covers parabolic as well as hyperbolic and elliptic regions.

$$\Omega_\beta = \{w : \Re w > \beta|w - 1|,\ \beta \geq 0\}.$$

For different values of β, the conic domain Ω_β, represents different image domains. For $\beta = 0$, this represents the right half plane, whereas hyperbolic regions when $0 < \beta < 1$, parabolic region for $\beta = 1$ and elliptic regions when $\beta > 1$. For further investigation, we refer to [15,16]. Another breakthrough occurred in this field when Noor and Malik [17] further generalized this domain Ω_β. They introduced the domain

$$\begin{aligned} \Omega_\beta[A,B] &= \Big\{u + iv : \left[\left(B^2 - 1\right)\left(u^2 + v^2\right) - 2(AB - 1)u + \left(A^2 - 1\right)\right]^2 \\ &\quad > \beta^2\left[\left(-2(B+1)\left(u^2 + v^2\right) + 2(A + B + 2)u - 2(A+1)\right)^2 + 4(A - B)^2 v^2\right]\Big\}. \end{aligned} \tag{3}$$

The class of functions given in the following definition takes all values from the above domain $\Omega_\beta[A,B]$, $-1 \leq B < A \leq 1$, $\beta \geq 0$. For more details, we refer to [17].

Definition 1. *A function $p(z)$ is said to be in the class $\beta - P[A,B]$, if and only if,*

$$p(z) \prec \frac{(A+1)\widetilde{p}_\beta(z) - (A-1)}{(B+1)\widetilde{p}_\beta(z) - (B-1)}, \quad -1 \leq B < A \leq 1,\ \beta \geq 0, \tag{4}$$

where $\widetilde{p}_{\beta}(z)$ *is defined by*

$$\widetilde{p}_{\beta}(z)=\begin{cases}\frac{1+z}{1-z}, & \beta=0,\\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta=1,\\ 1+\frac{2}{1-\beta^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\beta\right)\operatorname{arctanh}\sqrt{z}\right], & 0<\beta<1,\\ 1+\frac{1}{\beta^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{1}{\beta^2-1}, & \beta>1,\end{cases} \tag{5}$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t\in(0,1)$, $z\in\mathcal{U}$ *and* z *is chosen such that* $\beta=\cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ *is the Legendre's complete elliptic integral of the first kind, and* $R'(t)$ *is complementary integral of* $R(t)$. *For more details about the function* $\widetilde{p}_{\beta}(z)$, *we refer the readers to [15,16].*

It may be noted that if we restrict the domain as $\Omega_{\beta}[1,-1]=\Omega_{\beta}$, then it becomes the conic domain defined by Kanas and Wiśniowska [15,16]. With the help of this important fact, we notice the following important connections of different well-known classes of analytic functions.

1. $\beta-P[A,B]\subset P\left(\frac{2\beta+1-A}{2\beta+1-B}\right)$, the class of functions with real part greater than $\frac{2\beta+1-A}{2\beta+1-B}$.
2. $\beta-P[1,-1]=\mathcal{P}\left(\widetilde{p}_{\beta}\right)$, the well-known class introduced by Kanas and Wiśniowska [15,16].
3. $0-P[A,B]=P[A,B]$, the well-known class introduced by Janowski [18].

We now include the two very important classes $\beta-UCV[A,B]$ of $\beta-$uniformly Janowski functions and $\beta-ST[A,B]$ of corresponding $\beta-$Janowski starlike functions which are used in Section 2 of this paper. These are introduced in [17] and defined as follows.

Definition 2. *A function* $f\in\mathcal{A}$ *is said to be in the class* $\beta-UCV[A,B]$, $\beta\geq 0$, $-1\leq B<A\leq 1$, *if and only if* ,

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}\right)>\beta\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}-1\right|,$$

or equivalently,

$$\frac{(zf'(z))'}{f'(z)}\in\beta-P[A,B]. \tag{6}$$

Definition 3. *A function* $f\in\mathcal{A}$ *is said to be in the class* $\beta-ST[A,B]$, $\beta\geq 0$, $-1\leq B<A\leq 1$, *if and only if* ,

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right)>\beta\left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}-1\right|,$$

or equivalently,

$$\frac{zf'(z)}{f(z)}\in\beta-P[A,B]. \tag{7}$$

It can easily be seen that $f(z)\in\beta-UCV[A,B] \iff zf'(z)\in\beta-ST[A,B]$. It is clear that $\beta-UCV[1,-1]=\beta-UCV$ and $\beta-ST[1,-1]=\beta-ST$, the well-known classes of β-uniformly convex and corresponding β-starlike functions respectively, introduced by Kanas and Wiśniowska [15,16].

As it is mentioned earlier that a number of well known researchers contributed in the development of this area of study, to mark the importance of our work in this stream of work, we take a quick review of what is done so far. In 1994, Ma and Minda [6] found the maximum bound of Fekete-Szegö functional $\left|a_3-\lambda a_2^2\right|$ for the class UCV of uniformly convex functions whereas Kanas [19] solved the

Fekete-Szegö problem for the functions of class $\mathcal{P}\left(\widetilde{p}_{\beta}\right)$. Further, for the functions of classes $\beta - UCV$ and $\beta - ST$, the same problem was studies by Mishra and Gochhayat [20]. Keeping in view the ongoing research, our aim for this paper is to solve the classical Fekete-Szegö problem for the functions of classes $\beta - P\left[A, B\right]$, $\beta - UCV\left[A, B\right]$ and $\beta - ST\left[A, B\right]$. To prove our results, we need the following lemmas. For the proofs, one may study the reference [6].

Lemma 1. *If $p\left(z\right) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$, then, for any complex number μ,*

$$\left|p_2 - \mu p_1^2\right| \leq 2\max\left\{1, \left|2\mu - 1\right|\right\}$$

and the result is sharp for the functions

$$p_0\left(z\right) = \frac{1+z}{1-z} \quad \text{or} \quad p_*\left(z\right) = \frac{1+z^2}{1-z^2}, \qquad \left(z \in \mathcal{U}\right).$$

Lemma 2. *If $p\left(z\right) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$, then, for any real number v,*

$$\left|p_2 - v p_1^2\right| \leq \begin{cases} -4v + 2, & v \leq 0, \\ 2, & 0 \leq v \leq 1, \\ 4v - 2, & v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p\left(z\right)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then, the equality holds if and only if $p\left(z\right) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if,

$$p\left(z\right) = \left(\frac{1+\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\eta}{2}\right)\frac{1-z}{1+z} \quad \left(0 \leq \eta \leq 1\right),$$

or one of its rotations. If $v = 1$, then, the equality holds if and only if $p\left(z\right)$ is reciprocal of one of the function such that equality holds in the case of $v = 0$. Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$\left|p_2 - v p_1^2\right| + \left|p_1\right|^2 \leq 2 \qquad \left(0 < v \leq \frac{1}{2}\right)$$

and

$$\left|p_2 - v p_1^2\right| + \left(1 - v\right)\left|p_1\right|^2 \leq 2 \quad \left(\frac{1}{2} < v \leq 1\right).$$

2. Main Results

Theorem 1. *Let $p \in \beta - P\left[A, B\right], -1 \leq B < A \leq 1, 0 < \beta < 1$, and of the form $p\left(z\right) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for a complex number μ, we have*

$$\left|p_2 - \mu p_1^2\right| \leq \frac{\left(A-B\right)T^2}{1-\beta^2} \cdot \max\left(1, \left|\frac{\left(B+1\right)T^2}{\left(1-\beta^2\right)} + \mu\frac{\left(A-B\right)T^2}{\left(1-\beta^2\right)} - \frac{T^2}{3} - \frac{2}{3}\right|\right) \tag{8}$$

and for real number μ, we have

$$\left|p_2 - \mu p_1^2\right| \leq \frac{(A-B)T^2}{1-\beta^2}\begin{cases} \frac{2}{3} + \frac{T^2}{3} - \frac{(B+1)T^2}{1-\beta^2} - \frac{\mu(A-B)T^2}{1-\beta^2}, & \mu \leq -\frac{1-\beta^2}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}, \\ 1, & -\frac{1-\beta^2}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)} \leq \mu \\ & \leq \frac{5\left(1-\beta^2\right)}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}, \\ -\frac{2}{3} - \frac{T^2}{3} + \frac{(B+1)T^2}{1-\beta^2} + \frac{\mu(A-B)T^2}{1-\beta^2}, & \mu \geq \frac{5\left(1-\beta^2\right)}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}, \end{cases} \tag{9}$$

where $T = T(\beta) = \frac{2}{\pi}\arccos(\beta)$ *and the equality in (8) holds for the functions*

$$p_1(z) = \frac{\frac{A+1}{1-\beta^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\beta\right)\operatorname{arctanh}\sqrt{z}\right]+1}{\frac{B+1}{1-\beta^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\beta\right)\operatorname{arctanh}\sqrt{z}\right]+1} \tag{10}$$

or

$$p_2(z) = \frac{\frac{A+1}{1-\beta^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\beta\right)\operatorname{arctanh}(z)\right]+1}{\frac{B+1}{1-\beta^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\beta\right)\operatorname{arctanh}(z)\right]+1}. \tag{11}$$

When $\mu < -\frac{1-\beta^2}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}$ *or* $\mu > \frac{5(1-\beta^2)}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}$, *the equality in (9) for the function* $p_1(z)$ *or one of its rotations. If* $-\frac{1-\beta^2}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)} < \mu < \frac{5(1-\beta^2)}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}$, *then the equality in (9) holds for the function* $p_2(z)$ *or one of its rotations. If* $\mu = -\frac{1-\beta^2}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}$, *the equality in (9) holds for the function*

$$p_3(z) = \left(\frac{1+\eta}{2}\right)p_1(z) + \left(\frac{1-\eta}{2}\right)p_1(-z), \quad (0 \le \eta \le 1), \tag{12}$$

or one of its rotations. If $\mu = \frac{5(1-\beta^2)}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}$, *then, the equality in (9) holds for the function* $p(z)$ *which is reciprocal of one of the function such that equality holds in the case for* $\mu = -\frac{1-\beta^2}{3(A-B)T^2} - \frac{B+1}{A-B} + \frac{1-\beta^2}{3(A-B)}$.

Proof. For $h \in P$ and of the form $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, we consider

$$h(z) = \frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. It follows easily that

$$\begin{aligned} w(z) &= \frac{h(z)-1}{h(z)+1} \\ &= \frac{(1+c_1z+c_2z^2+c_3z^3+\cdots)-1}{(1+c_1z+c_2z^2+c_3z^3+\cdots)+1} \\ &= \frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_2c_1 + \frac{1}{8}c_1^3\right)z^3 + \cdots. \end{aligned} \tag{13}$$

Now, if $\widetilde{p}_\beta(w(z)) = 1 + R_1(\beta)w(z) + R_2(\beta)w^2(z) + R_3(\beta)w^3(z) + \cdots$, then from (13), one may have

$$\begin{aligned} \widetilde{p}_\beta(w(z)) &= 1 + R_1(\beta)w(z) + R_2(\beta)w^2(z) + R_3(\beta)w^3(z) + \cdots, \\ &= 1 + R_1(\beta)\left(\frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_2c_1 + \frac{1}{8}c_1^3\right)z^3 + \cdots\right) + \\ &\quad R_2(\beta)\left(\frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_2c_1 + \frac{1}{8}c_1^3\right)z^3 + \cdots\right)^2 + \\ &\quad R_3(\beta)\left(\frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_2c_1 + \frac{1}{8}c_1^3\right)z^3 + \cdots\right)^3 + \cdots, \end{aligned}$$

where $R_1(\beta)$, $R_2(\beta)$ and $R_3(\beta)$ are given by

$$R_1(\beta) = \frac{2T^2}{1-\beta^2},$$

$$R_2(\beta) = \frac{2T^2}{3(1-\beta^2)}\left(2+T^2\right),$$

$$R_3(\beta) = \frac{2T^2}{9(1-\beta^2)}\left(\frac{23}{5}+4T^2+\frac{2}{5}T^4\right),$$

and $T = T(\beta) = \frac{2}{\pi}\arccos(\beta)$, $0<\beta<1$, see [19]. Using these, the above series reduces to

$$\begin{aligned}\widetilde{p}_\beta(w(z)) &= 1+\frac{T^2}{1-\beta^2}c_1z+\frac{T^2}{1-\beta^2}\left(\left(T^2-1\right)\frac{1}{6}c_1^2+c_2\right)z^2+ \\ &\quad \frac{T^2}{1-\beta^2}\left(\frac{1}{9}\left(\frac{2}{5}-\frac{1}{2}T^2+\frac{1}{10}T^4\right)c_1^3-\frac{1}{3}\left(1-T^2\right)c_2c_1+c_3\right)z^3+\cdots. \end{aligned} \tag{14}$$

Since $p\in\beta-P[A,B]$, $0<\beta<1$, so from relations (2), (4) and (14), one may have

$$\begin{aligned} p(z) &= \frac{(A+1)\widetilde{p}_\beta(w(z))-(A-1)}{(B+1)\widetilde{p}_\beta(w(z))-(B-1)} \\ &= 1+\frac{(A-B)}{2}\frac{T^2}{1-\beta^2}c_1z+\frac{(A-B)}{2}\frac{T^2}{1-\beta^2}\left(\frac{T^2c_1^2}{6}-\frac{1}{6}c_1^2-\frac{(B+1)T^2}{2(1-\beta^2)}c_1^2+c_2\right)z^2+\cdots. \end{aligned} \tag{15}$$

If $p(z)=1+\sum_{n=1}^{\infty}p_nz^n$, then equating coefficients of like powers of z, we have

$$\begin{aligned} p_1 &= \frac{(A-B)}{2}\frac{T^2}{1-\beta^2}c_1, \\ p_2 &= \frac{(A-B)}{2}\frac{T^2}{1-\beta^2}\left(\frac{T^2c_1^2}{6}-\frac{1}{6}c_1^2-\frac{(B+1)T^2}{2(1-\beta^2)}c_1^2+c_2\right). \end{aligned}$$

Now for complex number μ, consider

$$p_2-\mu p_1^2 = \frac{(A-B)}{2}\frac{T^2}{1-\beta^2}\left(\frac{T^2c_1^2}{6}-\frac{1}{6}c_1^2-\frac{(B+1)T^2}{2(1-\beta^2)}c_1^2+c_2\right)-\mu\frac{(A-B)^2T^4}{4(1-\beta^2)^2}c_1^2.$$

This implies that

$$\left|p_2-\mu p_1^2\right| = \frac{(A-B)T^2}{2(1-\beta^2)}\left|c_2-c_1^2\left(\frac{1}{6}-\frac{T^2}{6}+\frac{(B+1)T^2}{2(1-\beta^2)}+\mu\frac{(A-B)T^2}{2(1-\beta^2)}\right)\right|. \tag{16}$$

Now using Lemma 1, we have

$$\left|p_2-\mu p_1^2\right| \le \frac{(A-B)T^2}{2(1-\beta^2)}\cdot 2\max(1,|2v-1|),$$

where

$$v=\frac{1}{6}-\frac{T^2}{6}+\frac{(B+1)T^2}{2(1-\beta^2)}+\mu\frac{(A-B)T^2}{2(1-\beta^2)}.$$

This leads us to the required inequality (8) and applying Lemma 2 to the expression (16) for real number μ, we get the required inequality (9). □

For $A=1$, $B=-1$, the above result reduces to the following form.

Corollary 1. *Let $p \in \beta - P[1,-1] = \mathcal{P}\left(\widetilde{p}_\beta\right), 0 < \beta < 1$, and of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for a complex number μ, we have*

$$\left|p_2 - \mu p_1^2\right| \leq \frac{2T^2}{1-\beta^2} \cdot \max\left(1, \left|\mu \frac{2T^2}{(1-\beta^2)} - \frac{T^2}{3} - \frac{2}{3}\right|\right) \tag{17}$$

and for real number μ, we have

$$\left|p_2 - \mu p_1^2\right| \leq \frac{T^2}{1-\beta^2} \begin{cases} \frac{4}{3} + \frac{2}{3}T^2 - \frac{4\mu T^2}{1-\beta^2}, & \mu < -\frac{1-\beta^2}{6T^2} + \frac{(1-\beta^2)}{6}, \\ 2, & -\frac{(1-\beta^2)}{6T^2} + \frac{(1-\beta^2)}{6} \leq \mu \leq \frac{5(1-\beta^2)}{6T^2} + \frac{1-\beta^2}{6}, \\ -\frac{4}{3} - \frac{2}{3}T^2 + \frac{4\mu T^2}{1-\beta^2}, & \mu > \frac{5(1-\beta^2)}{6T^2} + \frac{1-\beta^2}{6}. \end{cases} \tag{18}$$

These results are sharp.

In [3,19], Kanas studied the class $\mathcal{P}\left(\widetilde{p}_\beta\right)$ which consists of functions who take all values from the conic domain Ω_β. Kanas [19] found the bound of Fekete-Szegö functional for the class $\mathcal{P}\left(\widetilde{p}_\beta\right)$ whose particular case for $0 < \beta < 1$ is as follows:

Let $p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \in \mathcal{P}\left(\widetilde{p}_\beta\right), 0 < \beta < 1$. Then, for real number μ, we have

$$\left|b_2 - \mu b_1^2\right| \leq \frac{2T^2}{1-\beta^2} \begin{cases} 1 - \mu\frac{2T^2}{1-\beta^2}, & \mu \leq 0, \\ 1, & \mu \in (0,1], \\ 1 + (\mu - 1)\frac{2T^2}{1-\beta^2}, & \mu \geq 1. \end{cases} \tag{19}$$

For certain values of β and μ, we have the following bounds for $\left|p_2 - \mu p_1^2\right|$, shown in Table 1.

Table 1. Comparison of Fekete-Szegö inequalities.

β	μ	Bound from (18)	Bound from (19)
0.3	3	4.8652	5.51463
0.3	2	2.82267	3.47193
0.5	2	1.84841	2.5939
0.5	−1	2.37422	2.5939
0.7	3	2.28155	3.03221
0.7	−1	1.7698	2.01932

We observe that Corollary 1 gives more refined bounds of Fekete-Szegö functional $\left|p_2 - \mu p_1^2\right|$ for the functions of class $\mathcal{P}\left(\widetilde{p}_\beta\right), 0 < \beta < 1$ as compared to that from (19) as can be seen from above table.

Theorem 2. *Let $f \in \beta - UCV[A,B], -1 \leq B < A \leq 1, 0 \leq \beta < 1$ and of the form (1), then for a real number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)T^2}{12(1-\beta^2)} \begin{cases} \frac{4}{3} + \frac{2T^2}{3} - \frac{2(B+1)T^2}{1-\beta^2} + (2-3\mu)\frac{(A-B)T^2}{1-\beta^2}, & \mu \leq \frac{2}{3} - \frac{2(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}, \\ 2, & \frac{2}{3} - \frac{2(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)} \leq \mu \\ & \leq \frac{2}{3} + \frac{10(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}, \\ -\frac{4}{3} - \frac{2T^2}{3} + \frac{2(B+1)T^2}{1-\beta^2} - (2-3\mu)\frac{(A-B)T^2}{1-\beta^2}, & \mu \geq \frac{2}{3} + \frac{10(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}. \end{cases} \tag{20}$$

Proof. If $f(z) \in \beta - UCV[A,B]$, $-1 \leq B < A \leq 1$, $0 \leq \beta < 1$, then it follows from relations (2), (4), and (6) that

$$\frac{(zf'(z))'}{f'(z)} = \frac{(A+1)\,\widetilde{p}_\beta(w(z)) - (A-1)}{(B+1)\,\widetilde{p}_\beta(w(z)) - (B-1)}.$$

This implies by using (15) that

$$\frac{(zf'(z))'}{f'(z)} = 1 + \frac{(A-B)}{2}\frac{T^2}{1-\beta^2}c_1 z + \frac{(A-B)}{2}\frac{T^2}{1-\beta^2}\left(\frac{T^2c_1^2}{6} - \frac{1}{6}c_1^2 - \frac{(B+1)\,T^2}{2(1-\beta^2)}c_1^2 + c_2\right)z^2 + \cdots. \tag{21}$$

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then one may have

$$\frac{(zf'(z))'}{f'(z)} = 1 + 2a_2 z + \left(6a_3 - 4a_2^2\right)z^2 + \left(12a_4 - 18a_2a_3 + 8a_2^3\right)z^3 + \cdots. \tag{22}$$

From (21) and (22), comparison of like powers of z gives

$$a_2 = \frac{(A-B)\,T^2}{4(1-\beta^2)}c_1, \tag{23}$$

and

$$a_3 = \frac{(A-B)\,T^2}{12(1-\beta^2)}\left(c_2 - \left(\frac{1}{6} - \frac{T^2}{6} + \frac{(B+1)\,T^2}{2(1-\beta^2)} - \frac{(A-B)\,T^2}{2(1-\beta^2)}\right)c_1^2\right). \tag{24}$$

Now, for a real number μ, we consider

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &= \frac{(A-B)\,T^2}{12(1-\beta^2)}\left|c_2 - \left(\frac{1}{6} - \frac{T^2}{6} + \frac{(B+1)\,T^2}{2(1-\beta^2)} - \frac{(A-B)\,T^2}{2(1-\beta^2)}\right)c_1^2 - \mu\frac{3(A-B)}{4}\frac{T^2}{1-\beta^2}c_1^2\right| \\ &= \frac{(A-B)\,T^2}{12(1-\beta^2)}\left|c_2 - \left(\frac{1}{6} - \frac{T^2}{6} + \frac{(B+1)\,T^2}{2(1-\beta^2)} - \frac{(A-B)\,T^2}{2(1-\beta^2)} + \mu\frac{3(A-B)\,T^2}{4(1-\beta^2)}\right)c_1^2\right|.\end{aligned}$$

Now applying Lemma 2, we have the required result. The inequality (20) is sharp and equality holds for $\mu < \frac{2}{3} - \frac{2(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}$ or $\mu > \frac{2}{3} + \frac{10(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}$ when $f(z)$ is $f_1(z)$ or one of its rotations, where $f_1(z)$ is defined such that $\frac{(zf_1'(z))'}{f_1'(z)} = p_1(z)$. If $\frac{2}{3} - \frac{2(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)} < \mu < \frac{2}{3} + \frac{10(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}$, then, the equality holds for the function $f_2(z)$ or one of its rotations, where $f_2(z)$ is defined such that $\frac{(zf_2'(z))'}{f_2'(z)} = p_2(z)$. If $\mu = \frac{2}{3} - \frac{2(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}$, the equality holds for the function $f_3(z)$ or one of its rotations, where $f_3(z)$ is defined such that $\frac{(zf_3'(z))'}{f_3'(z)} = p_3(z)$. If $\mu = \frac{2}{3} + \frac{10(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}$, then, the equality holds for $f(z)$, which is such that $\frac{(zf'(z))'}{f'(z)}$ is reciprocal of one of the function such that equality holds in the case of $\mu = \frac{2}{3} - \frac{2(1-\beta^2)}{9(A-B)T^2} - \frac{2(B+1)}{3(A-B)} + \frac{2(1-\beta^2)}{9(A-B)}$. □

For $A = 1$, $B = -1$, the above result takes the following form which is proved by Mishra and Gochhayat [20].

Corollary 2. *Let $f \in \beta - UCV[1,-1] = \beta - UCV$, $0 \leq \beta < 1$ and of the form (1), then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{T^2}{6(1-\beta^2)}\begin{cases} \frac{4}{3} + \frac{2T^2}{3} + (4-6\mu)\frac{T^2}{1-\beta^2}, & \mu \leq \frac{2}{3} - \frac{1-\beta^2}{9T^2} + \frac{1-\beta^2}{9}, \\ 2, & \frac{2}{3} - \frac{1-\beta^2}{9T^2} + \frac{1-\beta^2}{9} \leq \mu \leq \frac{2}{3} + \frac{5(1-\beta^2)}{9T^2} + \frac{1-\beta^2}{9}, \\ -\frac{4}{3} - \frac{2T^2}{3} - (4-6\mu)\frac{T^2}{1-\beta^2}, & \mu \geq \frac{2}{3} + \frac{5(1-\beta^2)}{9T^2} + \frac{1-\beta^2}{9}. \end{cases}$$

Theorem 3. *If $f(z) \in \beta - ST[A,B]$, $-1 \le B < A \le 1$, $0 < \beta < 1$ and of the form (1), then for a real number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)T^2}{2(1-\beta^2)} \begin{cases} \frac{2}{3} + \frac{T^2}{3} - \frac{(B+1)T^2}{1-\beta^2} + (1-2\mu)\frac{(A-B)T^2}{1-\beta^2}, & \mu \le \frac{1}{2} - \frac{1-\beta^2}{6T^2(A-B)} - \frac{B+1}{2(A-B)} + \frac{1-\beta^2}{6(A-B)}, \\ 1, & \frac{1}{2} - \frac{1-\beta^2}{6T^2(A-B)} - \frac{B+1}{2(A-B)} + \frac{1-\beta^2}{6(A-B)} \le \mu \\ & \le \frac{1}{2} + \frac{5(1-\beta^2)}{6(A-B)T^2} - \frac{B+1}{2(A-B)} + \frac{1-\beta^2}{6(A-B)}, \\ -\frac{2}{3} - \frac{T^2}{3} + \frac{(B+1)T^2}{1-\beta^2} - (1-2\mu)\frac{(A-B)T^2}{1-\beta^2}, & \mu \ge \frac{1}{2} + \frac{5(1-\beta^2)}{6(A-B)T^2} - \frac{B+1}{2(A-B)} + \frac{1-\beta^2}{6(A-B)}. \end{cases}$$

This result is sharp.

Proof. The proof follows similarly as in Theorem 2. □

For $A = 1$, $B = -1$, the above result takes the following form which is proved by Mishra and Gochhayat [20].

Corollary 3. *Let $f \in \beta - ST[1,-1] = \beta - ST$, $0 < \beta < 1$ and of the form (1). Then, for a real number μ,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{T^2}{1-\beta^2} \begin{cases} \frac{2}{3} + \frac{T^2}{3} + (1-2\mu)\frac{2T^2}{1-\beta^2}, & \mu \le \frac{1}{2} - \frac{1-\beta^2}{12T^2} + \frac{1-\beta^2}{12}, \\ 1, & \frac{1}{2} - \frac{1-\beta^2}{12T^2} + \frac{1-\beta^2}{12} \le \mu \le \frac{1}{2} + \frac{5(1-\beta^2)}{12T^2} + \frac{1-\beta^2}{12}, \\ -\frac{2}{3} - \frac{T^2}{3} - (1-2\mu)\frac{2T^2}{1-\beta^2}, & \mu \ge \frac{1}{2} + \frac{5(1-\beta^2)}{12T^2} + \frac{1-\beta^2}{12}. \end{cases}$$

Now we consider the inverse function $\mathcal{F}$ which maps regions presented by (3) to the open unit disk $\mathcal{U}$, defined as $\mathcal{F}(w) = \mathcal{F}(f(z)) = z, z \in \mathcal{U}$ and we find the following coefficient bound for inverse functions. The functions of classes $\beta - UCV[A,B]$ and $\beta - ST[A,B]$ have inverses as they are univalent too.

Theorem 4. *Let $w = f(z) \in \beta - UCV[A,B]$, $-1 \le B < A \le 1$, $0 \le \beta < 1$ and $\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$. Then,*

$$|d_n| \le \frac{(A-B)T^2}{2(1-\beta^2)} \qquad (n = 2,3).$$

Proof. Since $\mathcal{F}(w) = \mathcal{F}(f(z)) = z$, so it is easy to see that

$$d_2 = -a_2, \qquad d_3 = 2a_2^2 - a_3, \qquad d_4 = -a_4 + 5a_2a_3 - 5a_2^3.$$

By using (23) and (24), one can have

$$d_2 = -\frac{(A-B)T^2}{4(1-\beta^2)}c_1 \tag{25}$$

and

$$\begin{aligned} d_3 &= \frac{(A-B)T^2}{12(1-\beta^2)}\left[\left(\frac{1}{6} - \frac{T^2}{6} + \frac{(B+1)T^2}{2(1-\beta^2)} + \frac{(A-B)T^2}{1-\beta^2}\right)c_1^2 - c_2\right] \\ &= \frac{(A-B)T^2}{12(1-\beta^2)}\left(\frac{1}{6} - \frac{T^2}{6} + \frac{(B+1)T^2}{2(1-\beta^2)} + \frac{(A-B)T^2}{1-\beta^2}\right)(c_1^2 - c_2) \\ &\quad - \frac{(A-B)T^2}{12(1-\beta^2)}\left(\frac{11}{6} + \frac{T^2}{6} - \frac{(B+1)T^2}{2(1-\beta^2)} - \frac{(A-B)T^2}{1-\beta^2}\right)c_2 + \frac{(A-B)T^2}{12(1-\beta^2)}c_2. \end{aligned} \tag{26}$$

Now, from (25) and (26), one can have

$$|d_2| \leq \frac{(A-B)\,T^2}{2\,(1-\beta^2)}$$

and

$$\begin{aligned} |d_3| &\leq \frac{(A-B)\,T^2}{12\,(1-\beta^2)}\left|\frac{1}{6}-\frac{T^2}{6}+\frac{(B+1)\,T^2}{2\,(1-\beta^2)}+\frac{(A-B)\,T^2}{1-\beta^2}\right|\left|c_2-c_1^2\right| \\ &\quad +\frac{(A-B)\,T^2}{12\,(1-\beta^2)}\left|\frac{11}{6}+\frac{T^2}{6}-\frac{(B+1)\,T^2}{2\,(1-\beta^2)}-\frac{(A-B)\,T^2}{1-\beta^2}\right||c_2|+\frac{(A-B)\,T^2}{12\,(1-\beta^2)}|c_2| \\ &= \frac{(A-B)\,T^2}{12\,(1-\beta^2)}\left\{|\lambda_1|\left|c_2-c_1^2\right|+|\lambda_2|\,|c_2|+|c_2|\right\}, \end{aligned}$$

where $\lambda_1 = \frac{1}{6}-\frac{T^2}{6}+\frac{(B+1)T^2}{2(1-\beta^2)}+\frac{(A-B)T^2}{(1-\beta^2)}$ and $\lambda_2 = \frac{11}{6}+\frac{T^2}{6}-\frac{(B+1)T^2}{2(1-\beta^2)}-\frac{(A-B)T^2}{(1-\beta^2)}$. We see that $\lambda_i \geq 0$; $i=1,2$ for $-1 \leq B < A \leq 1, 0 \leq \beta < 1$. Thus, the application of bounds $\left|c_2-c_1^2\right| \leq 2$ and $|c_2| \leq 2$ (see Lemma 2 for $v=1$ and $v=0$) gives

$$\begin{aligned} |d_3| &\leq \frac{(A-B)\,T^2}{6\,(1-\beta^2)}\{\lambda_1+\lambda_2+1\} \\ &= \frac{(A-B)\,T^2}{2\,(1-\beta^2)} \end{aligned}$$

□

Theorem 5. *Let $w = f(z) \in \beta - UCV\,[A,B]$, $-1 \leq B < A \leq 1, 0 \leq \beta < 1$ and $\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$. Then, for a real number μ, we have*

$$\left|d_3-\mu d_2^2\right| \leq \frac{(A-B)\,T^2}{12\,(1-\beta^2)}\begin{cases} \frac{4}{3}+\frac{2T^2}{3}-\frac{2(B+1)T^2}{1-\beta^2}-(4-3\mu)\,\frac{(A-B)T^2}{1-\beta^2}, & \mu \geq \frac{4}{3}+\frac{2(1-\beta^2)}{9(A-B)T^2}-\frac{2(1-\beta^2)}{9(A-B)}+\frac{2(B+1)}{3(A-B)}, \\ 2, & \frac{4}{3}-\frac{10(1-\beta^2)}{9(A-B)T^2}-\frac{2(1-\beta^2)}{9(A-B)}+\frac{2(B+1)}{3(A-B)} \leq \mu \\ & \leq \frac{4}{3}+\frac{2(1-\beta^2)}{9(A-B)T^2}-\frac{2(1-\beta^2)}{9(A-B)}+\frac{2(B+1)}{3(A-B)}, \\ -\frac{4}{3}-\frac{2T^2}{3}+\frac{2(B+1)T^2}{1-\beta^2}+(4-3\mu)\,\frac{(A-B)T^2}{1-\beta^2}, & \mu \leq \frac{4}{3}-\frac{10(1-\beta^2)}{9(A-B)T^2}-\frac{2(1-\beta^2)}{9(A-B)}+\frac{2(B+1)}{3(A-B)}. \end{cases}$$

This result is sharp.

Proof. The proof follows directly from (25), (26), and Lemma 2. □

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