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Variational Methods for an Impulsive Fractional Differential Equations with Derivative Term

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Abstract: This paper is devoted to studying the existence of solutions to a class of impulsive fractional differential equations with derivative dependence. The used technical approach is based on variational methods and iterative methods. In addition, an example is given to demonstrate the main results.

Keywords: impulsive fractional differential equations; mountain pass theorem; variational methods; critical points; iterative methods

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1. Introduction and Main Results

In this paper we are interested in the solvability of solutions for the following impulsive fractional differential equations with derivative dependence

$$\begin{cases} {}_tD_T^{\alpha}(a(t){}_0^cD_t^{\alpha}u(t)) + b(t)u(t) = f(t,u(t),{}_0^cD_t^{\alpha}u(t)), \ \ t \neq t_j, a.e\ t \in [0,T] \\ \Delta(a(t){}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j)) = I_j(u(t_j)), \ \ j = 1,2,\cdots,l, \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where $\alpha \in (\frac{1}{2},1]$, $a \in C^1([0,T],\mathbb{R})$ with $a_0 := \operatorname{ess\,inf}_{[0,T]} a(t) > 0$, and ${}_tD_T^{\alpha}$ denotes the right Riemann–Loiuville fractional derivative of order α; $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, the operator Δ is defined as $\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j)) = {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^-)$, where ${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+) = \lim_{t\to t_j^+}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t))$, ${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^-) = \lim_{t\to t_j^-}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t))$, and ${}_tD_T^{\alpha-1}$ is the right Riemann–Liouville fractional derivative of order $1-\alpha$; ${}_0^cD_t^{\alpha}$ is the left Caputo fractional derivatives of order α. Suppose that:

(C1) $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and $I_j (j = 1,2,\cdots,l) : \mathbb{R} \to \mathbb{R}$ are continuous functions, $b \in C([0,T])$ and there exist positive constants b_1, b_2 such that $0 < b_1 \leq b(t) \leq b_2$.

Fractional calculus is a generalization of the traditional calculus to arbitrary noninteger order. Fractional differential equations (FDEs) have played an important role in various fields [1,2] such as electricity, biology, electrical networks, mechanics, chemistry, rheology and probability, etc., With the help of fractional calculus, the natural phenomena and mathematical model can be more accurately described. As a consequence there was a rapid development of the theory and application concern with fractional differential equations. In particular, the solvability, attractivity, and multiplicity of solutions for FDEs have been greatly discussed. We refer to the monographs of Podlubny [1], Kilbas et al. [2], Diethelm [3], Zhou [4], the papers [5–19] and the references therein.

More recently, starting with the pioneering work of Jiao and Zhou [20], the variational methods have been applied to investigate the existence and multiplicity of solutions for fractional differential equations, which possess the variational structures in some suitable functional spaces under certain boundary conditions in many papers, see [21–30] and the references therein. For instance, Sun and Zhang [21] by establishing a variational structure and applying Mountain Pass theorem and iterative technique, investigated the solvability of solutions to the following nonlinear fractional differential equations

$$\begin{cases} \frac{d}{dt}(p\,{}_0D_t^{-\alpha}(u'(t)) + q\,{}_tD_1^{-\alpha}(u'(t))) + f(t,u(t)) = 0, \;\; t \in [0,1], \\ u(0) = u(1) = 0, \end{cases}$$

where $\alpha \in (0,1], 0 < p = 1-q < 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, ${}_0D_t^{-\alpha}$ and ${}_tD_1^{-\alpha}$ denote left and right Riemann-Loiuville fractional integrals of order α respectively. In case $\alpha \in (\frac{1}{2}, 1]$, Galewski and Molica Bisci in [22] by using variational methods, proved that the following fractional boundary problems

$$\begin{cases} \frac{d}{dt}\left({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\right) + f(t,u(t)) = 0, \text{ a.e. } t \in [0,T], \\ u(0) = u(T) = 0 \end{cases}$$

has at least a nontrivial solution under some suitable conditions.

On the other hand, boundary value problems for impulsive differential equations are intensively discussed. Such problems arising from the real world appear in mathematical models with sudden and discontinuous changes of their states in biology, population dynamics, physics, engineering, etc. [31,32]. For their significance, it is very important and interesting to discuss the solvability of solutions for impulsive differential equations. Recently, the existence and multiplicity of solutions for impulsive FDEs are treated by using topological methods, critical point theory and the coincidence degree theory, for example see [33–43] and the references therein. Taking an impulsive fractional Dirichlet problem as a model, Bonanno et al. [33], and Rodríguez-López and Tersian [34] by applying variational methods, investigated the existence results of at least one and three solutions for the following impulsive fractional boundary value problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u), \;\; 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), \;\; j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases}$$

where $\lambda, \mu \in (0, +\infty)$.

Motivated by [21,33,44], in this paper we shall deal with the solvability of solutions for the problem (1) by using the variational methods and iterative methods. The characteristic of problem (1) is the presence of fractional derivative in the nonlinearity term. To the best of our knowledge, there is no result concerned with the solvability of solutions for impulsive FDEs, such as problem (1), by applying the variational methods and iterative methods. We know, contrary to those equations in [33,34,39,40,42,43,45], the problem (1) is of no the variational structure and it cannot be studied by directly using the well-developed critical point theory. Furthermore, due to the appearance of left and right Riemann–Liouville fractional integral and impulsive effect, the calculation of problem (1) will be more complicated.

Throughout the paper, we assume that $f(t,u,v)$ and $I_j(j = 1,2,\cdots,l)$ satisfies the following conditions:

(C2) $\lim\limits_{u\to 0} \frac{f(t,u,v)}{|u|} = 0$ uniformly for all $t \in [0,T]$ and $v \in \mathbb{R}$ and $f(t,u,0) \neq 0$ for $t \in [0,T]$ and $u \in \mathbb{R}$.

(C3) There exists a constant $\vartheta > 2$ such that $\lim\limits_{u\to +\infty} \frac{f(t,u,v)}{|u|^{\vartheta-1}} = 0$ uniformly for all $t \in [0,T]$ and $v \in \mathbb{R}$.

(C4) There are constants $\mu > 2$ and $\zeta > 0$ such that

$$0 < \mu F(t,u,v) \leq uf(t,u,v), \forall t \in [0,T], |u| \geq \zeta, v \in \mathbb{R}.$$

(C5) There exists two constants $k_1, k_2 > 0$ such that

$$F(t,u,v) := \int_0^u f(t,s,v)ds \geq k_1|u|^\mu - k_2,\ \forall t \in [0,T], u, v \in \mathbb{R}.$$

(I1) There is a positive constant $\beta < \mu$ such that

$$0 < uI_j(u) \leq \beta \int_0^{u(t_j)} I_j(s)ds,\ \ \forall u \in \mathbb{R} \setminus \{0\},\ j = 1, 2, \cdots, l.$$

Remark 1. *According to assumptions (C2) and (C3), it is easy to obtain that for given $\varepsilon > 0$, there exists a positive constant $k(\varepsilon)$ independent of ω, such that*

$$|f(t,u,v)| \leq 2\varepsilon|u| + k(\varepsilon)\vartheta|u|^{\vartheta-1},\ \forall t \in [0,T], u \in \mathbb{R}.$$

Due to the fact that problem (1) is not variational, according to the idea be borrowed from [21,44], we will deal with a family of impulsive fractional boundary value problem without the fractional derivative of the solution; that is, we consider the following problems:

$$\begin{cases} {}_tD_T^\alpha(a(t){}_0^cD_t^\alpha u(t)) + b(t)u(t) = f(t,u(t),{}_0^cD_t^\alpha\omega(t)),\ \ t \neq t_j, a.e\ t \in [0,T] \\ \Delta({}_tD_T^{\alpha-1}(a(t_j){}_0^cD_t^\alpha u)(t_j)) = I_j(u(t_j)),\ \ j = 1, 2, \cdots, l, \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

For each $\omega \in E_0^\alpha$, where the space E_0^α will be introduced in Section 2. Obviously, problem (2) is of the variational structure and can be solved by applying the variational methods. Hence, for any $\omega \in E_0^\alpha$, we can deduce a unique solution $u_\omega \in E_0^\alpha$ with some bounds. Furthermore, we can prove that there exists a solution for problem (1) via iterative methods. Now let us give the preliminary result of the present paper:

Theorem 1. *Let $\omega \in E_0^\alpha$. Suppose that the hypotheses (C1)–(C5) and (I1) are satisfied; then there exist positive constants A_1 and A_2 independent of ω such that problem (2) has at least one solution u_ω satisfying $A_1 \leq \|u_\omega\|_\alpha \leq A_2$.*

We will established the main results of the paper by an iterative method which depends on the solvability of problem (2). To obtain the solvability of problem (1), we also need the following assumptions:

(C6) There exist constants $L_1, L_2 > 0$ and $\xi > 0$ such that the function f satisfies the following Lipschitz conditions:

$$|f(t,u_2,v_2) - f(t,u_1,v_1)| \leq L_1|u_2 - u_1| + L_2|v_2 - v_1|,\ \ \forall t \in [0,T], u_1, u_2 \in [-\xi, \xi], v_1, v_2 \in \mathbb{R}.$$

(I2) There exist constants $\rho_j > 0, j = 1, 2, \cdots, l$ such that

$$|I_j(x) - I_j(y)| \leq \rho_j|x - y|,\ \forall x, y \in [-\xi, \xi].$$

Theorem 2. *Suppose that the hypotheses of Theorem 1 are satisfied. In addition, if (C6) and (I2) hold with $L^* < 1$, we can obtain the solution u_ω of problem (2) is unique in E_0^α, where*

$$L^* := \frac{L_1 T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2} + \frac{T^{(2\alpha-1)}}{[\Gamma(\alpha)]^2 a_0(2\alpha-1)} \cdot \sum_{j=1}^{l} \rho_j < 1.$$

Theorem 3. *Assume conditions (C1)–(C6) and (I1), (I2) hold. Then problem (1) has at least one nontrivial solution provided*

$$\widetilde{L} := \frac{L_2 T^{\alpha}(2\alpha-1)\Gamma(\alpha+1)[\Gamma(\alpha)]^2}{a_0(2\alpha-1)[\Gamma(\alpha)\Gamma(\alpha+1)]^2 - L_1 T^{2\alpha}(2\alpha-1)[\Gamma(\alpha)]^2 - T^{2\alpha-1}[\Gamma(\alpha+1)]^2\sum_{j=1}^{l}\rho_j} \in (0,1).$$

The article is organized as follows. In Section 2, we shall give some definitions and lemmas that will be helpful to discuss our main results. In Section 3, we will prove the solvability of the problem (2) and the existence of at least one nontrivial solution to the problem (1).

2. Preliminaries

In this paper we need the following definitions and properties of the fractional calculus. Let ${}_0D_t^{-\gamma}u(t)$ and ${}_tD_T^{-\gamma}u(t)$ be the left and right fractional integrals of order γ as follows

$${}_0D_t^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u(s)ds, \qquad {}_tD_T^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_t^T (t-s)^{\gamma-1}fu(s)ds, \ \ \gamma > 0.$$

Definition 1 (see [2,4]). *Let f be a function defined on $[a,b]$. Then the left and right Riemann–Liouville fractional derivatives of order γ for function f denoted by ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$, are represented by*

$${}_aD_t^{\gamma}f(t) = \frac{d^n}{dt^n}{}_aD_t^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\gamma-1}f(s)ds,$$

and

$${}_tD_b^{\gamma}f(t) = (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\gamma-n}f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_t^b (t-s)^{n-\gamma-1}f(s)ds,$$

for every $t \in [a,b]$, where $n-1 \leq \gamma < n$ and $n \in \mathbf{N}$.

From [2,4], we have

Proposition 1 (See [2,4]). *If $f \in L^p([a,b],\mathbb{R}^N), g \in L^q([a,b],\mathbb{R}^N)$ and $p \leq 1, q \leq 1, 1/p + 1/q \leq 1+\gamma$ or $p \neq 1, q \neq 1, 1/p+1/q = 1+\gamma$. Then*

$$\int_a^b [{}_aD_t^{-\gamma}f(t)]g(t)dt = \int_a^b [{}_tD_b^{-\gamma}g(t)]f(t)dt, \qquad \gamma > 0.$$

For any fixed $t \in [0,T]$ and $1 \leq p < \infty$, let

$$\|x\|_{\infty} = \max_{t\in[0,T]}|x(t)|, \quad \|x\|_{L^p} = \left(\int_0^T |x(s)|^p ds\right)^{1/p}. \tag{3}$$

Definition 2. *Let $0 < \alpha \leq 1$. Then the fractional derivative space E_0^{α} is defined by the closure of $C_0^{\infty}([0,T],\mathbb{R})$ that is*

$$E_0^{\alpha} = \overline{C_0^{\infty}([0,T],\mathbb{R})}$$

with respect to the weighted norm

$$\|u\|_{\alpha} = \left(\int_0^T a(t)|{}_0^cD_t^{\alpha}u(t)|^2dt + \int_0^T |u(t)|^2dt\right)^{1/2}, \ \ \forall u \in E_0^{\alpha}. \tag{4}$$

From [20], E_0^{α} is a reflexive and a separable Banach space. Furthermore, E_0^{α} is the space of functions $u \in L^2([0,T],\mathbb{R})$ with an α-order Caputo fractional derivative ${}_0^cD_t^{\alpha}u \in L^2([0,T],\mathbb{R})$ and $u(0) = u(T) = 0$. For $u \in E_0^{\alpha}$, we have (see [8,33])

$${}_0^cD_t^{\alpha}u(t) = {}_0D_t^{\alpha}u(t), \ \ {}_t^cD_T^{\alpha}u(t) = {}_tD_T^{\alpha}u(t).$$

Lemma 1 (See [24]). *Let $0 < \alpha \leq 1$. For any $u \in E_0^{\alpha}$, one has*

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)\sqrt{a_0}} \left(\int_0^T a(t)|{}_0^cD_t^\alpha u(t)|^2 dt\right)^{1/2}, \tag{5}$$

moreover, if $\alpha > \frac{1}{2}$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}} \left(\int_0^T a(t)|{}_0^cD_t^\alpha u(t)|^2\right)^{1/2}. \tag{6}$$

Note that if $b \in C([0,T])$ is such that $0 < b_1 \leq b(t) \leq b_2$, and by (i) of Lemma 1, we can consider E_0^α with the following norm

$$\|u\|_{b,\alpha}^2 = \int_0^T (a(t)|{}_0^cD_t^\alpha u(t)|^2 + b(t)|u(t)|^2)dt, \quad \forall u \in E_0^\alpha, \tag{7}$$

which is equivalent to (4) and we still denote by $\|\cdot\|_\alpha$ for short.

Proposition 2 ([4], Proposition 5.6)**.** *Assume that $\frac{1}{2} < \alpha \leq 1$ and the sequence $\{u_n\}$ converges weakly to u in E_0^α, i.e., $u_n \rightharpoonup u$. Then $u_n \to u$ in $C([0,T])$, that is, $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.*

Definition 3. *A function $u \in E_0^\alpha$ is called a solution of problem (1), if*

- *(i) the limits ${}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+)$, ${}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-), j = 1,\cdots,l$, exist and satisfy the following impulsive condition*

$$\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j)) = {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-) = I_j(u(t_j)).$$

- *(ii) u satisfies the Equation (1) a.e. on $[0,T] \setminus \{t_1, t_2, \cdots, t_l\}$, and the boundary condition $u(0) = u(T) = 0$;*

Definition 4. *A function $u \in E_0^\alpha$ is said to be a weak solution of problem (1), if*

$$\int_0^T (a(t){}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha x(t) + b(t)u(t)x(t))dt + \sum_{j=1}^l I_j(u(t_j))x(t_j) - \int_0^T f(t,u(t),{}_0^cD_t^\alpha u(t))x(t)dt = 0$$

for every $x \in E_0^\alpha$.

Associated to the boundary value problem (2) for given $\omega \in E_0^\alpha$ we have the functional $\Phi_\omega : E_0^\alpha \to \mathbb{R}$ defined by

$$\Phi_\omega(u) = \frac{1}{2}\int_0^T (a(t)|{}_0^cD_t^\alpha u(t)|^2 + b(t)|u(t)|^2)dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u(t),{}_0^cD_t^\alpha \omega(t))dt, \tag{8}$$

where $F(t,u,v) = \int_0^u f(t,s,v)ds$ and $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Obviously, using the hypothesis (C1) we deduce that Φ_ω is continuous, differentiable and

$$< \Phi'_\omega(u), x > = \int_0^T (a(t){}_0^cD_t^\alpha u(t){}_0^cD_t^\alpha x(t) + b(t)u(t)x(t))dt + \sum_{j=1}^l I_j(u(t_j))x(t_j) - \int_0^T f(t,u(t),{}_0^cD_t^\alpha \omega(t))x(t)dt \tag{9}$$

for any $x \in E_0^\alpha$. Moreover, the critical point of Φ_ω is a solution of the problem (2).

Lemma 2 (see [46])**.** *Let E be a real Banach space. If any sequence $\{u_n\} \subset E$ for which $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence of $\{u_n\}$. Then we say Φ satisfies Palais-Smale(PS) condition in E.*

Lemma 3 (see [46])**.** *Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. Suppose that $\Phi(0) = 0$ and*

- *(i) there exist constants $\rho, \xi_0 > 0$ such that $\Phi|_{\partial B_\rho(0)} \geq \xi_0$, and*
- *(ii) there exists an $e \in E \backslash B_\rho(0)$ such that $\Phi(e) \leq 0$.*

Then, Φ possesses a critical value $c \geq \xi_0$. Moreover, c can be characterized as

$$c = \inf_{\gamma \in \Lambda} \max_{s \in [0,1]} \Phi(\gamma(s)),$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0 and $\Lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$.

3. Proof of Theorems 1–3

Proof of Theorem 1. The proof will be divided into four steps. We prove that the energy functional Φ_ω has the mountain pass geometric structure, that it is satisfies the (PS)-condition and finally that the obtained solutions have the uniform bounds.

- **(I)** *For $\omega \in E_0^\alpha$, we show that there exist positive numbers ρ and ξ_0 such that for $\|u\|_\alpha = \rho, \Phi_\omega(u) \geq \xi_0 > 0$ uniformly for $\omega \in E_0^\alpha$.*

In fact, By (C2), (C3) and Remark 1, we have for any $u \in E_0^\alpha$

$$|F(t, u, v)| \leq \varepsilon |u|^2 + k(\varepsilon)|u|^\vartheta. \tag{10}$$

Thanks to (I1), one has

$$\sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s) ds \geq 0. \tag{11}$$

Thus for any $u \in E_0^\alpha$, by (10), (11) and Lemma 1, one has

$$\begin{aligned}
\Phi_\omega(u) &= \frac{1}{2}\|u\|_\alpha^2 + \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s) ds - \int_0^T F(t, u(t), {}_0^cD_t^\alpha \omega(t)) dt \\
&\geq \frac{1}{2}\|u\|_\alpha^2 - \frac{\varepsilon T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2} \int_0^T a(t) |{}_0^cD_t^\alpha u(t)|^2 dt \\
&\quad - \frac{k(\varepsilon) T^{(\alpha-\frac{1}{2})\vartheta+1}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta} \left(\int_0^T a(t) |{}_0^cD_t^\alpha u(t)|^2 dt \right)^{\vartheta/2} \\
&\geq \left(\frac{1}{2} - \frac{\varepsilon T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2} \right) \|u\|_\alpha^2 - \frac{k(\varepsilon) T^{(\alpha-\frac{1}{2})\vartheta+1}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta} \|u\|_\alpha^\vartheta \\
&= \|u\|_\alpha^2 \left(\frac{1}{2} - \frac{\varepsilon T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2} - \frac{k(\varepsilon) T^{(\alpha-\frac{1}{2})\vartheta+1}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta} \|u\|_\alpha^{\vartheta-2} \right).
\end{aligned} \tag{12}$$

Choosing $\varepsilon = a_0[\Gamma(\alpha+1)]^2/(4T^{2\alpha}) := \varepsilon_0$, and let $\|u\|_\alpha = \rho > 0$. We may take ρ sufficiently small such that

$$\frac{1}{4} - \frac{k(\varepsilon_0) T^{(\alpha-\frac{1}{2})\vartheta+1} \rho^{\vartheta-2}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta} =: \vartheta^* > 0.$$

Hence $\Phi_\omega(u) \geq \rho^2 \vartheta^* := \xi_0 > 0$. This implies that Φ_ω satisfies assumption (i) of Lemma 3.

(II) *Fix $\omega \in E_0^\alpha$. We show that there exists $e \in E_0^\alpha$ such that $\|e\|_\alpha > \rho$ and $\Phi_\omega(e) < 0$, where ρ is given in (I).*

Using (I2), we obtain that there is $\beta_0 > 0$ such that the following inequalities

$$\int_0^{u(t_j)} I_j(s)ds \le \beta_0 |u|^\beta, \;\; \forall u \in \mathbb{R}, \; j = 1, 2, \cdots, l, \tag{13}$$

hold. In fact, for any $x \in \mathbb{R} \setminus \{0\}$ and set $\varphi(t) = I^*(tx) = \int_0^{tx} I(s)ds$, then

$$\varphi'(t) = I(tx)x = \frac{1}{t} I(tx)(tx) \le \frac{\beta}{t} \int_0^{tx} I(s)ds = \frac{\beta}{t}\varphi(t),$$

which implies that

$$\int_1^t \frac{d\varphi}{\varphi(s)} \le \beta \int_1^t \frac{ds}{s}.$$

So we have

$$\varphi(t) \le |t|^\beta \int_0^x I(s)ds,$$

and

$$\int_0^x I(s)ds = I^*(x) = I^*(|x| \cdot \frac{x}{|x|}) \le I^*(\frac{x}{|x|})|x|^\beta < \beta_0 |x|^\beta,$$

where $\beta_0 := \sup_{x \in \mathbb{R}\setminus\{0\}} I^*(\frac{x}{|x|})$. This implies (13) is satisfied.

From (C5) and (13), we obtain that for $\tau > 1$ and $u^* \in E_\alpha$

$$\begin{aligned} \Phi_\omega(\tau u^*) &= \frac{\tau^2}{2}\|u^*\|_\alpha^2 + \sum_{j=1}^{l} \int_0^{\tau u^*(t_j)} I_j(s)ds - \int_0^T F(t, \tau u^*(t), {}_0^cD_t^\alpha \omega(t))dt \\ &\le \frac{\tau^2}{2} + l\beta_0 |\tau|^\beta \|u^*\|_\infty^\beta - k_1|\tau|^\mu \int_0^T |u^*(t)|^\mu dt - k_2 T \\ &\le \frac{\tau^2}{2} + K_1|\tau|^\beta - K_2|\tau|^\mu - k_2 T, \end{aligned} \tag{14}$$

where K_1, K_2 are positive constants independent of ω. Choosing $u^* \in E_0^\alpha$ with $\|u^*\|_\alpha = 1$. Since $\mu > \beta$, (14) implies that there is large enough $\tau_1 \neq 0$ such that $\|e\|_\alpha > \rho$ and $\Phi_\omega(e) < 0$ if we take $e = \tau_1 u^*$. So Φ_ω satisfies assumption (ii) of Lemma 3. The energy functional Φ_ω has the mountain pass geometric structure.

(III) *Fix $\omega \in E_0^\alpha$. We prove that Φ_ω satisfies the Palais-Smale condition on the space E_0^α.*

For any sequence $\{u_n\}_n \subset E_0^\alpha$ such that $\{\Phi_\omega(u_n)\}$ is a bounded sequence and $\Phi'_\omega(u_n) \to 0$ as $n \to \infty$. Then, there are two positive constants $K_3, K_4 > 0$ such that for n sufficiently large

$$|\Phi_\omega(u_n)| \le K_3, \;\; |\Phi'_\omega(u_n)| \le K_4.$$

Thus, it follows from (C4) and (I1) that

$$\begin{aligned} \left(\tfrac{1}{2} - \tfrac{1}{\mu}\right)\|u_n\|_\alpha^2 \;&=\; \Phi_\omega(u_n) - \tfrac{1}{\mu}\Phi'_\omega(u_n)u_n - \sum_{j=1}^{l}\left(\int_0^{u_n(t_j)} I_j(s)ds - \tfrac{1}{\mu} I_j(u_n(t_j))u_n(t_j)\right) \\ &\quad + \int_0^T \left(F(t, u_n(t), {}_0^cD_t^\alpha\omega(t)) - \tfrac{1}{\mu} f(t, u_n(t), {}_0^cD_t^\alpha\omega(t))u_n(t)\right) dt \\ &\le\; \Phi_\omega(u_n) - \tfrac{1}{\mu}\Phi'_\omega(u_n)u_n - \sum_{j=1}^{l}\left(\int_0^{u_n(t_j)} I_j(s)ds - \tfrac{1}{\beta} I_j(u_n(t_j))u_n(t_j)\right) \\ &\quad + \int_{\{|u_n| \ge \zeta\}} \left(F(t, u_n(t), {}_0^cD_t^\alpha\omega(t)) - \tfrac{1}{\mu} f(t, u_n(t), {}_0^cD_t^\alpha\omega(t))u_n(t)\right) dt \\ &\quad + \int_{\{|u_n| \le \zeta\}} \left(F(t, u_n(t), {}_0^cD_t^\alpha\omega(t)) - \tfrac{1}{\mu} f(t, u_n(t), {}_0^cD_t^\alpha\omega(t))u_n(t)\right) dt \\ &\le\; K_5 + \frac{T^{\alpha - \frac{1}{2}} K_4}{\mu\Gamma(\alpha)\sqrt{a_0(2\alpha - 1)}}\|u_n\|_\alpha, \end{aligned}$$

where K_5 is a positive constant independent of ω and n. Therefore, $\{u_n\}$ is bounded in E_0^α.

Since E_0^α is a reflexive Banach space. It follows from Lemma 1 and Proposition 2 that $\{u_n\}$ is bounded in $C([0,T])$, and $\lim_{n\to\infty}\|u_n-u\|_\infty=0$. Hence, we can assume that there exists some $u\in E_0^\alpha$ such that the sequence $u_n \rightharpoonup u$ in E_0^α, and $u_n\to u$ in $L^2(0,T)$ and

$$\{u_n\} \text{ converges uniformly to } u \text{ on } [0,T].$$

Notice that

$$(\Phi'_\omega(u_n)-\Phi'_\omega(u_m))(u_n-u_m)=\|u_n-u_m\|_\alpha^2+\sum_{j=1}^{l}[I_j(u_n(t_j))-I_j(u_m(t_j))](u_n(t_j)-u_m(t_j)) \\ -\int_0^T[f(t,u_n(t),{}_0^cD_t^\alpha\omega(t))-f(t,u_m(t),{}_0^cD_t^\alpha\omega(t))]dt. \tag{15}$$

Since

$$\begin{aligned} I_{n,m}: &= [I_j(u_n(t_j))-I_j(u_m(t_j))](u_n(t_j)-u_m(t_j)) \\ &= [(I_j(u_n(t_j))-I_j(u^*))-(I_j(u_m(t_j))-I_j(u^*))](u_n(t_j)-u_m(t_j)) \\ &\le [|I_j(u_n(t_j))-I_j(u^*)|+|I_j(u_m(t_j))-I_j(u^*)|]\cdot\|u_n-u_m\|_\infty, \end{aligned}$$

which implies the second term of (15)

$$[I_j(u_n(t_j))-I_j(u_m(t_j))](u_n(t_j)-u_m(t_j))\to 0$$

as $n,m\to\infty$. According to Remark 1, we get

$$\begin{aligned} A_{n,m}: &= \int_0^T[f(t,u_n(t),{}_0^cD_t^\alpha\omega(t))-f(t,u_m(t),{}_0^cD_t^\alpha\omega(t))](u_n(t_j)-u(t_j))dt \\ &\le \int_0^T[|f(t,u_n(t),{}_0^cD_t^\alpha\omega(t))|+|f(t,u_m(t),{}_0^cD_t^\alpha\omega(t))|]|u_n(t_j)-u_m(t_j)|dt \\ &\le k_0\int_0^T\left(2\varepsilon|u_n|+k(\varepsilon)\vartheta|u_n|^{\vartheta-1}+2\varepsilon|u_m|+k(\varepsilon)\vartheta|u_m|^{\vartheta-1}\right)|u_n(t_j)-u_m(t_j)|dt\to 0 \end{aligned}$$

as $n,m\to\infty$. Thus, the third term of (15)

$$\int_0^T[f(t,u_n(t),{}_0^cD_t^\alpha\omega(t))-f(t,u_m(t),{}_0^cD_t^\alpha\omega(t))]dt\to 0$$

as $n,m\to\infty$. Since

$$\begin{aligned} (\Phi'_\omega(u_n)-\Phi'_\omega(u_m))(u_n-u_m) &= \Phi'_\omega(u_n)(u_n-u_m)-\Phi'_\omega(u))(u_n-u_m) \\ &\le |\Phi'_\omega(u_n)|\|u_n-u_m\|_\infty-\Phi'_\omega(u_m))(u_n-u_m)\to 0 \end{aligned}$$

as $n,m\to\infty$.

Consequently,

$$\|u_n-u_m\|_\alpha=(\Phi'_\omega(u_n)-\Phi'_\omega(u_m))(u_n-u_m)-I_{n,m}+A_{n,m}\to 0$$

as $n,m\to\infty$. That is, $\{u_n\}$ is a Cauchy sequence in E_0^α. This implies that $\{u_n\}$ has a convergent sequence in E_0^α. Thus Φ_ω satisfies (PS) condition.

Obviously, $\Phi_\omega(0)=0$. Therefore, applying Lemma 3, we deduce that Φ_ω admits a nontrivial critical points u_ω in E^α with

$$\Phi'_\omega(u_\omega) = 0, \;\; c_\omega = \Phi_\omega(u_\omega) = \inf_{\gamma\in\Lambda}\max_{s\in[0,1]} \Phi_\omega(\gamma(s)) > \Phi_\omega(0) = 0, \tag{16}$$

where $\Lambda = \{\gamma \in C([0,1], E^\alpha) : \gamma(0) = 0, \gamma(1) = e\}$ and $e = \tau_1 u^*$ has been given in (II). So problem (2) has at least one weak solution $u_\omega \neq 0$ for any $\omega \in E^\alpha$.

(IV) *Fix* $\omega \in E_0^\alpha$. *We prove that there exist positive constants* A_1 *and* A_2 *independent of* ω *such that* $A_1 \le \|u_\omega\|_\alpha \le A_2$.

Since u_ω is the solution of problem (2), then one has

$$\|u_\omega\|_\alpha^2 + \sum_{j=1}^{l} I_j(u_\omega(t_j))u_\omega(t_j) = \int_0^T f(t, u_\omega(t), {}_0^cD_t^\alpha \omega(t))u_\omega(t)dt.$$

By Remark 1, (I1) and Lemma 1, we have

$$\begin{aligned}\|u_\omega\|_\alpha^2 &\le \|u_\omega\|_\alpha^2 + \sum_{j=1}^{l} I_j(u_\omega(t_j))u_\omega(t_j)\\ &= \int_0^T f(t, u_\omega(t), {}_0^cD_t^\alpha \omega(t))u_\omega(t)dt\\ &\le 2\varepsilon \int_0^T |u_\omega(t)|^2 dt + k(\varepsilon)\int_0^T \vartheta |u_\omega(t)|^\vartheta dt\\ &\le \frac{2\varepsilon T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2}\int_0^T a(t)|{}_0^cD_t^\alpha u_\omega(t)|^2 dt + \frac{k(\varepsilon)\vartheta T^{(\alpha-\frac{1}{2})\vartheta+1}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta}\left(\int_0^T a(t)|{}_0^cD_t^\alpha u_\omega(t)|^2 dt\right)^{\vartheta/2}\\ &\le \frac{2\varepsilon T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2}\|u_\omega\|_\alpha^2 + \frac{k(\varepsilon)\vartheta T^{(\alpha-\frac{1}{2})\vartheta+1}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta}\|u_\omega\|_\alpha^{\vartheta/2},\end{aligned}$$

for any $\varepsilon > 0$. So

$$\left(1 - \frac{2\varepsilon T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2}\right)\|u_\omega\|_\alpha^2 \le \frac{k(\varepsilon)\vartheta T^{(\alpha-\frac{1}{2})\vartheta+1}}{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta}\|u_\omega\|_\alpha^{\vartheta/2}.$$

Combined with $\vartheta > 2$, by choosing $\varepsilon > 0$ small enough such that $a_0[\Gamma(\alpha+1)]^2 - 2\varepsilon T^{2\alpha} >$, we obtain

$$\|u_\omega\|_\alpha \ge \left(\frac{[\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}]^\vartheta(a_0[\Gamma(\alpha+1)]^2 - 2\varepsilon T^{2\alpha})}{a_0 k(\varepsilon)\vartheta T^{(\alpha-\frac{1}{2})\vartheta+1}[\Gamma(\alpha+1)]^2}\right)^{1/(\vartheta-2)} := A_1 > 0. \tag{17}$$

Notice that u_ω satisfying (16), then taking a special pass $\gamma^*(s) = su^*$, we have

$$\begin{aligned}\left(\frac{\mu}{2} - 1\right)\|u_\omega\|_\alpha^2 &\le \mu\Phi_\omega - < \Phi'_\omega(u_\omega), u_\omega > + K_6\\ &= \mu \inf_{\gamma\in\Lambda}\max_{s\in[0,1]} \Phi_\omega(\gamma^*(s)) + K_6\\ &\le \mu \max_{s\in[0,1]} \Phi_\omega(su^*) + K_6\\ &\le \mu\left(\frac{s^2}{2} + \sum_{j=1}^{l}\int_0^{su^*(t_j)} I_j(t)dt - k_1|s|^\mu \int_0^T |u^*|^\mu dt + k_2 T\right) + K_6\\ &\le \mu\left(\frac{s^2}{2} + l\beta_0\|u^*\|_\infty^\beta |s|^\beta - k_1|s|^\mu\int_0^T |u^*|^\mu dt\right) + K_7,\end{aligned} \tag{18}$$

where K_6, K_7 denote positive constants. Let

$$h(t) = \frac{t^2}{2} + l\beta_0 \|u^*\|_\infty^\beta |t|^\beta - k_1|t|^\mu \int_0^T |u^*|^\mu d\tau, \ \ t \geq 0. \tag{19}$$

Since $\mu > \beta$, then the function $h(t)$ can achieve its maximum at some $t_0 > 0$ and the value $\mu h(t_0) + K_7$ can be taken as $A_* > 0$. Obviously it is independent of ω. Then (18) implies that there exists $A_2 := \sqrt{2A_*/(\mu-2)}$ independent of ω such that $\|u_\omega\|_\alpha \leq A_2$. Therefore, this completes the proof of Theorem 1. □

Proof of Theorem 2. It follows from Theorem 1 that there exists at least one weak solution u_ω of problem (2). Next, fix $\omega \in E_0^\alpha$ we show that the solution of problem (2) is unique. In fact, if there are two different solutions u_1 and u_2 satisfying the first equation in problem (2) a.e.$t \in [0, T]$. Then

$$\begin{aligned}&\int_0^T [a(t)\,{}_0^cD_t^\alpha u_2(t)\,{}_0^cD_t^\alpha (u_2-u_1) + b(t)u_2(t)(u_2-u_1)]dt \\ &= \int_0^T f(t, u_2(t), {}_0^cD_t^\alpha \omega(t))(u_2-u_1)dt - \sum_{j=1}^l I_j(u_2(t_j))(u_2(t_j)-u_1(t_j)),\end{aligned}$$

and

$$\begin{aligned}&\int_0^T [a(t)\,{}_0^cD_t^\alpha u_1(t)\,{}_0^cD_t^\alpha (u_2-u_1) + b(t)u_1(t)(u_2-u_1)]dt \\ &= \int_0^T f(t, u_1(t), {}_0^cD_t^\alpha \omega(t))(u_2-u_1)dt - \sum_{j=1}^l I_j(u_1(t_j))(u_2(t_j)-u_1(t_j)).\end{aligned}$$

Combining with the condition (C6), (I2) and Lemma 1, we have

$$\begin{aligned}\|u_2-u_1\|_\alpha^2 &\leq \int_0^T |f(t,u_2(t),{}_0^cD_t^\alpha\omega(t)) - f(t,u_1(t),{}_0^cD_t^\alpha\omega(t))||u_2-u_1|dt \\ &\quad + \sum_{j=1}^l |I_j(u_2(t_j)) - I_j(u_1(t_j))||u_2(t_j)-u_1(t_j)| \\ &\leq L_1 \int_0^T |u_2-u_1|^2 dt + \sum_{j=1}^l \rho_j |u_2(t_j)-u_1(t_j)|^2 \\ &\leq \left(\frac{L_1 T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2} + \frac{T^{(2\alpha-1)}}{[\Gamma(\alpha)]^2 a_0(2\alpha-1)} \cdot \sum_{j=1}^l \rho_j \right) \|u_2-u_1\|_\alpha^2 \\ &= L^* \|u_2-u_1\|_\alpha^2.\end{aligned}$$

Since $0 < L^* < 1$, we can deduce that $\|u_2-u_1\|_\alpha = 0$ and $u_1 = u_2$. This ends the proof of Theorem 2. □

Proof of Theorem 3. According to Theorem 1, We can construct a iterative sequence $\{u_n\} \in E_0^\alpha$ as solutions of the following problem

$$\begin{cases} {}_tD_T^\alpha(a(t)\,{}_0^cD_t^\alpha u_n(t)) + b(t)u_n(t) = f(t, u_n(t), {}_0^cD_t^\alpha u_{n-1}(t)), \ \ t \neq t_j, a.e\ t \in [0,T] \\ \Delta({}_tD_T^{\alpha-1}(a(t_j)\,{}_0^cD_t^\alpha u_n)(t_j)) = I_j(u_n(t_j)), \ \ j = 1, 2, \cdots, l, \\ u_n(0) = u_n(T) = 0. \end{cases} \tag{20}$$

Obtained by the Mountain Pass theorem, starting with an arbitrary $u_0 \in E_0^\alpha$.

According to (IV) of Theorem 1, we have $\|u_n\|_\alpha \leq A_2$. It follows from (6) that

$$\|u_n\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}A_2}{\Gamma(\alpha)\sqrt{a_0(2\alpha-1)}} := \xi.$$

So by (9), $\Phi'_{u_n}(u_{n+1})(u_{n+1}-u_n)=0, \Phi'_{u_{n-1}}(u_n)(u_{n+1}-u_n)=0$, we have

$$\begin{aligned}&\textstyle\int_0^T[a(t){}_0^cD_t^\alpha u_n(t){}_0^cD_t^\alpha(u_{n+1}-u_n)+b(t)u_n(t)(u_{n+1}-u_n)]dt\\&\quad+\sum\limits_{j=1}^{l}I_j(u_n(t_j))(u_{n+1}(t_j)-u_n(t_j))=\textstyle\int_0^T f(t,u_n,{}_0^cD_t^\alpha u_{n-1})(u_{n+1}-u_n)dt,\end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_0^T[a(t){}_0^cD_t^\alpha u_{n+1}(t){}_0^cD_t^\alpha(u_{n+1}-u_n)+b(t)u_{n+1}(t)(u_{n+1}-u_n)]dt\\&\quad+\sum\limits_{j=1}^{l}I_j(u_{n+1}(t_j))(u_{n+1}(t_j)-u_n(t_j))=\textstyle\int_0^T f(t,u_{n+1},{}_0^cD_t^\alpha u_n)(u_{n+1}-u_n)dt.\end{aligned}$$

Hence, by (C6), (I2), and the Hölder inequality, we get

$$\begin{aligned}\|u_{n+1}-u_n\|_\alpha^2 &= \int_0^T[f(t,u_{n+1},{}_0^cD_t^\alpha u_n)-f(t,u_n,{}_0^cD_t^\alpha u_{n-1})](u_{n+1}-u_n)dt\\&\quad+\sum_{j=1}^{l}[I_j(u_{n+1}(t_j))-I_j(u_n(t_j))](u_{n+1}(t_j)-u_n(t_j))\\&\le L_1\int_0^T|u_{n+1}-u_n|^2dt+L_2\int_0^T|{}_0^cD_t^\alpha(u_n-u_{n-1})||u_{n+1}-u_n|dt\\&\quad+\sum_{j=1}^{l}\rho_j|u_{n+1}(t_j)-u_n(t_j)|^2\\&\le\left(\frac{L_1T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2}+\frac{T^{2\alpha-1}}{a_0(2\alpha-1)[\Gamma(\alpha)]^2}\cdot\sum_{j=1}^{l}\rho_j\right)\|u_{n+1}-u_n\|_\alpha^2\\&\quad+\frac{L_2T^\alpha}{a_0\Gamma(\alpha+1)}\|u_n-u_{n-1}\|_\alpha\cdot\|u_{n+1}-u_n\|_\alpha,\end{aligned}$$

which implies that

$$\|u_{n+1}-u_n\|_\alpha\le\widetilde{L}\|u_n-u_{n-1}\|_\alpha,$$

where

$$\widetilde{L}=\frac{L_2T^\alpha(2\alpha-1)\Gamma(\alpha+1)[\Gamma(\alpha)]^2}{a_0(2\alpha-1)[\Gamma(\alpha)\Gamma(\alpha+1)]^2-L_1T^{2\alpha}(2\alpha-1)[\Gamma(\alpha)]^2-T^{2\alpha-1}[\Gamma(\alpha+1)]^2\sum_{j=1}^{l}\rho_j}.$$

According to the condition of Theorem 3, $\widetilde{L}\in(0,1)$. The we know that $\{u_n\}$ is a Cauchy sequence in E_0^α. Therefore the sequence $\{u_n\}$ strongly converges in E_0^α to some $u\in E_0^\alpha$, Theorem 1 guarantees $u\neq 0$.

By (C6), we have, for any $x(t)\in E_0^\alpha$,

$$\begin{aligned}&\int_0^T|f(t,u_n(t),{}_0^cD_t^\alpha u_{n-1}(t))-f(t,u(t),{}_0^cD_t^\alpha u(t))|x(t)dt\\&\le L_1\int_0^T|u_n(t)-u(t)|x(t)dt+L_2\int_0^T|{}_0^cD_t^\alpha(u_{n-1}(t)-u(t))|x(t)dt\\&\le\left(\frac{L_1T^{2\alpha}}{a_0[\Gamma(\alpha+1)]^2}\|u_n-u\|_\alpha+\frac{L_2T^\alpha}{a_0\Gamma(\alpha+1)}\|u_{n-1}-u\|_\alpha\right)\|x\|_\alpha\to 0\end{aligned}$$

as $n\to+\infty$, which implies that u is the solution of problem (1). Hence, we obtain a nontrivial solution of problem (1). This completes the proof. □

Finally, in this paper, we present an explicit example to illustrate our main result.

Example 1. *Let* $\alpha = 0.75, T = 1, t_1 \in (0,1), a(t) = 1/48$, *and* $b(t) = (2-t)/48$. *Consider the following fractional boundary value problem:*

$$\begin{cases} {}_tD_1^{0.75}(a(t){}_0^cD_t^{0.75}u(t)) + b(t)u(t) = \frac{1}{20}(1+\sin^2({}_0^cD_t^{0.75}u(t)))u^5(t), a.e.\ t \in [0,1], t \neq t_1, \\ \Delta({}_tD_1^{-0.25}({}_0^cD_t^{0.75}u))(t_1) = \frac{1}{100}u^3(t_1), \\ u(0) = u(1) = 0. \end{cases} \tag{21}$$

Compared with problem (1), $f(t,u,v) = \frac{1}{20}(1+\sin^2 v)u^5, a_0 = \frac{1}{48}$, *and* $I_1(u(t_1)) = \frac{1}{100}u^3$. *By taking* $\vartheta > 6, \mu = 6$ *and* $k_1 = \frac{1}{120}, k_2 = \frac{1}{1000}, \beta = 4$ *and all* $\zeta > 0$. *Then by simple computation, it is easy to show that the function* f *satisfies the assumptions (C1)-(C5) and the function* I_1 *satisfies the hypotheses (I1).*

For the conditions (C6) and (I2), for all $t \in [0,1], u_1, u_2 \in [-\xi, \xi], v_1, v_2 \in \mathbb{R}$, it follows that

$$\begin{aligned} |f(t,u_2,v_2) - f(t,u_1,v_1)| &\leq \frac{1}{20}|u_2^5(1+\sin^2(v_2)) - u_1^5(1+\sin^2(v_2))| \\ &\quad + \frac{1}{20}|u_1^5(1+\sin^2(v_2)) - u_1^5(1+\sin^2(v_1))| \\ &\leq \frac{1}{20}|1+\sin^2(v_2)||u_2^5 - u_1^5| + \frac{1}{20}|u_1^5||\sin^2(v_2) - \sin^2(v_1)| \\ &\leq \frac{1}{2}\xi^5|u_2 - u_1| + \frac{1}{10}\xi^5|v_2 - v_1|, \end{aligned}$$

and

$$|I_1(u_2(t_1)) - I_1(u_1(t_1))| \leq \frac{3}{100}\xi^2|u_2 - u_1|.$$

Thus, we can choose $L_1 = \frac{1}{2}\xi^5, L_2 = \frac{1}{10}\xi^5$ and $\rho_1 = \frac{3}{100}\xi^2$, where $\xi = \sqrt{2}A_2/\Gamma(0.75)$. In this case, it suffices to verify that

$$\begin{aligned} \widetilde{L} &= \frac{L_2\Gamma(1.75)[\Gamma(0.75)]^2}{[\Gamma(0.75)\Gamma(1.75)]^2/16 - L_1[\Gamma(0.75)]^2 - 2[\Gamma(1.75)]^2\rho_1} \\ &= \frac{\xi^5\Gamma(1.75)[\Gamma(0.75)]^2}{10[\Gamma(0.75)\Gamma(1.75)]^2/16 - 5\xi^5[\Gamma(0.75)]^2 - 0.6[\Gamma(1.75)]^2\xi^2} \in (0,1). \end{aligned}$$

From (19), we estimate the value of $A_* = \mu h(t_0) + K_7$, where K_7 is dependent of ζ. Since

$$\int_0^1 |{}_0^cD_t^{0.75}(t^2 - t)|^2 dt = \frac{1}{[\Gamma(1.25)]^2} \cdot (\frac{2}{3} - \frac{96}{175}).$$

Then we may choose $u^*(t) = 4\sqrt{60/(1+20c_0)}(t^2 - t)$, where $c_0 = \frac{1}{[\Gamma(1.25)]^2}(2/3 - 96/175)$ such that $u^*(t) \in E_0^{0.75}$ with $\|u^*\|_{0.75} = 1$. By direct computation via Mathematica, we have $t_0 \approx 0.3547 \in (0,1)$, and

$$A_* = \mu h(t_0) + K_7 \approx 0.2522 + K_7.$$

According to the arbitrariness of K_7 and ζ, we may take enough small $K_7, \zeta > 0$, such that $A_* = 0.3$. Then $A_2 = \sqrt{2A_*/(\mu - 2)} \approx 0.3873, \xi \approx 1.1540A_2 \approx 0.4469$; we obtain

$$\widetilde{L} = \frac{\xi^5\Gamma(1.75)[\Gamma(0.75)]^2}{10[\Gamma(0.75)\Gamma(1.75)]^2/16 - 5\xi^5[\Gamma(0.75)]^2 - 0.6[\Gamma(1.75)]^2\xi^2} \approx 0.0597 \in (0,1).$$

Then all conditions in Theorem 1 are satisfied. Consequently the problem (21) admits at least one nontrivial solution.

4. Conclusions

In this work, we studied a class of impulsive fractional boundary value problems with nonlinear derivative dependence. Due to the fact that the studied problem (1) is of no the variational structure and it cannot be studied by directly using the well-developed critical point theory. First, we considered a family of impulsive fractional boundary value problem without the fractional derivative of the solution. Second, we give sufficient conditions of the existence of at least one nontrivial solution for problems (1). The used technical approach is based on variational methods and iterative methods. In future work, it is worth investigating multiplicity of solutions for the problem (1), and the existence of solutions to impulsive fractional differential equations involving p-Laplacian.

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