mathematics

Article

Mildly Inertial Subgradient Extragradient Method for Variational Inequalities Involving an Asymptotically Nonexpansive and Finitely Many Nonexpansive Mappings

Lu-Chuan Ceng [1], Xiaolong Qin [2], Yekini Shehu [3] and Jen-Chih Yao [4,*]

1 Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; zenglc@shnu.edu.cn
2 General Education Center, National Yunlin University of Science and Technology, Douliou 64002, Taiwan; qinxl@yuntech.edu.tw
3 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China; yekini.shehu@unn.edu.ng
4 Research Center for Interneural Computing, China Medical University Hospital, Taichung 40447, Taiwan
* Correspondence: yaojc@mail.cmu.edu.tw

Received: 13 July 2019; Accepted: 19 September 2019; Published: 22 September 2019

Abstract: In a real Hilbert space, let the notation VIP indicate a variational inequality problem for a Lipschitzian, pseudomonotone operator, and let CFPP denote a common fixed-point problem of an asymptotically nonexpansive mapping and finitely many nonexpansive mappings. This paper introduces mildly inertial algorithms with linesearch process for finding a common solution of the VIP and the CFPP by using a subgradient approach. These fully absorb hybrid steepest-descent ideas, viscosity iteration ideas, and composite Mann-type iterative ideas. With suitable conditions on real parameters, it is shown that the sequences generated our algorithms converge to a common solution in norm, which is a unique solution of a hierarchical variational inequality (HVI).

Keywords: inertial subgradient extragradient method; pseudomonotone variational inequality; asymptotically nonexpansive mapping; sequentially weak continuity

---

1. Introduction

Let C be a convex and closed nonempty set in a real Hilbert space $(H, \|\cdot\|)$ with inner product $\langle\cdot,\cdot\rangle$. Let $\mathrm{Fix}(S)$ indicate the fixed-point set of a non-self operator $S : C \to H$, i.e., $\mathrm{Fix}(S) = \{u \in C : u = Su\}$. One says that a self operator $T : C \to C$ is asymptotically nonexpansive if and only if $\|T^n u - T^n v\| \le (1+\theta_n)\|u - v\|\ \forall n \ge 1,\ u, v \in C$, where $\lim_{n\to\infty} \theta_n = 0$ is a real sequence. In the case of $\theta_n = 0\ \forall n \ge 1$, one says that T is nonexpansive. Both the class of nonexpansive operators and asymptotically nonexpansive operators via various iterative techniques have been studied recently; see, e.g., the works by the authors of [1–13]. Let $A : H \to H$ be a self operator. Consider the classical variational inequality problem (VIP) of consisting of $u^* \in C$ such that

$$\langle Au^*, v - u^*\rangle \ge 0\ \ \forall v \in C. \tag{1}$$

The set of solutions of problem (1) is indicated by VI(C, A). Recently, many authors studied the VIP via mean-valued and projection-based methods; see, e.g., the works by the authors of [14–21]. In 1976, Korpelevich [22] first designed and investigated an extragradient method for a solution of problem (1), that is, for arbitrarily given $u_0 \in C$, $\{u_n\}$ is the sequence constructed by

$$\begin{cases} v_n = P_C(u_n - \tau A u_n), \\ u_{n+1} = P_C(u_n - \tau A v_n) \quad \forall n \ge 0, \end{cases} \tag{2}$$

with $\tau \in (0, \frac{1}{L})$. If problem (1) has a solution, then he showed the weak convergence of $\{u_n\}$ constructed by (2) to a solution of problem (1). Since then, Korpelevich's extragradient method and its variants have been paid great attention to by many scholars, who improved it in various techniques and approaches; see, e.g., the works by the authors of [23–34].

Let $\{T_i\}_{i=1}^N$ be N nonexpansive mappings on H, such that $\Omega = \cap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let F be a κ-Lipschitzian, η-strongly monotone self-mapping on H, and f be a contractive map with constant $\delta \in (0,1)$. In 2015, Bnouhachem et al. [2] introduced an iterative algorithm for solving a hierarchical fixed point problem (HFPP) for a finite pool $\{T_i\}_{i=1}^N$, i.e., for arbitrarily given $x_0 \in H$, the sequence $\{x_n\}$ is constructed by

$$\begin{cases} y_n = (1-\beta_n)T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n + \beta_n x_n, \\ x_{n+1} = \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu F)y_n + \alpha_n\rho f(y_n), \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $T_{i,n} = (1-\delta_{i,n})I + \delta_{i,n}T_i$ and $\delta_{i,n} \in (0,1)$ for integer $i \in \{1,2,...,N\}$. Let the parameters satisfy $0 < \mu\kappa^2 < 2\eta$ and $0 \leq \rho\tau < \nu$, with $\nu = \mu(\eta - \frac{\mu\kappa^2}{2})$. Also, suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy the following requirements.

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\{\beta_n\} \subset [\sigma, 1)$ and $\lim_{n\to\infty} \beta_n = \beta < 1$;
(iii) $\limsup_{n\to\infty} \gamma_n < 1$ and $\liminf_{n\to\infty} \gamma_n > 0$;
(iv) $\lim_{n\to\infty} |\delta_{i,n-1} - \delta_{i,n}| = 0$ for $i = 1,2,...,N$.

They proved the strong convergence of $\{x_n\}$ to a point $x^* \in \Omega$, which is only a solution to the HFPP: $\langle(\mu F - \rho f)x^*, y - x^*\rangle \geq 0 \ \forall y \in \Omega$.

On the other hand, let the mappings $A_1, A_2 : C \to H$ be both inverse-strongly monotone and the mapping $T : C \to C$ be asymptotically nonexpansive one with $\{\theta_n\}$. In 2018, by the modified extragradient method, Cai et al. [35] designed a viscosity implicit method for computing a point in the common solution set Ω of the VIPs for A_1 and A_2 and the FPP of T, i.e., for arbitrarily given $x_1 \in C$, the sequence $\{x_n\}$ is constructed by

$$\begin{cases} v_n = t_n x_n + (1-t_n)u_n, \\ z_n = P_C(v_n - \mu A_2 v_n), \\ u_n = P_C(z_n - \lambda A_1 z_n), \\ x_{n+1} = P_C[(I - \alpha_n\rho F)T^n u_n + \alpha_n f(x_n)], \end{cases} \tag{4}$$

where $f : C \to C$ be a δ-contraction with $0 \leq \delta < 1$, and $\{\alpha_n\}, \{t_n\}$ are the sequences in $(0,1]$ satisfying

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $0 < \epsilon \leq t_n \leq 1$ and $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$;
(iv) $\sum_{n=1}^{\infty} \|T^{n+1}u_n - T^n u_n\| < \infty$.

They proved that $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, which is a unique solution to the VIP: $\langle(f - \rho F)x^*, y - x^*\rangle \leq 0 \ \forall y \in \Omega$.

Under the setting of extragradient approaches, we must calculate metric projections twice for every iteration. Without doubt, if C is a general convex and closed subset, the computation of the projection onto C might be prohibitively consuming-time. In 2011, motivated by Korpelevich's extragradient method, Censor et al. [5] first purposed the subgradient extragradient method, where a projection onto a half-space is used in place of the second projection onto C:

$$\begin{cases} v_n = P_C(u_n - \ell A u_n), \\ C_n = \{u \in H : \langle u_n - \ell A u_n - v_n, u - v_n\rangle \leq 0\}, \\ u_{n+1} = P_{C_n}(u_n - \ell A v_n) \quad \forall n \geq 0, \end{cases} \tag{5}$$

with $\ell \in (0, \frac{1}{L})$. In 2014, Kraikaew and Saejung [36] introduced the Halpern subgradient extragradient method for solving VIP (1) and proved that the sequence generated by the proposed method converges strongly to a solution of VIP (1).

In 2018, by virtue of the inertial technique, Thong and Hieu [37] first introduced the inertial subgradient extragradient method and proved weak convergence of the proposed method to a solution of VIP (1). Very recently, Thong and Hieu [37] introduced two inertial subgradient extragradient algorithms with the linesearch process to solve the VIP (1) for Lipschitzian, monotone operator A, and the FPP of quasi-nonexpansive operator S satisfying the demiclosedness in H.

Under mild assumptions, they proved that the sequences defined by the above algorithms converge to a point in $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ with the aid of dual spaces. Being motivated by the research work [2,37,38] and using the subgradient extragradient technique, this paper designs two mildly inertial algorithms with linesearch process to solve the VIP (1) for Lipschitzian, pseudomonotone operator, and the CFPP of an asymptotically nonexpansive mapping and finitely many nonexpansive mappings in H. Our algorithms fully absorb inertial subgradient extragradient approaches with linesearch process, hybrid steepest-descent algorithms, viscosity iteration techniques, and composite Mann-type iterative methods. Under suitable conditions, it is shown that the sequences constructed by our algorithms converge to a common solution of the VIP and CFPP in norm, which is only a solution of a hierarchical variational inequality (HVI). Finally, we apply our main theorems to deal with the VIP and CFPP in an illustrating example.

The outline of the article is arranged as follows. In Section 2, some concepts and preliminary conclusions are recalled for later use. In Section 3, the convergence criteria of the suggested algorithms are established. In Section 4, our main theorems are used to deal with the VIP and CFPP in an illustrating example. As our algorithms concern solving VIP (1) with Lipschitzian, pseudomonotone operator, and the CFPP of an asymptotically nonexpansive mapping and finitely many nonexpansive mappings, they are more advantageous and more subtle than Algorithms 1 and 2 in [37]. Our theorems strengthen and generalize the corresponding results announced in Bnouhachem et al. [2], Cai et al. [35], Kraikaew and Saejung [36], and Thong and Hieu [37,38].

Algorithm 1: of Thong and Hieu [37]

1 **Initial Step:** Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$.
2 **Iteration Steps:** Compute x_{n+1} in what follows,
Step 1. Put $u_n = x_n - \alpha_n(x_{n-1} - x_n)$ and calculate $y_n = P_C(u_n - \ell_n A u_n)$, where ℓ_n is chosen to be the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\mu\|u_n - y_n\| \geq \ell\|Au_n - Ay_n\|$.
Step 2. Calculate $z_n = P_{C_n}(u_n - \ell_n A y_n)$ with $C_n := \{u \in H : \langle u_n - \ell_n A u_n - y_n, u - y_n\rangle \leq 0\}$.
Step 3. Calculate $x_{n+1} = (1 - \beta_n)u_n + \beta_n S z_n$. If $u_n = z_n = x_{n+1}$ then $u_n \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. Put $n := n + 1$ and return to Step 1.

Algorithm 2: of Thong and Hieu [37]

1 **Initial Step:** Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$.
2 **Iteration Steps:** Compute x_{n+1} in what follows,
Step 1. Put $u_n = x_n - \alpha_n(x_{n-1} - x_n)$ and calculate $y_n = P_C(u_n - \ell_n A u_n)$, where ℓ_n is chosen to be the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\mu\|u_n - y_n\| \geq \ell\|Au_n - Ay_n\|$.
Step 2. Calculate $z_n = P_{C_n}(u_n - \ell_n A y_n)$ with $C_n := \{u \in H : \langle u_n - \ell_n A u_n - y_n, u - y_n\rangle \leq 0\}$.
Step 3. Calculate $x_{n+1} = (1 - \beta_n)x_n + \beta_n S z_n$. If $u_n = z_n = x_n = x_{n+1}$ then $x_n \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. Put $n := n + 1$ and return to Step 1.

2. Preliminaries

Given a sequence $\{u_n\}$ in H. We use the notations $u_n \to u$ and $u_n \rightharpoonup u$ to indicate the strong convergence of $\{u_n\}$ to u and weak convergence of $\{u_n\}$ to u, respectively. An operator $T : C \to H$ is said to be

(i) L-Lipschitz continuous (or L-Lipschitzian) iff $\exists L > 0$ s.t.

$$\|Tu - Tv\| \leq L\|u - v\| \ \forall u, v \in C;$$

(ii) monotone iff

$$\langle Tu - Tv, u - v\rangle \geq 0 \ \forall u, v \in C;$$

(iii) pseudomonotone iff

$$\langle Tu, v - u\rangle \geq 0 \Rightarrow \langle Tv, v - u\rangle \geq 0 \ \forall u, v \in C;$$

(iv) β-strongly monotone if $\exists \beta > 0$ s.t.

$$\langle Tu - Tv, u - v\rangle \geq \beta\|u - v\|^2 \ \forall u, v \in C;$$

(v) sequentially weakly continuous if $\forall \{u_n\} \subset C$, the relation holds: $u_n \rightharpoonup u \Rightarrow Tu_n \rightharpoonup Tu$.

It is clear that every monotone mapping is pseudomonotone but the converse is not valid; e.g., take $Tx := \frac{a}{a+x}, \ \ x, a \in (0, +\infty)$.

For every $u \in H$, we know that there is only a nearest point in C, indicated by $P_C u$, s.t. $\|u - P_C u\| \leq \|u - v\| \ \ \forall v \in C$. The operator P_C is said to be the metric projection from H to C.

Proposition 1. *The following hold in real Hilbert spaces:*
(i) $\langle u - v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2 \ \ \forall u, v \in H$;
(ii) $\langle u - P_C u, v - P_C u\rangle \leq 0 \ \ \forall u \in H, v \in C$;
(iii) $\|u - v\|^2 - \|u - P_C u\|^2 \geq \|v - P_C u\|^2 \ \ \forall u \in H, v \in C$;
(iv) $\|u - v\|^2 + 2\langle u - v, v\rangle = \|u\|^2 - \|v\|^2 \ \ \forall u, v \in H$;
(v) $\|\lambda u + (1 - \lambda)v\|^2 + \lambda(1 - \lambda)\|u - v\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|v\|^2 \ \ \forall u, v \in H, \lambda \in [0, 1]$.

An operator $S : H \to H$ is called an averaged one if $\exists \alpha \in (0, 1)$ s.t. $S = (1 - \alpha)I + \alpha T$, where I is the identity operator of H and $T : H \to H$ is a nonexpansive operator. In this case, S is also called α-averaged. It is clear that the averaged operator S is also nonexpansive and $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.

Lemma 1. [2] *If the mappings* $\{T_i\}_{i=1}^N$ *defined on* H *are averaged and have a common fixed point, then* $\cap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 \cdots T_N)$.

The next result immediately follows from the subdifferential inequality of the function $\|\cdot\|^2/2$.

Lemma 2. *The following inequality holds,*

$$\|u + v\|^2 - \|u\|^2 \leq 2\langle v, u + v\rangle \quad \forall u, v \in H.$$

Lemma 3. [39] *Assume that the mapping* A *is pseudomonotone and continuous on* C. *Given a point* $u \in C$. *Then the relation holds:* $\langle Au, v - u\rangle \geq 0 \ \forall v \in C \ \Leftrightarrow \ \langle Av, v - u\rangle \geq 0 \ \forall v \in C$.

Lemma 4. *[40] Let $\{t_n\}$ be a sequence in $[0, +\infty)$ satisfying the condition $t_{n+1} \le s_n b_n + (1 - s_n)t_n$ $\forall n \ge 1$, where $\{s_n\}$ and $\{b_n\}$ lie in $\mathbf{R} := (-\infty, \infty)$ s.t. (a) $\{s_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} s_n = \infty$, and (b) $\limsup_{n\to\infty} b_n \le 0$ or $\sum_{n=1}^{\infty} |s_n b_n| < \infty$. Then $t_n \to 0$ as $n \to \infty$.*

Definition 1. *An operator $S : C \to H$ is called ζ-strictly pseudocontractive iff $\exists \zeta \in [0,1)$ s.t. $\|Su - Sv\|^2 - \zeta\|(I - S)u - (I - S)v\|^2 \le \|u - v\|^2$ $\forall u, v \in C$.*

Lemma 5. *[41] Assume that $S : C \to H$ is ζ-strictly pseudocontractive. Define $T : C \to H$ by $Tu = \mu Su + (1 - \mu)u$ $\forall u \in C$. If $\mu \in [\zeta, 1)$, T is nonexpansive such that $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.*

Lemma 6. *[42] Let $\ell \in (0,1]$, $S : C \to H$ be nonexpansive, and $S^{\ell} : C \to H$ be defined as $S^{\ell}u := Su - \ell\mu F(Su)$ $\forall u \in C$, where F is κ-Lipschitzian and η-strongly monotone self-mapping on H. Then, S^{ℓ} is a contractive map provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e., $\|S^{\ell}u - S^{\ell}v\| \le (1 - \ell\tau)\|u - v\|$ $\forall u, v \in C$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.*

Lemma 7. *[43] Assume that the Banach space X admits a weakly continuous duality mapping; the subset $C \subset X$ is nonempty, convex, and closed; and the asymptotically nonexpansive mapping $S : C \to C$ has a fixed point. Then, $I - S$ is demiclosed at zero, i.e., if the sequence $\{u_n\} \subset C$ satisfies $u_n \rightharpoonup u \in C$ and $u_n - Su_n \to 0$, then $u \in \mathrm{Fix}(S)$.*

3. Main Results

In this section, we always suppose the following conditions.

- T is an asymptotically nonexpansive operator on H with $\{\theta_n\}$ and $\{T_i\}_{i=1}^N$ are N nonexpansive operators on H.
- A is L-Lipschitzian, pseudomonotone on H, and sequentially weakly continuous on C, s.t. $\Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \neq \emptyset$ with $T_0 := T$.
- f is a contractive map on H with coefficient $\delta \in [0,1)$, and F is κ-Lipschitzian, η-strongly monotone on H.
- $\nu\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\nu \ge 0$ and $\rho \in (0, \frac{2\eta}{\kappa^2})$.
- $T_{i,n} := (1 - \delta_{i,n})I + \delta_{i,n}T_i$ and $\delta_{i,n} \in (0,1)$ for $i = 1, 2, ..., N$.
- $\{\sigma_n\} \subset [0,1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ such that

  (i) $\sup_{n\ge1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;

  (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;

  (iii) $\{\beta_n\} \subset [\sigma, 1)$ and $\lim_{n\to\infty} \beta_n = \beta < 1$;

  (iv) $\limsup_{n\to\infty} \gamma_n < 1$, $\liminf_{n\to\infty} \gamma_n > 0$ and $\alpha_n + \gamma_n \le 1$ $\forall n \ge 1$. For example, take

$$\alpha_n = \frac{1}{n+1}, \sigma_n = \frac{1}{(n+1)^2} = \theta_n, \beta_n = \frac{n}{2(n+1)}, \gamma_n = \frac{n}{4(n+1)}.$$

Remark 1. *For Step 2 in Algorithm 3, the composite mapping $T_{N,n}T_{N-1,n}\cdots T_{1,n}$ with $T_{i,n} := (1 - \delta_{i,n})I + \delta_{i,n}T_i$ and $\delta_{i,n} \in (0,1)$ for $i = 1, 2, ..., N$, has the following property,*

$$\cap_{i=1}^N \mathrm{Fix}(T_i) = \cap_{i=1}^N \mathrm{Fix}(T_{i,n}) = \mathrm{Fix}(T_{N,n}T_{N-1,n}\cdots T_{1,n}) \quad \forall n \ge 1,$$

due to Lemmas 1 and 5.

Algorithm 3: MISEA I

1 **Initial Step:** Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$.

2 **Iteration Steps:** Compute x_{n+1} in what follows.

Step 1. Put $u_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $y_n = P_C(u_n - \ell_n Au_n)$, where ℓ_n is chosen to be the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\ell\|Au_n - Ay_n\| \le \mu\|u_n - y_n\|. \tag{6}$$

Step 2. Calculate $z_n = \beta_n x_n + (1-\beta_n)T_{N,n}T_{N-1,n}\cdots T_{1,n}P_{C_n}(u_n - \ell_n Ay_n)$ with $C_n := \{u \in H : \langle u_n - \ell_n Au_n - y_n, u - y_n\rangle \le 0\}$.

Step 3. Calculate

$$x_{n+1} = \alpha_n \nu f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\rho F)T^n z_n. \tag{7}$$

Update $n := n+1$ and return to Step 1.

Lemma 8. *The Armijo-like search rule (6) is defined well, and the following holds:* $\min\{\gamma, \frac{\mu l}{L}\} \le \ell_n \le \gamma$.

Proof. As A is L-Lipschitzian, we get $\frac{\mu}{L}\|Au_n - AP_C(u_n - \gamma l^m Au_n)\| \le \mu\|u_n - P_C(u_n - \gamma l^m Au_n)\|$. Therefore, (6) is valid for $\gamma l^m \le \frac{\mu}{L}$. This means that ℓ_n is defined well. It is clear that $\ell_n \le \gamma$. In the case of $\ell_n = \gamma$, the inequality holds. In the case of $\ell_n < \gamma$, from (6) it follows that $\|Au_n - AP_C(u_n - \frac{\ell_n}{l}Au_n)\| > \frac{\mu}{\frac{\ell_n}{l}}\|u_n - P_C(u_n - \frac{\ell_n}{l}Au_n)\|$. Thus, from the L-Lipschitzian property of A, we get $\ell_n > \frac{\mu l}{L}$. Consequently, the inequality holds. □

Lemma 9. *Let $\{u_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 3. Then*

$$\begin{aligned} \|z_n - \omega\|^2 &\le \beta_n\|x_n - \omega\|^2 + (1-\beta_n)\|u_n - \omega\|^2 \\ &\quad - (1-\beta_n)(1-\mu)[\|u_n - y_n\|^2 + \|v_n - y_n\|^2] \quad \forall \omega \in \Omega, n \ge 1, \end{aligned} \tag{8}$$

where $v_n := P_{C_n}(u_n - \ell_n Ay_n)$.

Proof. First, take an arbitrary $p \in \Omega \subset C \subset C_n$. We note that

$$\begin{aligned} 2\|v_n - p\|^2 &= 2\|P_{C_n}(u_n - \ell_n Ay_n) - P_{C_n}p\|^2 \le 2\langle v_n - p, u_n - \ell_n Ay_n - p\rangle \\ &= \|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - u_n\|^2 - 2\langle v_n - p, \ell_n Ay_n\rangle. \end{aligned}$$

So, it follows that $\|v_n - p\|^2 \le \|u_n - p\|^2 - \|v_n - u_n\|^2 - 2\langle v_n - p, \ell_n Ay_n\rangle$, which together with (6) and the pseudomonotonicity of A, deduces that $\langle Ay_n, y_n - p\rangle \ge 0$ and

$$\begin{aligned} \|v_n - p\|^2 &\le \|u_n - p\|^2 - \|v_n - u_n\|^2 + 2\ell_n(\langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - v_n\rangle) \\ &\le \|u_n - p\|^2 - \|v_n - u_n\|^2 + 2\ell_n\langle Ay_n, y_n - v_n\rangle \\ &= \|u_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle u_n - \ell_n Ay_n - y_n, v_n - y_n\rangle. \end{aligned} \tag{9}$$

As $v_n = P_{C_n}(u_n - \ell_n Ay_n)$ with $C_n := \{u \in H : \langle u_n - \ell_n Au_n - y_n, u - y_n\rangle \le 0\}$, we have $\langle u_n - \ell_n Au_n - y_n, v_n - y_n\rangle \le 0$, which together with (6), implies that

$$\begin{aligned} 2\langle u_n - \ell_n Ay_n - y_n, v_n - y_n\rangle &= 2\langle u_n - \ell_n Au_n - y_n, v_n - y_n\rangle + 2\ell_n\langle Au_n - Ay_n, v_n - y_n\rangle \\ &\le 2\mu\|u_n - y_n\|\|v_n - y_n\| \le \mu(\|u_n - y_n\|^2 + \|v_n - y_n\|^2). \end{aligned}$$

Therefore, substituting the last inequality for (9), we obtain

$$\|v_n - p\|^2 \le \|u_n - p\|^2 - (1-\mu)\|u_n - y_n\|^2 - (1-\mu)\|v_n - y_n\|^2 \quad \forall p \in \Omega, \tag{10}$$

which together with Algorithm 3 and $\mathrm{Fix}(T_{N,n}T_{N-1,n}\cdots T_{1,n}) = \cap_{i=1}^{N}\mathrm{Fix}(T_{i,n}) = \cap_{i=1}^{N}\mathrm{Fix}(T_i)$, due to Lemmas 1 and 5 implies that for all $\omega \in \Omega$,

$$\begin{aligned}
&\|z_n - \omega\|^2 \le \beta_n\|x_n - \omega\|^2 + (1-\beta_n)\|T_{N,n}T_{N-1,n}\cdots T_{1,n}v_n - \omega\|^2 \\
&\le \beta_n\|x_n - \omega\|^2 + (1-\beta_n)\|v_n - \omega\|^2 \\
&\le \beta_n\|x_n - \omega\|^2 + (1-\beta_n)[\|u_n - \omega\|^2 - (1-\mu)\|u_n - y_n\|^2 - (1-\mu)\|v_n - y_n\|^2] \\
&= \beta_n\|x_n - \omega\|^2 + (1-\beta_n)\|u_n - \omega\|^2 - (1-\beta_n)(1-\mu)[\|u_n - y_n\|^2 + \|v_n - y_n\|^2].
\end{aligned}$$

This completes the proof. □

Lemma 10. *Let $\{u_n\}, \{x_n\}, \{y_n\}$, and $\{z_n\}$ be bounded vector sequences generated by Algorithm 3. If $T^n x_n - T^{n+1}x_n \to 0$, $x_n - x_{n+1} \to 0$, $u_n - y_n \to 0$, $u_n - z_n \to 0$ and $\exists\{w_{n_k}\} \subset \{u_n\}$ such that $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. From Algorithm 3, we get $u_n - x_n = \sigma_n(x_n - x_{n-1})\ \forall n \ge 1$, and therefore $\|u_n - x_n\| = \sigma_n\|x_n - x_{n-1}\| \le \|x_n - x_{n-1}\|$. Utilizing the assumption $x_n - x_{n+1} \to 0$, we have $u_n - x_n \to 0$. So, it follows from the assumption $u_n - y_n \to 0$, that $\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - y_n\| \to 0\ (n \to \infty)$. Therefore, according to the assumption $u_n - z_n \to 0$, we get $\|x_n - z_n\| \le \|x_n - u_n\| + \|u_n - z_n\| \to 0\ (n \to \infty)$. Furthermore, in terms of Lemma 9 we deduce that for each $\omega \in \Omega$,

$$\begin{aligned}
&(1-\beta_n)(1-\mu)[\|u_n - y_n\|^2 + \|v_n - y_n\|^2] \\
&\le \beta_n\|x_n - \omega\|^2 + (1-\beta_n)\|u_n - \omega\|^2 - \|z_n - \omega\|^2 \\
&\le \beta_n\|x_n - \omega\|^2 + (1-\beta_n)(\|x_n - \omega\| + \|x_n - x_{n-1}\|)^2 - \|z_n - \omega\|^2 \\
&= \beta_n\|x_n - \omega\|^2 + (1-\beta_n)[\|x_n - \omega\|^2 + \|x_n - x_{n-1}\|(2\|x_n - \omega\| + \|x_n - x_{n-1}\|)] - \|z_n - \omega\|^2 \\
&= \|x_n - \omega\|^2 - \|z_n - \omega\|^2 + (1-\beta_n)\|x_n - x_{n-1}\|(2\|x_n - \omega\| + \|x_n - x_{n-1}\|) \\
&\le (\|x_n - \omega\| + \|z_n - \omega\|)\|x_n - z_n\| + \|x_n - x_{n-1}\|(2\|x_n - \omega\| + \|x_n - x_{n-1}\|).
\end{aligned}$$

As $\lim_{n\to\infty}(1-\beta_n) = (1-\beta) > 0$, $\mu \in (0,1)$, $x_n - x_{n+1} \to 0$ and $x_n - z_n \to 0$, from the boundedness of $\{x_n\}, \{z_n\}$ we get

$$\lim_{n\to\infty}\|u_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|v_n - y_n\| = 0.$$

Thus we obtain that $\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\| \to 0\ (n \to \infty)$. □

Now, according to (7) in Algorithm 3, we have

$$\begin{aligned}
x_{n+1} - x_n &= (1-\gamma_n)(T^n z_n - x_n) - \alpha_n\rho F T^n z_n + \alpha_n \nu f(x_n) \\
&= (1-\gamma_n)(T^n z_n - T^n x_n) + (1-\gamma_n)(T^n x_n - x_n) - \alpha_n\rho F T^n z_n + \alpha_n \nu f(x_n).
\end{aligned}$$

So it follows that

$$\begin{aligned}
&(1-\gamma_n)\|T^n x_n - x_n\| = \|x_{n+1} - x_n - \alpha_n \nu f(x_n) - (1-\gamma_n)(T^n z_n - T^n x_n) + \alpha_n\rho F T^n z_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n(\|\nu f(x_n)\| + \|\rho F T^n z_n\|) + (1-\gamma_n)\|T^n z_n - T^n x_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n(\|\nu f(x_n)\| + \|\rho F T^n z_n\|) + (1+\theta_n)\|z_n - x_n\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\gamma_n) > 0$, $\alpha_n \to 0$, $\theta_n \to 0$, $x_n - x_{n+1} \to 0$ and $x_n - z_n \to 0$, from the boundedness of $\{x_n\}, \{z_n\}$ and the Lipschitz continuity of f, F, T, we infer that

$$\lim_{n\to\infty}\|x_n - T^n x_n\| = 0. \tag{11}$$

Also, let the mapping $W : H \to H$ be defined as $Wx := \beta x + (1-\beta)T_{N,n}T_{N-1,n}\cdots T_{1,n}x$, where $\beta \in [\sigma, 1)$. By Lemma 5 we know that W is nonexpansive self-mapping on H and $\mathrm{Fix}(W) = \cap_{i=1}^{N}\mathrm{Fix}(T_i)$. We observe that

$$\begin{aligned}
&\|Wx_n - x_n\| \le \|Wx_n - z_n\| + \|z_n - x_n\| \\
&= \|(\beta-\beta_n)(x_n - T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n) + (1-\beta_n)(T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n - T_N^nT_{N-1}^n\cdots T_1^nv_n)\| \\
&\quad + \|z_n - x_n\| \\
&\le |\beta-\beta_n|\|x_n - T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n\| + (1-\beta_n)\|T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n - T_N^nT_{N-1}^n\cdots T_1^nv_n\| \\
&\quad + \|z_n - x_n\| \\
&\le |\beta-\beta_n|\|x_n - T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n\| + \|x_n - v_n\| + \|z_n - x_n\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded and the composite $T_{N,n}T_{N-1,n}\cdots T_{1,n}$ is nonexpansive, from $\lim_{n\to\infty}\beta_n = \beta$, $x_n - v_n \to 0$ and $x_n - z_n \to 0$ we deduce that

$$\lim_{n\to\infty}\|x_n - Wx_n\| = 0. \tag{12}$$

Noticing $y_n = P_C(u_n - \ell_n Au_n)$, we get $\langle u_n - \ell_n Au_n - y_n, x - y_n\rangle \le 0\ \forall x \in C$, and hence

$$\frac{1}{\ell_n}\langle u_n - y_n, x - y_n\rangle + \langle Au_n, y_n - u_n\rangle \le \langle Au_n, x - u_n\rangle \quad \forall x \in C. \tag{13}$$

Then, by the boundedness of $\{u_{n_k}\}$ and Lipschitzian property of A, we know that $\{Au_{n_k}\}$ is bounded. Also, from $u_n - y_n \to 0$, we have that $\{y_{n_k}\}$ is bounded as well. Observe that $\ell_n \ge \min\{\gamma, \frac{\mu l}{L}\}$. So, from (13), it follows that $\liminf_{k\to\infty}\langle Au_{n_k}, x - u_{n_k}\rangle \ge 0\ \forall x \in C$. Moreover, note that $\langle Ay_n, x - y_n\rangle = \langle Ay_n - Au_n, x - u_n\rangle + \langle Au_n, x - u_n\rangle + \langle Ay_n, u_n - y_n\rangle$. Since $u_n - y_n \to 0$, from L-Lipschitzian property of A we get $Au_n - Ay_n \to 0$, which together with (13) arrives at $\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \ge 0\ \forall x \in C$. We below claim that $x_n - Tx_n \to 0$. Indeed, observe that

$$\begin{aligned}
\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^nx_n\| + \|T^nx_n - x_n\| \\
&\le (1+\theta_1)\|x_n - T^nx_n\| + \|T^{n+1}x_n - T^nx_n\| + \|T^nx_n - x_n\| \\
&= (2+\theta_1)\|x_n - T^nx_n\| + \|T^{n+1}x_n - T^nx_n\|.
\end{aligned}$$

Therefore, from (11) and the assumption $T^nx_n - T^{n+1}x_n \to 0$, we get

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{14}$$

We now select a sequence $\{\epsilon_k\} \subset (0,1)$ s.t. $\epsilon_k \downarrow 0$ as $k \to \infty$. For every $k \ge 1$, we indicate by m_k the smallest natural number s.t.

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \epsilon_k \ge 0 \quad \forall j \ge m_k. \tag{15}$$

As $\{\epsilon_k\}$ is decreasing, $\{m_k\}$ obviously is increasing. Considering that $\{y_{m_k}\} \subset C$ ensures $Ay_{m_k} \neq 0\ \forall k \ge 1$, we put $\nu_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we have $\langle Ay_{m_k}, \nu_{m_k}\rangle = 1\ \forall k \ge 1$. Therefore, from (15), we have $\langle Ay_{m_k}, x + \epsilon_k\nu_{m_k} - y_{m_k}\rangle \ge 0\ \forall k \ge 1$. Also, from the pseudomonotonicity of A we get $\langle A(x + \epsilon_k\nu_{m_k}), x + \epsilon_k\nu_{m_k} - y_{m_k}\rangle \ge 0\ \forall k \ge 1$. This means that

$$\langle Ax, x - y_{m_k}\rangle \ge \langle Ax - A(x + \epsilon_k\nu_{m_k}), x + \epsilon_k\nu_{m_k} - y_{m_k}\rangle - \epsilon_k\langle Ax, \nu_{m_k}\rangle \quad \forall k \ge 1. \tag{16}$$

We show that $\lim_{k\to\infty}\epsilon_k\nu_{m_k} = 0$. In fact, from $u_{n_k} \rightharpoonup z$ and $u_n - y_n \to 0$, we get $y_{n_k} \rightharpoonup z$. Hence, $\{y_n\} \subset C$ ensures $z \in C$. Also, since A is sequentially weakly continuous, we infer that $Ay_{n_k} \rightharpoonup Az$. So, we get $Az \neq 0$ (otherwise, z is a solution). Utilizing the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, we have $0 < \|Az\| \le \liminf_{k\to\infty}\|Ay_{n_k}\|$. Since $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \downarrow 0$ as $k \to \infty$, we deduce that $0 \leq \limsup_{k\to\infty} \|\epsilon_k \nu_{m_k}\| = \limsup_{k\to\infty} \frac{\epsilon_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0$. Thus we have $\epsilon_k \mu_{m_k} \to 0$.

Finally, we claim $z \in \Omega$. In fact, from $u_n - x_n \to 0$ and $u_{n_k} \rightharpoonup z$, we have $x_{n_k} \rightharpoonup z$. By (14) we get $x_{n_k} - Tx_{n_k} \to 0$. Because Lemma 7 ensures the demiclosedness of $I - T$ at zero, we have $z \in \mathrm{Fix}(T)$. Moreover, using $u_n - x_n \to 0$ and $u_{n_k} \rightharpoonup z$, we have $x_{n_k} \rightharpoonup z$. Using (12) we get $x_{n_k} - Wx_{n_k} \to 0$. Using Lemma 7 we deduce that $I - W$ has the demiclosedness at zero. So, we have $(I - W)z = 0$, i.e., $z \in \mathrm{Fix}(W) = \cap_{i=1}^N \mathrm{Fix}(T_i)$. In addition, taking $k \to \infty$, we conclude that the right hand side of (16) tends to zero according to the Lipschitzian property of A, the boundedness of $\{y_{m_k}\}, \{\nu_{m_k}\}$ and the limit $\lim_{k\to\infty} \epsilon_k \nu_{m_k} = 0$. Consequently, we get $\langle Ay, y - z\rangle = \liminf_{k\to\infty} \langle Ay, y - y_{m_k}\rangle \geq 0 \ \forall y \in C$. By Lemma 3 we have $z \in \mathrm{VI}(C, A)$. So, $z \in \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) = \Omega$.

Remark 2. *It is clear that the boundedness assumption of the generated sequences in Lemma 10 can be disposed with when T is the identity.*

Theorem 1. *Assume that the sequence $\{x_n\}$ constructed by Algorithm 3 satisfies $T^n x_n - T^{n+1} x_n \to 0$. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - T_{N,n} T_{N-1,n} \cdots T_{1,n} x_n \to 0 \end{cases}$$

where $x^* \in \Omega$ is only a solution to the HVI: $\langle (\nu f - \rho F)x^*, \omega - x^*\rangle \leq 0 \ \forall \omega \in \Omega$.

Proof. We first note that $\limsup_{n\to\infty} \gamma_n < 1$ and $\liminf_{n\to\infty} \gamma_n > 0$. Then, we may suppose that $\{\gamma_n\} \subset [a, b] \subset (0, 1)$. We show that $P_\Omega(\nu f + I - \rho F)$ is a contractive map. In fact, using Lemma 6 we get

$$\begin{aligned} \|P_\Omega(\nu f + I - \rho F)u - P_\Omega(\nu f + I - \rho F)v\| &\leq \nu \|f(u) - f(v)\| + \|(I - \rho F)u - (I - \rho F)v\| \\ &\leq \nu\delta \|u - v\| + (1 - \tau)\|u - v\| = [1 - (\tau - \nu\delta)]\|u - v\| \ \ \forall u, v \in H. \end{aligned}$$

This means that $P_\Omega(\nu f + I - \rho F)$ has only a fixed point $x^* \in H$, i.e., $x^* = P_\Omega(\nu f + I - \rho F)x^*$. Accordingly, there is only a solution $x^* \in \Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ to the VIP

$$\langle (\nu f - \rho F)x^*, \omega - x^*\rangle \leq 0 \quad \forall \omega \in \Omega. \tag{17}$$

It is now easy to see that the necessity of the theorem is valid. Indeed, if $x_n \to x^* \in \Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$, then $T_1 x^* = x^*, ..., T_N x^* = x^*$, which together with $\cap_{i=1}^N \mathrm{Fix}(T_i) = \cap_{i=1}^N \mathrm{Fix}(T_{i,n}) = \mathrm{Fix}(T_N^n T_{N-1}^n \cdots T_1^n)$ (due to Lemmas 1 and 5), imply that $\|x_n - x_{n+1}\| \leq \|x_n - x^*\| + \|x_{n+1} - x^*\| \to 0 \ (n \to \infty)$, and

$$\begin{aligned} \|x_n - T_N^n T_{N-1}^n \cdots T_1^n x_n\| &\leq \|x_n - x^*\| + \|T_N^n T_{N-1}^n \cdots T_1^n x_n - x^*\| \\ &\leq \|x_n - x^*\| + \|x_n - x^*\| = 2\|x_n - x^*\| \to 0 \quad (n \to \infty). \end{aligned}$$

We below claim the sufficiency of the theorem. For this purpose, we suppose $\lim_{n\to\infty}(\|x_n - x_{n+1}\| + \|x_n - T_{N,n} T_{N-1,n} \cdots T_{1,n} x_n\|) = 0$ and prove the sufficiency by the following steps. □

Step 1. We claim the boundedness of $\{x_n\}$. In fact, noticing $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, we know that $\theta_n \leq \frac{\alpha_n(\tau - \nu\delta)}{2} \ \forall n \geq n_0$ for some $n_0 \geq 1$. Therefore, we have that for all $n \geq n_0$,

$$\alpha_n \nu\delta + \gamma_n + (1 - \gamma_n - \alpha_n \tau)(1 + \theta_n) \leq 1 - \alpha_n(\tau - \nu\delta) + \theta_n \leq 1 - \frac{\alpha_n(\tau - \nu\delta)}{2}.$$

Let p be an arbitrary point in $\Omega = \cap_{i=0}^{N}\mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$. Then $Tp = p,\ T_i p = p,\ i = 1, ..., N$, and (10) is true, that is,

$$\|v_n - p\|^2 + (1-\mu)\|u_n - y_n\|^2 + (1-\mu)\|v_n - y_n\|^2 \le \|u_n - p\|^2. \tag{18}$$

Thus, we obtain

$$\|v_n - p\| \le \|u_n - p\| \quad \forall n \ge 1. \tag{19}$$

From the definition of u_n, we have

$$\|u_n - p\| \le \|x_n - p\| + \sigma_n\|x_n - x_{n-1}\| = \|x_n - p\| + \alpha_n \cdot \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|. \tag{20}$$

From $\sup_{n\ge1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\sup_{n\ge1} \|x_n - x_{n-1}\| < \infty$, we infer that $\sup_{n\ge1} \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| < \infty$, which immediately yields that $\exists M_1 > 0$ s.t.

$$\frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1 \quad \forall n \ge 1. \tag{21}$$

Using (19)–(21), we obtain

$$\|v_n - p\| \le \|u_n - p\| \le \|x_n - p\| + \alpha_n M_1 \quad \forall n \ge 1. \tag{22}$$

Accordingly, by Algorithm 3, Lemma 6 and (22) we conclude that for all $n \ge n_0$,

$$\begin{aligned} &\|z_n - p\| = \|(1-\beta_n)(T_{N,n}T_{N-1,n}\cdots T_{1,n}v_n - p) + \beta_n(x_n - p)\| \\ &\le (1-\beta_n)\|T_{N,n}T_{N-1,n}\cdots T_{1,n}v_n - p\| + \beta_n\|x_n - p\| \\ &\le (1-\beta_n)(\|x_n - p\| + \alpha_n M_1) + \beta_n\|x_n - p\| \le \|x_n - p\| + \alpha_n M_1, \end{aligned} \tag{23}$$

and therefore

$$\begin{aligned} \|x_{n+1} - p\| &= \|\gamma_n(x_n - p) + \alpha_n(\nu f(x_n) - \rho F p) + ((1-\gamma_n)I - \alpha_n\rho F)T^n z_n \\ &\quad - ((1-\gamma_n)I - \alpha_n\rho F)p\| \\ &\le \alpha_n\nu\delta\|x_n - p\| + \alpha_n\|(\nu f - \rho F)p\| + \gamma_n\|x_n - p\| \\ &\quad + \|((1-\gamma_n)I - \alpha_n\rho F)T^n z_n - ((1-\gamma_n)I - \alpha_n\rho F)p\| \\ &= \alpha_n\nu\delta\|x_n - p\| + \alpha_n\|(\nu f - \rho F)p\| + \gamma_n\|x_n - p\| \\ &\quad + (1-\gamma_n)\|(I - \tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n z_n - (I - \tfrac{\alpha_n}{1-\gamma_n}\rho F)p\| \\ &\le \alpha_n\nu\delta\|x_n - p\| + \alpha_n\|(\nu f - \rho F)p\| + \gamma_n\|x_n - p\| \\ &\quad + (1-\gamma_n)(1 - \tfrac{\alpha_n}{1-\gamma_n}\tau)(1+\theta_n)\|z_n - p\| \\ &\le \alpha_n\nu\delta\|x_n - p\| + \alpha_n\|(\nu f - \rho F)p\| \\ &\quad + \gamma_n\|x_n - p\| + (1-\gamma_n-\alpha_n\tau)(1+\theta_n)(\|x_n - p\| + \alpha_n M_1) \\ &= [\alpha_n\nu\delta + \gamma_n + (1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n - p\| \\ &\quad + (1-\gamma_n-\alpha_n\tau)(1+\theta_n)\alpha_n M_1 + \alpha_n\|(\nu f - \rho F)p\| \\ &\le [1 - \tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(\tau-\nu\delta)}{2} \cdot \tfrac{2(M_1 + \|(\nu f - \rho F)p\|)}{\tau-\nu\delta} \\ &\le \max\{\tfrac{2(M_1 + \|(\nu f - \rho F)p\|)}{\tau-\nu\delta}, \|x_n - p\|\}. \end{aligned}$$

By induction, we conclude that $\|x_n - p\| \le \max\{\frac{2(M_1 + \|(\rho F - \nu f)p\|)}{\tau-\nu\delta}, \|x_{n_0} - p\|\}\ \forall n \ge n_0$. Therefore, we get the boundedness of vector sequence $\{x_n\}$.

Step 2. We claim that $\exists M_4 > 0$ s.t. $\forall n \ge n_0$,

$$(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n - y_n\|^2 + \|v_n - y_n\|^2] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.$$

In fact, using Lemma 6, Lemma 9, and the convexity of $\|\cdot\|^2$, from $\alpha_n+\gamma_n\le 1$, we obtain that for all $n\ge n_0$,

$$\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\alpha_n\nu(f(x_n)-f(p))+\gamma_n(x_n-p)+((1-\gamma_n)I-\alpha_n\rho F)T^n z_n \\
&\quad -((1-\gamma_n)I-\alpha_n\rho F)p+\alpha_n(\nu f-\rho F)p\|^2 \\
&\le \|\alpha_n\nu(f(x_n)-f(p))+\gamma_n(x_n-p)+((1-\gamma_n)I-\alpha_n\rho F)T^n z_n \\
&\quad -((1-\gamma_n)I-\alpha_n\rho F)p\|^2+2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\
&= \|\alpha_n\nu(f(x_n)-f(p))+\gamma_n(x_n-p)+(1-\gamma_n)[(I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n z_n \\
&\quad -(I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)p]\|^2+2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\
&\le [\alpha_n\nu\delta\|x_n-p\|+\gamma_n\|x_n-p\|+(1-\gamma_n)(1-\tfrac{\alpha_n}{1-\gamma_n}\tau)(1+\theta_n)\|z_n-p\|]^2 \\
&\quad +2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\
&\le \alpha_n\nu\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\|z_n-p\|^2 \\
&\quad +2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\
&\le \alpha_n\nu\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\beta_n\|x_n-p\|^2 \\
&\quad +(1-\beta_n)\|u_n-p\|^2-(1-\beta_n)(1-\mu)(\|u_n-y_n\|^2+\|v_n-y_n\|^2)]+\alpha_n M_2,
\end{aligned} \tag{24}$$

where $\sup_{n\ge1}2\|(\nu f-\rho F)p\|\|x_{n+1}-p\|\le M_2$ for some $M_2>0$. Also, from (22), we get

$$\|u_n-p\|^2\le\|x_n-p\|^2+\alpha_n(2M_1\|x_n-p\|+\alpha_n M_1^2)\le\|x_n-p\|^2+\alpha_n M_3, \tag{25}$$

where $\sup_{n\ge1}\{2M_1\|x_n-p\|+\alpha_n M_1^2\}\le M_3$ for some $M_3>0$. Note that $\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\le 1-\frac{\alpha_n(\tau-\nu\delta)}{2}$ for all $n\ge n_0$. Substituting (25) for (24), we deduce that for all $n\ge n_0$,

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le \alpha_n\nu\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\beta_n\|x_n-p\|^2 \\
&\quad +(1-\beta_n)(\|x_n-p\|^2+\alpha_n M_3)-(1-\beta_n)(1-\mu)(\|u_n-y_n\|^2+\|v_n-y_n\|^2)]+\alpha_n M_2 \\
&\le [\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n-p\|^2 \\
&\quad +(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\alpha_n M_3-(1-\beta_n)(1-\mu)(\|u_n-y_n\|^2+\|v_n-y_n\|^2)]+\alpha_n M_2 \\
&\le (1-\tfrac{\alpha_n(\tau-\nu\delta)}{2})\|x_n-p\|^2-(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)\times \\
&\quad \times[\|u_n-y_n\|^2+\|v_n-y_n\|^2]+\alpha_n M_2+\alpha_n M_3 \\
&\le \|x_n-p\|^2-(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2]+\alpha_n M_4,
\end{aligned}$$

where $M_4:=M_2+M_3$. This immediately implies that for all $n\ge n_0$,

$$(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2]\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4. \tag{26}$$

Step 3. We claim that $\exists M>0$ s.t. $\forall n\ge n_0$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 \quad &\le [1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(\tau-\nu\delta)}{2}[\tfrac{4}{\tau-\nu\delta}\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\
&\quad +\tfrac{\sigma_n}{\alpha_n}\cdot\tfrac{2M}{\tau-\nu\delta}\|x_n-x_{n-1}\|].
\end{aligned}$$

In fact, we get

$$\|u_n-p\|^2\le(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)^2\le\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M, \tag{27}$$

with $\sup_{n\ge1}\{2\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|\}\le M$ for some $M>0$. Note that $\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\le 1-\frac{\alpha_n(\tau-\nu\delta)}{2}$ for all $n\ge n_0$. Thus, combining (24) and (27), we have that for all $n\ge n_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\nu\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2\\ &\quad+\sigma_n\|x_n-x_{n-1}\|M]+2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle\\ &=[\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n-p\|^2\\ &\quad+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\sigma_n\|x_n-x_{n-1}\|M+2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle\\ &\le[1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(\tau-\nu\delta)}{2}[\tfrac{4\langle(\nu f-\rho F)p,x_{n+1}-p\rangle}{\tau-\nu\delta}+\tfrac{\sigma_n}{\alpha_n}\cdot\tfrac{\|x_n-x_{n-1}\|2M}{\tau-\nu\delta}].\end{aligned}\tag{28}$$

Step 4. We claim that $x_n\to x^*\in\Omega$, which is only a solution to the VIP (17). In fact, setting $p=x^*$, we obtain from (28) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le[1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-x^*\|^2+\tfrac{\alpha_n(\tau-\nu\delta)}{2}[\tfrac{4}{\tau-\nu\delta}\langle(\nu f-\rho F)x^*,x_{n+1}-x^*\rangle\\ &\quad+\tfrac{\sigma_n}{\alpha_n}\cdot\tfrac{2M}{\tau-\nu\delta}\|x_n-x_{n-1}\|].\end{aligned}\tag{29}$$

According to Lemma 4, it is sufficient to prove that $\limsup_{n\to\infty}\langle(\nu f-\rho F)x^*,x_{n+1}-x^*\rangle\le 0$. As $x_n-x_{n+1}\to 0$, $\alpha_n\to 0$, $\beta_n\to\beta<1$ and $\theta_n\to 0$, from (26) and $\{\gamma_n\}\subset[a,b]\subset(0,1)$, we have

$$\begin{aligned}&\limsup_{n\to\infty}(1-b-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2]\\ &\le\limsup_{n\to\infty}(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2]\\ &\le\limsup_{n\to\infty}(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|=0.\end{aligned}$$

This immediately implies that

$$\lim_{n\to\infty}\|u_n-y_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|v_n-y_n\|=0.\tag{30}$$

In addition, it is clear that $\|u_n-x_n\|=\sigma_n\|x_n-x_{n-1}\|\le\|x_n-x_{n-1}\|\to 0$ $(n\to\infty)$, and hence $\|x_n-y_n\|\le\|x_n-u_n\|+\|u_n-y_n\|\to 0$ $(n\to\infty)$. So it follows from (30) that $\|x_n-v_n\|\le\|x_n-y_n\|+\|y_n-v_n\|\to 0$ $(n\to\infty)$. Thus, from Algorithm 3 and the assumption $x_n-T_N^nT_{N-1}^n\cdots T_1^nx_n\to 0$, we obtain

$$\begin{aligned}&\|z_n-x_n\|=(1-\beta_n)\|T_{N,n}T_{N-1,n}\cdots T_{1,n}v_n-x_n\|\le\|T_{N,n}T_{N-1,n}\cdots T_{1,n}v_n-x_n\|\\ &\le\|T_{N,n}T_{N-1,n}\cdots T_{1,n}v_n-T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n\|+\|T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n-x_n\|\\ &\le\|v_n-x_n\|+\|T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n-x_n\|\to 0\quad(n\to\infty).\end{aligned}\tag{31}$$

As $x_n-y_n\to 0$, $x_n-z_n\to 0$ and $u_n-x_n\to 0$, we deduce that as $n\to\infty$,

$$\|u_n-y_n\|\le\|u_n-x_n\|+\|x_n-y_n\|\to 0\quad\text{and}\quad\|u_n-z_n\|\le\|u_n-x_n\|+\|x_n-z_n\|\to 0.\tag{32}$$

On the other hand, from the boundedness of $\{x_n\}$, it follows that $\exists\{x_{n_k}\}\subset\{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle(\nu f-\rho F)x^*,x_n-x^*\rangle=\lim_{k\to\infty}\langle(\nu f-\rho F)x^*,x_{n_k}-x^*\rangle.\tag{33}$$

Utilizing the reflexivity of H and the boundedness of $\{x_n\}$, one may suppose that $x_{n_k}\rightharpoonup\tilde{x}$. Therefore, one gets from (33),

$$\limsup_{n\to\infty}\langle(\nu f-\rho F)x^*,x_n-x^*\rangle=\langle(\nu f-\rho F)x^*,\tilde{x}-x^*\rangle.\tag{34}$$

It is easy to see from $u_n-x_n\to 0$ and $x_{n_k}\rightharpoonup\tilde{x}$ that $w_{n_k}\rightharpoonup\tilde{x}$. Since $T^nx_n-T^{n+1}x_n\to 0$, $x_n-x_{n+1}\to 0$, $u_n-y_n\to 0$, $u_n-z_n\to 0$ and $w_{n_k}\rightharpoonup\tilde{x}$, from Lemma 10 we get $\tilde{x}\in\Omega$. Therefore, from (17) and (34), we infer that

$$\limsup_{n\to\infty}\langle(\nu f-\rho F)x^*,x_n-x^*\rangle=\langle(\nu f-\rho F)x^*,\tilde{x}-x^*\rangle\le 0,$$

which together with $x_n - x_{n+1} \to 0$, implies that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle(\nu f-\rho F)x^*, x_{n+1}-x^*\rangle \\
&= \limsup_{n\to\infty}[\langle(\nu f-\rho F)x^*, x_{n+1}-x_n\rangle + \langle(\nu f-\rho F)x^*, x_n-x^*\rangle] \\
&= \langle(\nu f-\rho F)x^*, \tilde{x}-x^*\rangle \le 0.
\end{aligned} \tag{35}$$

Observe that $\{\frac{\alpha_n(\tau-\nu\delta)}{2}\} \subset [0,1]$, $\sum_{n=1}^{\infty}\frac{\alpha_n(\tau-\nu\delta)}{2} = \infty$, and

$$\limsup_{n\to\infty}[\frac{4}{\tau-\nu\delta}\langle(\nu f-\rho F)x^*, x_{n+1}-x^*\rangle + \frac{\sigma_n}{\alpha_n}\cdot\frac{2M}{\tau-\nu\delta}\|x_n-x_{n-1}\|] \le 0. \tag{36}$$

Consequently, by Lemma 4 we obtain from (29) that $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Next, we introduce another mildly inertial subgradient extragradient algorithm with line-search process.

It is remarkable that Lemmas 8 and 9 remain true for Algorithm 4.

Algorithm 4: MISEA II

1 **Initial Step:** Given $x_0, x_1 \in H$ arbitrary. Let $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$.
2 **Iteration Steps:** Compute x_{n+1} in what follows:

Step 1. Put $u_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $y_n = P_C(u_n - \ell_n A u_n)$, where ℓ_n is chosen to be the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\ell\|Au_n - Ay_n\| \le \mu\|u_n - y_n\|. \tag{37}$$

Step 2. Calculate $z_n = \beta_n x_n + (1-\beta_n)T_{N,n}T_{N-1,n}\cdots T_{1,n}P_{C_n}(u_n - \ell_n A y_n)$ with $C_n := \{u \in H : \langle u_n - \ell_n A u_n - y_n, u - y_n\rangle \le 0\}$.

Step 3. Calculate

$$x_{n+1} = \gamma_n u_n + ((1-\gamma_n)I - \alpha_n\rho F)T^n z_n + \alpha_n \nu f(x_n). \tag{38}$$

Update $n := n+1$ and return to Step 1.

Theorem 2. *Assume that the sequence $\{x_n\}$ constructed by Algorithm 4 satisfies $T^n x_n - T^{n+1}x_n \to 0$. Then,*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n \to 0 \end{cases}$$

where $x^* \in \Omega$ is only a solution to the HVI: $\langle(\nu f - \rho F)x^*, \omega - x^*\rangle \le 0 \ \forall \omega \in \Omega$.

Proof. Using the similar inference to that in the proof of Theorem 1, we obtain that there is only a solution $x^* \in \Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ to the HVI (17), and that the necessity of the theorem is true.

We claim the sufficiency of the theorem below. For this purpose, we suppose $\lim_{n\to\infty}(\|x_n - x_{n+1}\| + \|x_n - T_{N,n}T_{N-1,n}\cdots T_{1,n}x_n\|) = 0$ and prove the sufficiency by the following steps.

Step 1. We claim the boundedness of $\{x_n\}$. In fact, using the similar reasoning to that in Step 1 for the proof of Theorem 1, we know that inequalities (18)–(23) hold. Taking into account $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$, we know that $\theta_n \le \frac{\alpha_n(\tau-\nu\delta)}{2} \ \forall n \ge n_0$ for some $n_0 \ge 1$. Hence we deduce that for all $n \ge n_0$,

$$\alpha_n\nu\delta + \gamma_n + (1-\gamma_n-\alpha_n\tau)(1+\theta_n) \le 1 - \alpha_n(\tau-\nu\delta) + \theta_n \le 1 - \frac{\alpha_n(\tau-\nu\delta)}{2}.$$

Also, from Algorithm 4, Lemma 6, and (22) and (23) we obtain

$$\begin{aligned}\|x_{n+1}-p\| &= \|\gamma_n(u_n-p)+\alpha_n(\nu f(x_n)-\rho Fp)+((1-\gamma_n)I-\alpha_n\rho F)T^n z_n \\ &\quad -((1-\gamma_n)I-\alpha_n\rho F)p\| \\ &\le \alpha_n\nu\delta\|x_n-p\|+\alpha_n\|(\nu f-\rho F)p\|+\gamma_n\|u_n-p\| \\ &\quad +(1-\gamma_n)\|(I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n z_n-(I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)p\| \\ &\le \alpha_n\nu\delta\|x_n-p\|+\alpha_n\|(\nu f-\rho F)p\|+\gamma_n\|u_n-p\| \\ &\quad +(1-\gamma_n)(1-\tfrac{\alpha_n}{1-\gamma_n}\tau)(1+\theta_n)\|z_n-p\| \\ &\le \alpha_n\nu\delta\|x_n-p\|+\alpha_n\|(\nu f-\rho F)p\| \\ &\quad +\gamma_n(\|x_n-p\|+\alpha_n M_1)+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)(\|x_n-p\|+\alpha_n M_1) \\ &= [\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n-p\| \\ &\quad +[\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\alpha_n M_1+\alpha_n\|(\nu f-\rho F)p\| \\ &\le [1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|+\tfrac{\alpha_n(\tau-\nu\delta)}{2}\cdot\tfrac{2(M_1+\|(\nu f-\rho F)p\|)}{\tau-\nu\delta} \\ &\le \max\{\tfrac{2(M_1+\|(\nu f-\rho F)p\|)}{\tau-\nu\delta},\|x_n-p\|\}.\end{aligned}$$

By induction, we conclude that $\|x_n-p\|\le\max\{\frac{2(M_1+\|(\rho F-\nu f)p\|)}{\tau-\nu\delta},\|x_{n_0}-p\|\}\ \forall n\ge n_0$. Therefore, we obtain the boundedness of vector sequence $\{x_n\}$.

Step 2. One claims $\exists M_4>0$ s.t. $\forall n\ge n_0$,

$$(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2]\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4.$$

In fact, using Lemma 6, Lemma 9, and the convexity of $\|\cdot\|^2$, from $\alpha_n+\gamma_n\le 1$, we obtain that for all $n\ge n_0$,

$$\begin{aligned}&\|x_{n+1}-p\|^2=\|\gamma_n(u_n-p)+\alpha_n\nu(f(x_n)-f(p))+((1-\gamma_n)I-\alpha_n\rho F)T^n z_n \\ &\quad -((1-\gamma_n)I-\alpha_n\rho F)p+\alpha_n(\nu f-\rho F)p\|^2 \\ &\le \|\alpha_n\nu(f(x_n)-f(p))+\gamma_n(u_n-p)+((1-\gamma_n)I-\alpha_n\rho F)T^n z_n \\ &\quad -((1-\gamma_n)I-\alpha_n\rho F)p\|^2+2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\ &\le [\alpha_n\nu\delta\|x_n-p\|+\gamma_n\|u_n-p\|+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\|z_n-p\|]^2 \\ &\quad +2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\ &\le \alpha_n\nu\delta\|x_n-p\|^2+\gamma_n\|u_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\|z_n-p\|^2 \\ &\quad +2\alpha_n\langle(\nu f-\rho F)p,x_{n+1}-p\rangle \\ &\le \alpha_n\nu\delta\|x_n-p\|^2+\gamma_n\|u_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\beta_n\|x_n-p\|^2 \\ &\quad +(1-\beta_n)\|u_n-p\|^2-(1-\beta_n)(1-\mu)(\|u_n-y_n\|^2+\|v_n-y_n\|^2)]+\alpha_n M_2,\end{aligned} \tag{39}$$

where $\sup_{n\ge1}2\|(\nu f-\rho F)p\|\|x_{n+1}-p\|\le M_2$ for some $M_2>0$. Also, from (22) we have

$$\|u_n-p\|^2\le\|x_n-p\|^2+\alpha_n M_3, \tag{40}$$

where $\sup_{n\ge1}\{2M_1\|x_n-p\|+\alpha_n M_1^2\}\le M_3$ for some $M_3>0$. Note that $\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\le 1-\frac{\alpha_n(\tau-\nu\delta)}{2}$ for all $n\ge n_0$. Substituting (40) for (39), we deduce that for all $n\ge n_0$,

$$\begin{aligned}&\|x_{n+1}-p\|^2 \\ &\le \gamma_n(\|x_n-p\|^2+\alpha_n M_3)+\alpha_n\nu\delta\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2 \\ &\quad +\alpha_n M_3-(1-\beta_n)(1-\mu)(\|u_n-y_n\|^2+\|v_n-y_n\|^2)]+\alpha_n M_2 \\ &= [\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n-p\|^2+\gamma_n\alpha_n M_3 \\ &\quad +(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\alpha_n M_3-(1-\beta_n)(1-\mu)(\|u_n-y_n\|^2+\|v_n-y_n\|^2)]+\alpha_n M_2 \\ &\le (1-\tfrac{\alpha_n(\tau-\nu\delta)}{2})\|x_n-p\|^2-(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)\times \\ &\quad \times[\|u_n-y_n\|^2+\|v_n-y_n\|^2]+\alpha_n M_2+\alpha_n M_3 \\ &\le \|x_n-p\|^2-(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2]+\alpha_n M_4,\end{aligned}$$

where $M_4 := M_2 + M_3$. This immediately implies that for all $n \geq n_0$,

$$(1-\gamma_n-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|u_n-y_n\|^2+\|v_n-y_n\|^2] \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4. \tag{41}$$

Step 3. One claims that $\exists M > 0$ s.t. $\forall n \geq n_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(\tau-\nu\delta)}{2}[\tfrac{4}{\tau-\nu\delta}\langle(\nu f-\rho F)p, x_{n+1}-p\rangle \\ &\quad+\tfrac{\sigma_n}{\alpha_n}\cdot\tfrac{2M}{\tau-\nu\delta}\|x_n-x_{n-1}\|].\end{aligned}$$

In fact, we get

$$\|u_n-p\|^2 \leq \|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M, \tag{42}$$

where $\sup_{n\geq1}\{2\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|\} \leq M$ for some $M > 0$. Observe that $\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n) \leq 1-\frac{\alpha_n(\tau-\nu\delta)}{2}$ for all $n \geq n_0$. Thus, combining (39) and (42), we have that for all $n \geq n_0$,

$$\begin{aligned}&\|x_{n+1}-p\|^2 \\ &\leq \gamma_n(\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M)+\alpha_n\nu\delta\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\beta_n\|x_n-p\|^2 \\ &\quad+(1-\beta_n)(\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M)]+2\alpha_n\langle(\nu f-\rho F)p, x_{n+1}-p\rangle \\ &\leq \gamma_n(\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M)+\alpha_n\nu\delta\|x_n-p\|^2+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2 \\ &\quad+\sigma_n\|x_n-x_{n-1}\|M]+2\alpha_n\langle(\nu f-\rho F)p, x_{n+1}-p\rangle \\ &= [\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n-p\|^2+\gamma_n\sigma_n\|x_n-x_{n-1}\|M \\ &\quad+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\sigma_n\|x_n-x_{n-1}\|M+2\alpha_n\langle(\nu f-\rho F)p, x_{n+1}-p\rangle \\ &\leq [1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M+2\alpha_n\langle(\nu f-\rho F)p, x_{n+1}-p\rangle \\ &= [1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(\tau-\nu\delta)}{2}[\tfrac{4\langle(\nu f-\rho F)p, x_{n+1}-p\rangle}{\tau-\nu\delta}+\tfrac{\sigma_n}{\alpha_n}\cdot\tfrac{\|x_n-x_{n-1}\|2M}{\tau-\nu\delta}].\end{aligned} \tag{43}$$

Step 4. One claims that $x_n \to x^* \in \Omega$, which is only a solution to the VIP (17). In fact, using the similar inference to that in Step 4 for the proof of Theorem 1, one derives the desired conclusion. □

Example 1. *We can get an example of T satisfying the condition assumed in Theorems 1 and 2. As a matter of fact, we put $H = \mathbf{R}$, whose inner product and induced norm are defined by $\langle a, b\rangle = ab$ and $\|\cdot\| = |\cdot|$ indicate, respectively. Let $T : H \to H$ be defined as $Tx := \sin(\frac{7}{8}x)$ $\forall x \in H$. Then T is a contraction with constant $\frac{7}{8}$, and hence a nonexpansive mapping. Thus, T is an asymptotically nonexpansive mapping. As*

$$\|T^n x - T^n y\| \leq \frac{7}{8}\|T^{n-1}x - T^{n-1}y\| \leq \cdots \leq (\frac{7}{8})^n\|x-y\| \quad \forall x, y \in H,$$

we know that for any sequence $\{x_n\} \subset H$,

$$\|T^{n+1}x_n - T^n x_n\| \leq (\frac{7}{8})^{n-1}\|T^2x_n - Tx_n\| = (\frac{7}{8})^{n-1}\|\sin(\frac{7}{8}Tx_n)-\sin(\frac{7}{8}x_n)\| \leq 2(\frac{7}{8})^{n-1} \to 0$$

as $n \to \infty$. That is, $T^n x_n - T^{n+1}x_n \to 0$ $(n \to \infty)$.

Remark 3. *Compared with the corresponding results in Bnouhachem et al. [2], Cai et al. [35], Kraikaew and Saejung [36], and Thong and Hieu [37,38], our results improve and extend them in what follows.*

(i) The problem of obtaining a point of $\mathrm{VI}(C, A)$ in the work by the authors of [36] is extendable to the development of our problem of obtaining a point of $\cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$, where $T_0 := T$ is asymptotically nonexpansive and $\{T_i\}_{i=1}^N$ is a pool of nonexpansive maps. The Halpern subgradient method for solving the VIP in the work by the authors of [36] is extendable to the development of our mildly inertial subgradient algorithms

with linesearch process for solving the VIP and CFPP.

(ii) The problem of obtaining a point of $\mathrm{VI}(C, A)$ *in the work by the authors of [37] is extendable to the development of our problem of finding a point of* $\cap_{i=0}^{N}\mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$*, where* $T_0 := T$ *is asymptotically nonexpansive and* $\{T_i\}_{i=1}^{N}$ *is a pool of nonexpansive maps. The inertial subgradient method with weak convergence for solving the VIP in the work by the authors of [37] is extendable to the development of our mildly inertial subgradient algorithms with linesearch process (which are convergent in norm) for solving the VIP and CFPP.*

(iii) The problem of obtaining a point of $\mathrm{VI}(C, A) \cap \mathrm{Fix}(T)$ *(where* A *is monotone and* T *is quasi-nonexpansive) in the work by the authors of [38] is extendable to the development of our problem of obtaining a point of* $\cap_{i=0}^{N}\mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$*, where* $T_0 := T$ *is asymptotically nonexpansive and* $\{T_i\}_{i=1}^{N}$ *is a pool of nonexpansive maps. The inertial subgradient extragradient method with linesearch (which is weakly convergent) for solving the VIP and FPP in the work by the authors of [38] is extendable to the development of our mildly inertial subgradient algorithms with linesearch process (which are convergent in norm) for solving the VIP and CFPP. It is worth mentioning that the inertial subgradient method with linesearch process in the work by the authors of [38] combines the inertial subgradient approaches [37] with the Mann method.*

(iv) The problem of obtaining a point in the common fixed-point set $\cap_{i=1}^{N}\mathrm{Fix}(T_i)$ *of* N *nonexpansive mappings* $\{T_i\}_{i=1}^{N}$ *in the work by the authors of [2], is extendable to the development of our problem of obtaining a point of* $\cap_{i=0}^{N}\mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$*, where* $T_0 := T$ *is asymptotically nonexpansive and* $\{T_i\}_{i=1}^{N}$ *is a pool of nonexpansive maps. The iterative algorithm for hierarchical FPPs for finitely many nonexpansive mappings in the work by the authors of [2] (i.e., iterative scheme (3) in this paper), is extendable to the development of our mildly inertial subgradient algorithms with linesearch process for solving the VIP and CFPP. Meantime, the restrictions* $\limsup_{n\to\infty}\gamma_n < 1$*,* $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\lim_{n\to\infty}|\delta_{n-1}^{i} - \delta_{i,n}| = 0$ *for* $i = 1, ..., N$ *imposed on (3), are dropped, where* $0 < \liminf_{n\to\infty}\gamma_n < \limsup_{n\to\infty}\gamma_n < 1$ *is weakened to the condition* $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$.

(v) The problem of obtaining a point in the common solution set Ω *of the VIPs for two inverse-strongly monotone mappings and the FPP of an asymptotically nonexpansive mapping in the work by the authors of [35], is extendable to the development of our problem of obtaining a point of* $\cap_{i=0}^{N}\mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ *where* $T_0 := T$ *is asymptotically nonexpansive and* $\{T_i\}_{i=1}^{N}$ *is a pool of nonexpansive maps. The viscosity implicit rule involving a modified extragradient method in the work by the authors of [35] (i.e., iterative scheme (4) in this paper), is extendable to the development of our mildly inertial subgradient algorithms with linesearch process for solving the VIP and CFPP. Moreover, the conditions* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty}\|T^{n+1}y_n - T^n y_n\| < \infty$ *imposed on (4), are deleted where* $\sum_{n=1}^{\infty}\|T^{n+1}y_n - T^n y_n\| < \infty$ *is weakened to the assumption* $\|T^{n+1}x_n - T^n x_n\| \to 0$ $(n \to \infty)$.

4. Applications

In this section, our main theorems are used to deal with the VIP and CFPP in an illustrating example. The initial point $x_0 = x_1$ is randomly chosen in $\mathbf{R}$. Take $\nu f(x) = F(x) = \frac{1}{2}x$, $\gamma = l = \mu = \frac{1}{2}$, $\sigma_n = \alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{2}$, $\nu = \frac{3}{4}$, $f = \frac{2}{3}I$ and $\rho = 2$. Then, we know that $\alpha_n + \gamma_n \le 1\ \forall n \ge 1$, $\nu\delta = \kappa = \eta = \frac{1}{2}$, and

$$\tau = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} = 1 - \sqrt{1 - 2(2 \cdot \frac{1}{2} - 2(\frac{1}{2})^2)} = 1 \in (0, 1].$$

We first provide an example of a Lipschitzian, pseudomonotone operator A, asymptotically nonexpansive operator T, and nonexpansive operator T_1 with $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $C = [-1, 3]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Let $A, T, T_1, T_1^n : H \to H$ be defined as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$, $Tx := \frac{4}{5}\sin x$, $Tx := \sin x$ and

$T_1^n x := \frac{3}{8}x + \frac{5}{8}\sin x \; \forall x \in H, n \geq 1$. Then it is clear that T_1 is a nonexpansive mapping on H. Moreover, from Lemma 5 we know that $\mathrm{Fix}(T_1^n) = \mathrm{Fix}(T_1) = \{0\} \; \forall n \geq 1$. Now, we first show that A is Lipschitzian, pseudomonotone operator with $L = 2$. In fact, for all $x, y \in H$ we get

$$\begin{aligned} \|Ax - Ay\| &= |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|\sin y\|} + \tfrac{1}{1+\|y\|}| \\ &\leq |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|\sin y\|}| + |\tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|y\|}| \\ &= |\tfrac{\|\sin y\| - \|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}| + |\tfrac{\|y\| - \|x\|}{(1+\|x\|)(1+\|y\|)}| \\ &\leq \|\sin x - \sin y\| + \|x - y\| \\ &\leq 2\|x - y\|. \end{aligned}$$

This means that A is Lipschitzian with $L = 2$. We below claim that A is pseudomonotone. For any given $x, y \in H$, it is clear that the relation holds:

$$\langle Ax, y - x\rangle = (\frac{1}{1+|\sin x|} - \frac{1}{1+|x|})(y - x) \geq 0 \Rightarrow \langle Ay, y - x\rangle = (\frac{1}{1+|\sin y|} - \frac{1}{1+|y|})(y - x) \geq 0.$$

Furthermore, it is easy to see that T is asymptotically nonexpansive with $\theta_n = (\frac{4}{5})^n \; \forall n \geq 1$, such that $\|T^{n+1}x_n - T^n x_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\|T^n x - T^n y\| \leq \frac{4}{5}\|T^{n-1}x - T^{n-1}y\| \leq \cdots \leq (\frac{4}{5})^n \|x - y\| \leq (1 + \theta_n)\|x - y\|,$$

and

$$\|T^{n+1}x_n - T^n x_n\| \leq (\frac{4}{5})^{n-1}\|T^2 x_n - Tx_n\| = (\frac{4}{5})^{n-1}\|\frac{4}{5}\sin(Tx_n) - \frac{4}{5}\sin x_n\| \leq 2(\frac{4}{5})^n \to 0 \; (n \to \infty).$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = \lim_{n\to\infty} \frac{(4/5)^n}{1/(n+1)} = 0.$$

Therefore, $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. In this case, Algorithm 3 can be rewritten as follows,

$$\begin{cases} u_n = x_n + \frac{1}{n+1}(x_n - x_{n-1}), \\ y_n = P_C(u_n - \ell_n A u_n), \\ z_n = \frac{1}{3}x_n + \frac{2}{3}T_1^n P_{C_n}(u_n - \ell_n A y_n), \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2}x_n + \frac{1}{2}x_n + (\frac{n}{n+1} - \frac{1}{2})T^n z_n \quad \forall n \geq 1, \end{cases} \tag{44}$$

where for every $n \geq 1$, C_n and ℓ_n are picked up as in Algorithm 3. Then, by Theorem 1, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A)$ if and only if $|x_n - x_{n+1}| + |x_n - T_1^n x_n| \to 0$ as $n \to \infty$.

On the other hand, Algorithm 4 can be rewritten as follows,

$$\begin{cases} u_n = x_n + \frac{1}{n+1}(x_n - x_{n-1}), \\ y_n = P_C(u_n - \ell_n A u_n), \\ z_n = \frac{1}{3}x_n + \frac{2}{3}T_1^n P_{C_n}(u_n - \ell_n A y_n), \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2}x_n + \frac{1}{2}u_n + (\frac{n}{n+1} - \frac{1}{2})T^n z_n \quad \forall n \geq 1, \end{cases} \tag{45}$$

where for every $n \geq 1$, C_n and ℓ_n are picked up as in Algorithm 4. Then, by Theorem 2, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A)$ if and only if $|x_n - x_{n+1}| + |x_n - T_1^n x_n| \to 0$ as $n \to \infty$.

Author Contributions: These authors contributed equally to this work.

Funding: The first author was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002), and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ceng, L.C.; Petruşel, A.; Yao, J.C.; Yao, Y. Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. *Fixed Point Theory* **2018**, *19*, 487–501. [CrossRef]
2. Bnouhachem, A.; Ansari, Q.H.; Yao, J.C. An iterative algorithm for hierarchical fixed point problems for a finite family of nonexpansive mappings. *Fixed Point Theory Appl.* **2015**, *2015*, 111. [CrossRef]
3. Ceng, L.C.; Petruşel, A.; Yao, J.C.; Yao, Y. Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. *Fixed Point Theory* **2019**, *20*, 113–133. [CrossRef]
4. Qin, X.; An, N.T. Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets. *Comput. Optim. Appl.* **2019**. [CrossRef]
5. Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **2011**, *148*, 318–335. [CrossRef] [PubMed]
6. Takahahsi, W.; Yao, J.C. The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces. *J. Nonlinear Convex Anal.* **2019**, *20*, 173–195.
7. Cho, S.Y.; Li, W.; Kang, S.M. Convergence analysis of an iterative algorithm for monotone operators. *J. Inequal. Appl.* **2013**, *2013*, 199. [CrossRef]
8. Chang, S.S.; Wen, C.F.; Yao, J.C. Zero point problem of accretive operators in Banach spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 105–118. [CrossRef]
9. Nguyen, L.V.; Qin, X. Some results on strongly pseudomonotone quasi-variational inequalities. *Set-Valued Var. Anal.* **2019**. [CrossRef]
10. Ceng, L.C.; Postolache, M.; Wen, C.F.; Yao, Y. Variational inequalities approaches to minimization problems with constraints of generalized mixed equilibria and variational inclusions. *Mathematics* **2019**, *7*, 270. [CrossRef]
11. Ceng, L.C.; Ansari, Q.H.; Yao, J.C. Some iterative methods for finding fixed points and for solving constrained convex minimization problems. *Nonlinear Aanl.* **2011**, *74*, 5286–5302. [CrossRef]
12. Chen, J.; Kobis, E.; Yao, J.C. Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints. *J. Optim. Theory Appl.* **2019**, *181*, 411–436. [CrossRef]
13. Liu, L.; Qin, X.; Agarwal, R.P. Iterative methods for fixed points and zero points of nonlinear mappings with applications. *Optimization* **2019**. [CrossRef]
14. Cho, S.Y. Strong convergence analysis of a hybrid algorithm for nonlinear operators in a Banach space. *J. Appl. Anal. Comput.* **2018**, *8*, 19–31.
15. Ansari, Q.H.; Babu, F.; Yao, J.C. Regularization of proximal point algorithms in Hadamard manifolds. *J. Fixed Point Theory Appl.* **2019**, *21*, 25. [CrossRef]
16. Zhao, X. Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. *Appl. Math. Optim.* **2018**, *78*, 613–641. [CrossRef]
17. Cho, S.Y.; Kang, S.M. Approximation of common solutions of variational inequalities via strict pseudocontractions. *Acta Math. Sci.* **2012**, *32*, 1607–1618. [CrossRef]
18. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker: New York, NY, USA, 1984.
19. Chang, S.S.; Wen, C.F.; Yao, J.C. Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces. *Optimization* **2018**, *67*, 1183–1196. [CrossRef]
20. Ceng, L.C.; Ansari, Q.H.; Yao, J.C. An extragradient method for solving split feasibility and fixed point problems. *Comput. Math. Appl.* **2012**, *64*, 633–642. [CrossRef]
21. Ceng, L.C.; Liu, Z.; Yao, J.C.; Yao, Y. Optimal control of feedback control systems governed by systems of evolution hemivariational inequalities. *Filomat* **2018**, *32*, 5205–5220. [CrossRef]
22. Korpelevich, G.M. The extragradient method for finding saddle points and other problems. *Ekonomikai Matematicheskie Metody* **1976**, *12*, 747–756.

23. Takahashi, W.; Wen, C.F.; Yao, J.C. The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space. *Fixed Point Theory* **2018**, *19*, 407–419. [CrossRef]
24. Vuong, P.T. On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. *J. Optim. Theory Appl.* **2018**, *176*, 399–409. [CrossRef]
25. Vuong, P.T.; Shehu, Y. Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. *Numer. Algorithms* **2019**, *81*, 269–291. [CrossRef]
26. Ceng, L.C.; Ansari, Q.H.; Wong, M.M.; Yao, J.C. Mann type hybrid extragradient method for variational inequalities, variational inclusions and fixed point problems. *Fixed Point Theory* **2012**, *13*, 403–422.
27. Ceng, L.C.; Wen, C.F. Systems of variational inequalities with hierarchical variational inequality constraints for asymptotically nonexpansive and pseudocontractive mappings. *RACSAM* **2019**, *113*, 2431–2447. [CrossRef]
28. Shehu, Y.; Dong, Q.; Jiang, D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. *Optimization* **2019**, *68*, 385–409. [CrossRef]
29. Shehu, Y.; Iyiola, O.S. Strong convergence result for monotone variational inequalities. *Numer. Algorithms* **2017**, *76*, 259–282. [CrossRef]
30. Ceng, L.C.; Hadjisavvas, N.; Wong, N.C. Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems. *J. Glob. Optim.* **2010**, *46*, 635–646. [CrossRef]
31. Ceng, L.C.; Teboulle, M.; Yao, J.C. Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems. *J. Optim. Theory Appl.* **2010**, *146*, 19–31. [CrossRef]
32. Ceng, L.C.; Ansari, Q.H.; Wong, N.C.; Yao, J.C. An extragradient-like approximation method for variational inequalities and fixed point problems. *Fixed Point Theory Appl.* **2011**, *2011*, 22. [CrossRef]
33. Ceng, L.C.; Ansari, Q.H.; Yao, J.C. Relaxed extragradient iterative methods for variational inequalities. *Appl. Math. Comput.* **2011**, *218*, 1112–1123. [CrossRef]
34. Ceng, L.C.; Ansari, Q.H.; Schaible, S. Hybrid extragradient-like methods for generalized mixed equilibrium problems, systems of generalized equilibrium problems and optimization problems. *J. Glob. Optim.* **2012**, *53*, 69–96. [CrossRef]
35. Cai, G.; Shehu, Y.; Iyiola, O.S. Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules. *Numer. Algorithms* **2018**, *77*, 535–558. [CrossRef]
36. Kraikaew, R.; Saejung, S. Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.* **2014**, *163*, 399–412. [CrossRef]
37. Thong, D.V.; Hieu, D.V. Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. *Numer. Algorithms* **2019**, *80*, 1283–1307. [CrossRef]
38. Thong, D.V.; Hieu, D.V. Modified subgradient extragradient method for variational inequality problems. *Numer. Algorithms* **2018**, *79*, 597–610. [CrossRef]
39. Cottle R.W.; Yao, J.C. Pseudo-monotone complementarity problems in Hilbert space. *J. Optim. Theory Appl.* **1992**, *75*, 281–295. [CrossRef]
40. Xue, Z.; Zhou, H.; Cho, Y.J. Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces. *J. Nonlinear Convex Anal.* **2000**, *1*, 313–320.
41. Zhou, H. Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces. *Nonlinear Anal.* **2008**, *69*, 456–462. [CrossRef]
42. Yamada, I. The hybrid steepest-descent method for variational inequalities problems over the intersection of the fixed point sets of nonexpansive mappings, In *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*; Butnariu, D., Censor, Y., Reich, S., Eds.; North-Holland: Amsterdam, The Netherlands, 2001; pp. 473–504.
43. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory (Cambridge Studies in Advanced Mathematics)*; Cambridge University Press: Cambridge, UK, 1990; Volume 28.