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Fixed Points for a Pair of *F*-Dominated Contractive Mappings in Rectangular *b*-Metric Spaces with Graph

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Abstract: Recently, George et al. (in Georgea, R.; Radenovicb, S.; Reshmac, K.P.; Shuklad, S. Rectangular b-metric space and contraction principles. *J. Nonlinear Sci. Appl.* 2015, 8, 1005–1013) furnished the notion of rectangular b-metric pace (RBMS) by taking the place of the binary sum of triangular inequality in the definition of a b-metric space ternary sum and proved some results for Banach and Kannan contractions in such space. In this paper, we achieved fixed-point results for a pair of *F*-dominated mappings fulfilling a generalized rational *F*-dominated contractive condition in the better framework of complete rectangular *b*-metric spaces complete rectangular *b*-metric spaces. Some new fixed-point results with graphic contractions for a pair of graph-dominated mappings on rectangular *b*-metric space have been obtained. Some examples are given to illustrate our conclusions. New results in ordered spaces, partial *b*-metric space, dislocated metric space, dislocated *b*-metric space, partial metric space, *b*-metric space, rectangular metric spaces, and metric space can be obtained as corollaries of our results.

Keywords: fixed point; generalized *F*-contraction; α_*-dominated mapping; graphic contractions

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1. Introduction and Preliminaries

Fixed-point theory is a basic tool in functional analysis. Banach [1] has shown significant result for contraction mappings. Due to its significance, a large number of authors have proved newsworthy of this result (see [1–28]). In the sequel George et al. [2] furnished the notion of rectangular *b*-metric space (RBMS) by taking the place of the binary sum of triangular inequality in the definition of a *b*-metric space ternary sum and proved some results for Banach and Kannan contractions in such space. Further recent results on rectangular *b*-metric spaces can be seen in [10,11]. In this paper, we achieved fixed-point results for a pair of α-dominated mappings fulfilling a generalized rational *F*-dominated contractive condition in complete rectangular *b*-metric spaces. Therefore, here, we investigate our results in a better framework of rectangular *b*-metric space. Some new fixed-point results with graphic contractions for a pair of graph-dominated mappings on rectangular *b*-metric space have been obtained. New results in ordered spaces, partial *b*-metric space, dislocated metric space, dislocated *b*-metric space, partial metric space, *b*-metric space, rectangular metric spaces, and metric space can be obtained as corollaries of our results. First, we give the precise definitions that we will use.

Definition 1 ([2])**.** *Let Z be a nonempty set and let $d_l : Z \times Z \to [0, \infty)$ be a function, called a rectangular b-metric (or simply d_l-metric), if there exists $b \geq 1$ such that the following conditions hold:*

(i) $d_l(g,p) = 0$, *if and only if* $g = p$;
(ii) $d_l(g,p) = d_l(p,g)$;
(iii) $d_l(g,p) \leq b[d_l(g,q) + d_l(q,h) + d_l(h,p)]$ *for all* $g,p \in Z$ *and all distinct points* $q,h \in Z \setminus \{g,p\}$.
The pair (Z,d_l) *is said a rectangular b-metric space (in short R.B.M.S) with coefficient b.*

Definition 2 ([2])**.** *Let* (Z,d_l) *be a R.B.M.S.*
(i) A sequence $\{g_n\}$ *in* (Z,d_l) *said to be Cauchy sequence if for each* $\varepsilon > 0$, *there corresponds* $n_0 \in N$ *such that for all* $n,m \geq n_0$ *we have* $d_l(g_m,g_n) < \varepsilon$ *or* $\lim_{n,m\to\infty} d_l(g_n,g_m) = 0$.
(ii) A sequence $\{g_n\}$ *rectangular b-converges (for short* d_l *-converges) to g if* $\lim_{n\to\infty} d_l(g_n,g) = 0$. *In this case, g is called a* d_l*-limit of* $\{g_n\}$.
(iii) (Z,d_l) *is complete if every Cauchy sequence in Z converges to a point* $g \in Z$ *for which* $d_l(g,g) = 0$.

Example 1 ([2])**.** *Let* $Z = N$ *define* $d : Z \times Z \to Z$ *such that* $d(u,v) = d(v,u)$ *for all* $u,v \in Z$ *and*

$$d(u,v) = \begin{cases} 0, \text{ if } u = v; \\ 10\alpha, \text{ if } u = 1,\ v = 2; \\ \alpha, \text{ if } u \in \{1,2\} \text{ and } v \in \{3\}; \\ 2\alpha, \text{ if } u \in \{1,2,3\} \text{ and } v \in \{4\}; \\ 3\alpha, \text{ if } u \text{ or } v \notin \{1,2,3,4\} \text{ and } u \neq v; \end{cases}$$

where $\alpha > 0$ *is a constant. Then* (Z,d) *is a R.B.M.S with coefficient* $b = 2 > 1$, *but* (Z,d) *does not be a rectangular metric, since*

$$d(1,2) = 10\alpha > 5\alpha = d(1,3) + d(3,4) + d(4,2).$$

Definition 3 ([26])**.** *Let* (Z,d_l) *be a metric space,* $S : Z \to P(Z)$ *be a multivalued mapping and* $\alpha : Z \times Z \to [0,+\infty)$. *Let* $A \subseteq Z$, *the mapping S is said semi* α_**-admissible on A, if* $\alpha(x,y) \geq 1$ *implies* $\alpha_\star(Sx,Sy) \geq 1$, *for all* $x \in A$, *where* $\alpha_*(Sx,Sy) = \inf\{\alpha(a,b) : a \in Sx, b \in Sy\}$. *When* $A = Z$, *we say that the S is* α_**-admissible on Z. In the case in which S is a single valued mapping, the previous definition becomes.*

Definition 4. *Let* (Z,d_l) *be a R.B.M.S. Let* $S : Z \to Z$ *be a mapping and* $\alpha : Z \times Z \to [0,+\infty)$. *If* $A \subseteq Z$, *we say that the S is* α*-dominated on A, whenever* $\alpha(i,Si) \geq 1$ *for all* $i \in A$. *If* $A = Z$, *we say that S is* α*-dominated.*

Definition 5 ([28])**.** *Let* (Z,d) *be a metric space. A mapping* $H : Z \to Z$ *is said to be an* A *−contraction if there exists* $\tau > 0$ *such that*

$$\forall j,k \in Z,\ d(Hj,Hk) > 0 \Rightarrow \tau + A\left(d(Hj,Hk)\right) \leq A\left(d(j,k)\right)$$

with $A : \mathbb{R}_+ \to \mathbb{R}$ *real function which satisfies three assumptions:*
(F1) A is strictly increasing
(F2) For any sequence $\{\alpha_n\}_{n=1}^{\infty}$ *of positive real numbers,* $\lim_{n\to\infty} \alpha_n = 0$ *is equivalent to* $\lim_{n\to\infty} A(\alpha_n) = -\infty$;
(F3) There is $k \in (0,1)$ *for which* $\lim \alpha \to 0^+ \alpha^k A(\alpha) = 0$.

Example 2 ([19])**.** *Let* $Z = \mathbb{R}$. *Define the mapping* $\alpha : Z \times Z \to [0,\infty)$ *by*

$$\alpha(x,y) = \left\{ \begin{array}{c} 1 \text{ if } x > y \\ \frac{1}{2} \text{ otherwise} \end{array} \right\}.$$

Define the self-mappings $S,T : Z \to Z$ *by* $Sx = \frac{x}{4}$, *and* $Ty = \frac{y}{2}$, *where* $x,y \in Z$. *Suppose* $x = 3$ *and* $y = 2$. *As* $3 > 2$, *then* $\alpha(3,2) \geq 1$. *Now,* $\alpha(S3,T2) = \frac{1}{2} \ngeq 1$, *this means the pair* (S,T) *is not* α*-admissible. Also,*

$\alpha(S3, S2) \ngeq 1$ and $\alpha(T3, T2) \ngeq 1$. This implies S and T are not α-admissible individually. Now, $\alpha(x, Sx) \geq 1$, for all $x \in Z$. Hence S is α-dominated mapping. Similarly it is clear that $\alpha(y, Ty) \geq 1$ for all $x \in Z$. Hence it is clear that S and T are α-dominated but not α-admissible.

2. Main Result

Theorem 1. *Let (Z, d_l) be a complete R.B.M.S with coefficient $b \geq 1$. Let $\alpha : Z \times Z \to [0, \infty)$ be a function and $S, T : Z \to Z$ be the α-dominated mappings on Z. Suppose that the following condition is satisfied:*

There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1$ and a continuous and strictly increasing real function F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{array}\right), \tag{1}$$

whenever $e, y \in \{g_n\}$, $\alpha(e, y) \geq 1$ and $d_l(Se, Ty) > 0$ "where the sequence g_n is defined by g_0 arbitrary in Z, $g_{2n+1} = S(TS)^n g_0$ and $g_{2n} = (TS)^{n+1} g_0$". Then $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{g_n\} \to u \in Z$. Also, if the inequality (1) holds for u and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have a common fixed point u in Z.

Proof. Chose a point g_0 in Z such that $g_1 = Sg_0$ and $g_2 = Tg_1$. Continuing this process we construct a sequence $\{g_n\}$ of points in Z such that $g_{2n+1} = Sg_{2n}$ and $g_{2n+2} = Tg_{2n+1}$for all for all $n \in \mathbb{N} \cup \{0\}$. Let $g_1, \cdots, g_j \in Z$ for some $j \in \mathbb{N}$. If j is odd, then $j = 2\imath + 1$ for some $\imath \in \mathbb{N}$. Since $S, T : Z \to Z$ be the α-dominated mappings on Z, so $\alpha(g_{2\imath}, Sg_{2\imath}) \geq 1$ and $\alpha(g_{2\imath+1}, Tg_{2\imath+1}) \geq 1$. As $\alpha(g_{2\imath}, Sg_{2\imath}) \geq 1$, this implies $\alpha(g_{2\imath}, Sg_{2\imath}) = \alpha(g_{2\imath}, g_{2\imath+1}) \geq 1$ where $g_{2\imath+1} = Sg_{2\imath}$. Now, by using inequality (1),

$$\begin{aligned}
\tau + F(d_l(g_{2\imath+1}, g_{2\imath+2})) &\leq \tau + F(d_l(Sg_{2\imath}, Tg_{2\imath+1})) \\
&\leq F\left[\begin{array}{c} \eta_1 d_l(g_{2\imath}, g_{2\imath+1}) + \eta_2 d_l(g_{2\imath}, Sg_{2\imath}) + \eta_3 d_l(g_{2\imath}, Tg_{2\imath+1}) \\ +\eta_4 \frac{d_l^2(g_{2\imath},Sg_{2\imath}).d_l(g_{2\imath+1},Tg_{2\imath+1})}{1+d_l^2(g_{2\imath},g_{2\imath+1})} \end{array}\right] \\
&\leq F\left[\begin{array}{c} \eta_1 d_l(g_{2\imath}, g_{2\imath+1}) + \eta_2 d_l(g_{2\imath}, g_{2\imath+1}) + b\eta_3 d_l(g_{2\imath}, g_{2\imath+1}) \\ +b\eta_3 d_l(g_{2\imath+1}, g_{2\imath+2}) + \eta_4 \frac{d_l^2(g_{2\imath},g_{2\imath+1}).d_l(g_{2\imath+1},g_{2\imath+2})}{1+d_l^2(g_{2\imath},g_{2\imath+1})} \end{array}\right] \\
&\leq F\left[(\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath}, g_{2\imath+1}) + (b\eta_3 + \eta_4) d_l(g_{2\imath+1}, g_{2\imath+2})\right].
\end{aligned}$$

This implies

$$F(d_l(g_{2\imath+1}, g_{2\imath+2})) < F\left[\begin{array}{c} (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath}, g_{2\imath+1}) \\ +(b\eta_3 + \eta_4) d_l(g_{2\imath+1}, g_{2\imath+2}) \end{array}\right]$$

As F is strictly increasing. Therefore, we have

$$d_l(g_{2\imath+1}, g_{2\imath+2}) < \left[\begin{array}{c} (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath}, g_{2\imath+1}) \\ +(b\eta_3 + \eta_4) d_l(g_{2\imath+1}, g_{2\imath+2}). \end{array}\right]$$

Which implies

$$\begin{aligned}
(1 - b\eta_3 - \eta_4) d_l(g_{2\imath+1}, g_{2\imath+2}) &< (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath}, g_{2\imath+1}) \\
d_l(g_{2\imath+1}, g_{2\imath+2}) &< \left(\frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4}\right) d_l(g_{2\imath}, g_{2\imath+1}).
\end{aligned}$$

Now, we note that by assumption of inequality (1) it immediately follows $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4} < 1$. Hence

$$d_l(g_{2\imath+1}, g_{2\imath+2}) < \lambda d_l(g_{2\imath}, g_{2\imath+1}) < \lambda^2 d_l(g_{2\imath-1}, g_{2\imath}) < \cdots < \lambda^{2i+1} d_l(g_0, g_1).$$

Similarly, if j is even, we have

$$d_l(g_{2i+2}, g_{2i+3}) < \lambda^{2i+2} d_l(g_0, g_1). \tag{2}$$

Now, we have

$$d_l(g_j, g_{j+1}) < \lambda^j d_l(g_0, g_1) \text{ for } j \in \mathbb{N}. \tag{3}$$

Also $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$d_l(g_n, g_{n+1}) < \lambda^n d_l(g_0, g_1) \text{ for all } n \in \mathbb{N}. \tag{4}$$

Now, for any positive integers m, n $(m > n)$, we have

$$\begin{aligned} d_l(g_n, g_m) &\leq b[d_l(g_n, g_{n+1}) + d_l(g_{n+1}, g_{n+2}) + d_l(g_{n+2}, g_m)] \\ &\leq b[d_l(g_n, g_{n+1}) + d_l(g_{n+1}, g_{n+2})] + b^2[d_l(g_{n+2}, g_{n+3}) \\ &\quad + d_l(g_{n+3}, g_{n+4}) + d_l(g_{n+4}, g_m)] \end{aligned}$$

$$\begin{aligned} &\leq b[\lambda^n + \lambda^{n+1}]d_l(g_0, g_1) + b^2[\lambda^{n+2} + \lambda^{n+3}]d_l(g_0, g_1) \\ &\quad + b^3[[\lambda^{n+4} + \lambda^{n+5}]d_l(g_0, g_1) + \cdots \\ &\quad + b^{2m-1}\lambda^{m-n}d_l(g_0, g_1), \qquad \text{(by (2.4))} \\ &\leq b\lambda^n[1 + b\lambda^2 + b^2\lambda^4 + \cdots]d_l(g_0, g_1) \\ &\quad + b\lambda^{n+1}[1 + b\lambda^2 + b^2\lambda^4 + \cdots]d_l(g_0, g_1) \\ &\leq \frac{1+\lambda}{1-b\lambda^2} b\lambda^n d_l(g_0, g_1). \end{aligned}$$

As $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, $b \geq 1$ and $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$, so $|b\lambda^2| < 1$. Then, we have

$$d_l(g_n, g_m) < \frac{1+\lambda}{1-b\lambda^2} b\lambda^n d_l(g_0, g_1) \to 0 \text{ as } n \to \infty.$$

Hence $\{g_n\}$ is a Cauchy sequence in Z. Since (Z, d_l) is a complete metric space, so there exist $u \in Z$ such that $\{g_n\} \to u$ as $n \to \infty$, then

$$\lim_{n\to\infty} d_l(g_n, u) = 0. \tag{5}$$

By assumption, $\alpha(u, g_n) \geq 1$. Suppose that $d_l(u, Su) > 0$, then there exists positive integer k such that $d_l(Tg_{2n+1}, Su) > 0$ for all $n \geq k$. For $n \geq k$, we have

$$\begin{aligned} d_l(u, Su) &\leq b[d_l(u, g_n) + d_l(g_n, g_{2n+2}) + d_l(g_{2n+2}, Su)] \\ &\leq b[d_l(u, g_n) + d_l(g_n, g_{2n+1}) + d_l(Tg_{2n+1}, Su)] \\ &\leq b[d_l(u, g_n) + d_l(g_n, g_{2n+1}) + d_l(Su, Tg_{2n+1})] \\ &< b\begin{bmatrix} d_l(u, g_n) + d_l(g_n, g_{2n+1}) + \eta_1 d_l(u, g_{2n+1}) \\ +\eta_2 d_l(u, Su) + \eta_3 d_l(g_{2n+1}, Tg_{2n+1}) \\ +\eta_4 \frac{d_l(u,Su).d_l^2(g_{2n+1},Tg_{2n+1})}{1+d_l^2(g_{2n+1},u)}. \end{bmatrix} \end{aligned}$$

Letting $n \to \infty$, and by using the inequalities (4) and (5) we get

$$d_l(u, Su) < \eta_3 d_l(u, Su) < d_l(u, Su),$$

which is a contradiction. So, our supposition is wrong. Hence $d_l(u, Su) = 0$. Similarly, by using the above inequlity

$$\begin{aligned} d_l(u, Tu) &\leq b[d_l(u, g_n) + d_l(g_n, g_{2n+1}) + d_l(g_{2n+1}, Tu)] \\ d_l(u, Tu) &\leq b[d_l(u, g_n) + d_l(g_n, g_{2n+1}) + d_l(Sg_{2n}, Tu)] \end{aligned}$$

we can get $d_l(u, Tu) = 0$. This shows that u is a common fixed point of S and T. □

Example 3. *Let $Z = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2, 3, 4, 5\}\}$ and $B = [1, 2]$. Define $d_l : Z \times Z \to [0, \infty)$ such that defined by $d_l(x, y) = d_l(y, x)$ for $x, y \in Z$ and*

$$\begin{cases} d_l(\frac{1}{2}, \frac{1}{3}) = d_l(\frac{1}{4}, \frac{1}{5}) = 0.03 \\ d_l(\frac{1}{2}, \frac{1}{5}) = d_l(\frac{1}{3}, \frac{1}{4}) = 0.02 \\ d_l(\frac{1}{2}, \frac{1}{4}) = d_l(\frac{1}{5}, \frac{1}{3}) = 0.6 \\ d_l(x, y) = |x - y|^2 \quad \text{otherwise.} \end{cases}$$

be the complete R.B.M.S with coefficient $b = 4 > 1$ but (Z, d_l) is neither a metric space nor a rectangular metric space. Take $\eta_1 = \frac{1}{10}, \eta_2 = \frac{1}{20}, \eta_3 = \frac{1}{60}, \eta_4 = \frac{1}{30}, \tau \in (0, \frac{12}{95}]$ then $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1$, $\lambda = \frac{11}{56}$ and $F(x) = \ln x$. Consider the mapping $\alpha : Z \times Z \to [0, \infty)$ by

$$\alpha(x, y) = \left\{ \begin{array}{cc} 1 & \text{if } x > y \\ \frac{1}{2} & \text{otherwise} \end{array} \right\}.$$

Let $S, T : Z \to Z$ be defined as

$$Sx = \begin{cases} \frac{1}{2} \text{ if } x \in A \\ \frac{x}{4} \text{ if } x \in B. \end{cases} \qquad Tx = \begin{cases} \frac{1}{3} \text{ if } x \in A \\ \frac{x}{4} \text{ if } x \in B. \end{cases}$$

As$\frac{1}{2}, \frac{1}{3} \in Z$, $\alpha(\frac{1}{2}, \frac{1}{3}) > 1$ taking $F(x) = \ln x$, for any $\tau \in (0, \frac{12}{95}]$. Then S and T satisfy the condition of Theorem 1.

If, we take $S = T$ in Theorem 1, then we are left with result.

Corollary 1. *Let (Z, d_l) be a complete R.B.M.S with coefficient $b \geq 1$. Let $\alpha : Z \times Z \to [0, \infty)$ be a function and $S : Z \to Z$ be the α-dominated mapping on Z. Suppose that the following condition is satisfied:*

There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1$ and a continuous and strictly increasing real function F such that

$$\tau + F(d_l(Se, Sy)) \leq F \left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Sy) + \eta_4 \frac{d_l^2(e, Se).d_l(y, Sy)}{1 + d_l^2(e, y)} \end{array} \right), \tag{6}$$

whenever $e, y \in \{g_n\}$, $\alpha(e, y) \geq 1$ and $d_l(Se, Sy) > 0$ "where the sequence g_n is defined by g_0 arbitrary in Z, $g_{2n+1} = S^{2n} g_0$". Then $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{g_n\} \to u \in Z$. Also, if the inequality (6) holds for u and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have a common fixed point u in Z.

If, we take $\eta_2 = 0$ in Theorem 1, then we are left with the result.

Corollary 2. *Let (Z, d_l) be a complete R.B.M.S with constant $b \geq 1$. Let $\alpha : Z \times Z \to [0, \infty)$ be a function and $S, T : Z \to Z$ be the α-dominated mappings on Z. Suppose that the following condition is satisfied:*

There exist $\tau, \eta_1, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + (1+b)b\eta_3 + \eta_4 < 1$ and a continuous and strictly increasing real function F such that

$$\tau + F(d_l(Se, Ty)) \leq F\begin{pmatrix} \eta_1 d_l(e,y) + \eta_3 d_l(e,Ty) \\ +\eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{pmatrix}, \tag{7}$$

whenever $e, y \in \{g_n\}$, $\alpha(e,y) \geq 1$ and $d_l(Se,Ty) > 0$ "where the sequence g_n is defined by g_0 arbitrary in Z, $g_{2n+1} = S(TS)^n g_0$ and $g_{2n} = (TS)^{n+1} g_0$". Then $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{g_n\} \to u \in Z$. Also, if the inequality (7) holds for u and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in Z.

If, we take $\eta_3 = 0$ in Theorem 1, then we are left with the result.

Corollary 3. *Let (Z, d_l) be a complete R.B.M.S with constant $b \geq 1$. Let $\alpha : Z \times Z \to [0, \infty)$ be a function and $S, T : Z \to Z$ be the α-dominated mappings on Z. Suppose that the following condition is satisfied: There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + \eta_4 < 1$ and a continuous and strictly increasing real function F such that*

$$\tau + F(d_l(Se, Ty)) \leq F\begin{pmatrix} \eta_1 d_l(e,y) + \eta_2 d_l(e,Se) \\ +\eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{pmatrix}, \tag{8}$$

whenever $e, y \in \{g_n\}$, $\alpha(e,y) \geq 1$ and $d_l(Se,Ty) > 0$ "where the sequence g_n is defined by g_0 arbitrary in Z, $g_{2n+1} = S(TS)^n g_0$ and $g_{2n} = (TS)^{n+1} g_0$". Then $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{g_n\} \to u \in Z$. Also, if the inequality (8) holds for u and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in Z.

If, we take $\eta_4 = 0$ in Theorem 1, then we are left with the result.

Corollary 4. *Let (Z, d_l) be a complete R.B.M.S with coefficient $b \geq 1$. Let $\alpha : Z \times Z \to [0, \infty)$ be a function and $S, T : Z \to Z$ be the α-dominated mappings on Z. Suppose that the following condition is satisfied:*

There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and a continuous and strictly increasing real function F such that

$$\tau + F(d_l(Se, Ty)) \leq F\begin{pmatrix} \eta_1 d_l(e,y) + \eta_2 d_l(e,Se) \\ +\eta_3 d_l(e,Ty) \end{pmatrix}, \tag{9}$$

whenever $e, y \in \{g_n\}$, $\alpha(e,y) \geq 1$ and $d_l(Se,Ty) > 0$ "where the sequence g_n is defined by g_0 arbitrary in Z, $g_{2n+1} = S(TS)^n g_0$ and $g_{2n} = (TS)^{n+1} g_0$", Then $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{g_n\} \to u \in Z$. Also, if the inequality (9) holds for u and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have a common fixed point u in Z.

3. Fixed Points for Graphic Contractions

Lastly, we give a realization of Theorem 1 in graph theory. Jachymski, [14], shown the particular case for contraction mappings on metric space with a graph. Hussain et al. [12], introduced the concept of graphic contractions and obtained a point fixed result. Further results on graphic contraction can be seen in [8,21,27]. Shang [25], discussed briefly basic notions of graph limit theory and fix some necessary notations and presented many interesting applications.

Definition 6. *Let Z be a nonempty set and $Q = (V(Q), W(Q))$ be a graph such that $V(Q) = Z$, $A \subseteq Z$. A mapping $S : Z \to Z$ is said to be a graph dominated on A if $(p, q) \in W(Q)$, for all $q \in Sp$ and $q \in A$.*

Theorem 2. *Let (Z, d_l) be a complete R.B.M.S endowed with a graph Q with coefficient $b \geq 1$. Let $S, T : Z \to Z$ be two self mappings. Suppose that the following satisfy:*

(i) S and T are graph dominated on Z.
(ii) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and a continuous and strictly increasing real function F such that

$$\tau + F(H_{d_l}(Sp, Tq)) \leq F\left(\begin{array}{c} \eta_1 d_l(p,q) + \eta_2 d_l(p, Sp) \\ +\eta_3 d_l(p, Tq) + \eta_4 \frac{d_l^2(p,Sp).d_l(q,Tq)}{1+d_l^2(p,q)} \end{array}\right), \tag{10}$$

whenever $p, q \in \{g_n\}$, $(p,q) \in W(Q)$ and $d_l(Sp, Tq) > 0$ "where the sequence g_n is defined by g_0 arbitrary in Z, $g_{2n+1} = S(TS)^n g_0$ and $g_{2n} = (TS)^{n+1} g_0$". Then $(g_n, g_{n+1}) \in W(Q)$ and $\{g_n\} \to m^*$. Also, if the inequality (10) holds for m^* and $(g_n, m^*) \in W(Q)$ or $(m^*, g_n) \in W(Q)$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point m^* in Z.

Proof. Define, $\alpha : Z \times Z \to [0, \infty)$ by

$$\alpha(p,q) = \begin{cases} 1, & \text{if } p \in Z,\ (p,q) \in W(Q) \\ 0, & \text{otherwise.} \end{cases}$$

As S and T are graph dominated on Z, then for $p \in Z$, $(p,q) \in W(Q)$ for all $q \in Sp$ and $(p,q) \in W(Q)$ for all $q \in Tp$. Therefore, $\alpha(p,q) = 1$ for all $q \in Sp$ and $\alpha(p,q) = 1$ for all $q \in Tp$. Hence $\alpha_*(p, Sp) = 1$, $\alpha_*(p, Tp) = 1$ for all $p \in Z$. Therefore, $S, T : Z \to Z$ are the α-dominated mappings on Z. Moreover, inequality (10) can be written as

$$\tau + F(H_{d_l}(Sp, Tq)) \leq F\left(\begin{array}{c} \eta_1 d_l(p,q) + \eta_2 d_l(p, Sp) \\ +\eta_3 d_l(p, Tq) + \eta_4 \frac{d_l^2(p,Sp).d_l(q,Tq)}{1+d_l^2(p,q)} \end{array}\right)$$

whenever $p, q \in \{g_n\}$, $\alpha(p,q) \geq 1$ and $d_l(Sp, Tq) > 0$. Also, (ii) holds. Then, by Theorem 1, we have $\{g_n\} \to s^* \in Z$. Now, $g_n, s^* \in Z$ and either $(g_n, s^*) \in W(Q)$ or $(s^*, g_n) \in W(Q)$ implies that either $\alpha(g_n, s^*) \geq 1$ or $\alpha(s^*, g_n) \geq 1$. Therefore, all the conditions of Theorem 1 are satisfied. Hence, by Theorem 1, S and T have a common fixed point s^* in Z and $d_l(s^*, s^*) = 0$. □

4. Conclusions

In the present work, we have achieved fixed-point results for new generalized F-contraction for a more general class of α-dominated mappings rather than α_*-admissible mappings and for a weaker class of strictly increasing mapping F rather than class of mappings F used by Wordowski [28]. We introduced the concept of a pair of graph-dominated mappings and given a fixed-point existence result of a fixed point for graphic contractions. Our results generalized and extended many recent fixed-point results of Rasham et al. [16,20], Wordowski's result [28], Ameer et al. [6] and many classical results in the current literature (see [4,7,9,13,17,18,23,24]).

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