mathematics

Article

Boundedness of Generalized Parametric Marcinkiewicz Integrals Associated to Surfaces

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Received: 22 August 2019; Accepted: 20 September 2019; Published: 23 September 2019

Abstract: In this article, the boundedness of the generalized parametric Marcinkiewicz integral operators $\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}$ is considered. Under the condition that Ω is a function in $L^q(\mathbf{S}^{n-1})$ with $q \in (1,2]$, appropriate estimates of the aforementioned operators from Triebel–Lizorkin spaces to L^p spaces are obtained. By these estimates and an extrapolation argument, we establish the boundedness of such operators when the kernel function Ω belongs to the block space $B^{0,\nu-1}_q(\mathbf{S}^{n-1})$ or in the space $L(\log L)^\nu(\mathbf{S}^{n-1})$. Our results represent improvements and extensions of some known results in generalized parametric Marcinkiewicz integrals.

Keywords: L^p boundedness; rough kernels; Marcinkiewicz integrals; Triebel–Lizorkin spaces; extrapolation

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1. Introduction

Throughout this work, we assume that $\mathbf{R}^n$ $(n \geq 2)$ is the n-dimensional Euclidean space and $x' = x/|x|$ for $x \in \mathbf{R}^n \setminus \{0\}$. In addition, we assume that $\mathbf{S}^{n-1}$ is the unit sphere in $\mathbf{R}^n$, which is equipped with the normalized Lebesgue surface measure $d\sigma$.

For $\rho = \tau + iv$ ($\tau, v \in \mathbf{R}$ with $\tau > 0$), let $K_{\Omega,h}$ be the kernel on $\mathbf{R}^n$ defined by

$$K_{\Omega,h}(u) = |u|^{\rho-n}\,\Omega(u')h(|u|),$$

where h is a measurable function on $\mathbf{R}^+$ and Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1}$$

For a suitable function $\phi : \mathbf{R}^+ \to \mathbf{R}$, we consider the generalized parametric Marcinkiewicz integral operator $\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}$ given by

$$\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f)(x) = \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|u|\leq t} f(x-\phi(|u|)u')K_{\Omega,h}(u)du\right|^r \frac{dt}{t}\right)^{1/r},$$

where $r > 1$ and $f \in \mathcal{S}(\mathbf{R}^n)$.

If $\phi(t) = t$, $h = 1$, $\rho = 1$, and $r = 2$, then the operator $\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}$, denoted by $\mathcal{M}_\Omega$, reduces to the classical Marcinkiewicz integral operator. The operator $\mathcal{M}_\Omega$ was introduced by Stein in [1] in which Stein established the L^p $(1 < p \leq 2)$ boundedness of $\mathcal{M}_\Omega$ provided that $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $0 < \alpha \leq 1$. This result was discussed and improved by many mathematicians. For example, the authors of [2] proved that, if $\Omega \in C^1(\mathbf{S}^{n-1})$, then the L^p boundedness of $\mathcal{M}_\Omega$ is satisfied for all $p \in (1,\infty)$. Later on, Al-Qassem and Al-Salman found in [3] that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for

$1 < p < \infty$ whenever $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ with $q > 1$. Moreover, they proved that the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ is optimal in the sense that the operator $\mathcal{M}_\Omega$ may lose the L^2 boundedness when Ω belongs to the space $\Omega \in B_q^{(0,-\frac{1}{2}-\varepsilon)}(\mathbf{S}^{n-1})$ for some $0 < \varepsilon < 1/2$. Walsh in [4] obtained that $\mathcal{M}_\Omega$ is bounded on $L^2(\mathbf{R}^n)$ if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$. Furthermore, he established the optimality of the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ in the sense that the exponent $1/2$ in $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ cannot be replaced by any smaller number.

Hörmander in [5] started studying the parametric Marcinkiewicz integral operator $\mathcal{M}^{(2)}_{\Omega,t,1,\rho}$. In fact, he proved the $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ boundedness of $\mathcal{M}^{(2)}_{\Omega,t,1,\rho}$ provided that $\rho > 0$ and $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $\alpha > 0$. Subsequently, the investigation of the L^p boundedness of the parametric Marcinkiewicz integrals under very various conditions on Ω, ϕ, and h has attracted the attention of many authors. For a sampling of studies of such operators, the readers are referred to [6–14] and the references therein.

Although some open problems related to the boundedness of the operators $\mathcal{M}^{(2)}_{\Omega,\phi,h,\rho}$ remain open, the investigation to determine the boundedness of the generalized parametric Marcinkiewicz integrals has been started. Historically, the operator $\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}$ was introduced by Chen, Fan and Ying in [15]; they showed that, if $h \equiv 1$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ and $1 < r < \infty$, then

$$\left\| \mathcal{M}^{(r)}_{\Omega,t,h,1} f \right\|_{L^p(\mathbf{R}^n)} \leq C \left\| f \right\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{2}$$

holds for all $1 < p < \infty$. However, Le in [16] improved this result. As a matter of fact, he found that the last result is still true for all $p \in (1, \infty)$ under the conditions that $\Omega \in L(\log L)(\mathbf{S}^{n-1})$, $1 < r < \infty$ and $h \in \Gamma_{\max\{r',2\}}(\mathbf{R}^+)$, where $\Gamma_s(\mathbf{R}^+)$ is the collection of all measurable functions $h : [0, \infty) \to \mathbf{C}$ satisfying

$$\|h\|_{\Gamma_s(\mathbf{R}^+)} = \sup_{k\in\mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |h(t)|^s \frac{dt}{t} \right)^{1/s} < \infty.$$

For the significance and recent advances on the study of such operators, readers may consult [14,17–20].

For $s \geq 1$, we let $\mathfrak{L}^s(\mathbf{R}^+)$ denote the set of all measurable functions $h : [0, \infty) \to \mathbf{C}$ that satisfy the condition

$$L_s(h) = \sup_{k\in\mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |h(t)| \left(\log(2 + |h(t)|)\right)^s \frac{dt}{t} \right) < \infty.$$

In addition, we let $\mathcal{N}^s(\mathbf{R}^+)$ denote the set of all measurable functions $h : [0, \infty) \to \mathbf{C}$ that satisfy the condition

$$N_s(h) = \sum_{k=1}^{\infty} 2^k k^s d_k(h) < \infty,$$

where $d_k(h) = \sup_{j\in\mathbf{Z}} 2^{-j} |E(j,k)|$ with $E(j,k) = \left\{ t \in (2^j, 2^{j+1}] : 2^{k-1} < |h(t)| \leq 2^k \right\}$ for $k \geq 2$ and $E(j,1) = \left\{ t \in (2^j, 2^{j+1}] : |h(t)| \leq 2 \right\}$.

It is obvious that $\Gamma_s\left(\mathbf{R}^+\right) \subset \mathcal{N}^\beta(\mathbf{R}^+) \subset \mathfrak{L}^\beta(\mathbf{R}^+)$ for any $s \geq 1$, $\beta > 0$; and also $\mathfrak{L}^{s+\beta}(\mathbf{R}^+) \subset \mathcal{N}^\beta(\mathbf{R}^+)$ for all $s > 1$, $\beta > 0$.

For $\nu > 0$, let $L(\log L)^\nu(\mathbf{S}^{n-1})$ denote the space of all measurable functions Ω on $\mathbf{S}^{n-1}$ that satisfy

$$\|\Omega\|_{L(\log L)^\nu(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(w)| \left(\log^\nu(2 + |\Omega(w)|\right) d\sigma(w) < \infty.$$

It is worth mentioning that $B_q^{(0,\delta)}(\mathbf{S}^{n-1})$ (for $q > 1$ and $\delta > -1$) is denoted for the special class of the block spaces, which was introduced by Jiang and Lu in [21].

Let us recall the definition of the Triebel–Lizorkin spaces. For $\alpha \in \mathbf{R}$ and $1 < p, r \leq \infty$ with $(p \neq \infty)$, the homogeneous Triebel–Lizorkin space $\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n)$ is defined by

$$\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n)} = \left\| \left(\sum_{j \in \mathbf{Z}} 2^{j\alpha r} \left| \Psi_j * f \right|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} < \infty \right\},$$

where $\mathcal{S}'$ denotes the tempered distribution class on $\mathbf{R}^n$, $\widehat{\Psi_j}(\zeta) = \Phi(2^{-j}\zeta)$ for $j \in \mathbf{Z}$ and Φ is a radial function satisfying the following conditions:

(a) $0 \leq \Phi \leq 1$;
(b) $supp\, \Phi \subset \left\{ \zeta : \frac{1}{2} \leq |\zeta| \leq 2 \right\}$;
(c) $\Phi(\zeta) \geq c > 0$ if $\frac{3}{5} \leq |\zeta| \leq \frac{5}{3}$;
(d) $\sum_{j \in \mathbf{Z}} \Phi(2^{-j}\zeta) = 1 \;\; (\zeta \neq 0)$.

The following properties of the Triebel–Lizorkin space are well known:

(i) $\mathcal{S}'(\mathbf{R}^n)$ is dense in $\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n)$;
(ii) $\dot{F}^{0}_{p,2}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$ for $1 < p < \infty$, and $\dot{F}^{0}_{\infty,2}(\mathbf{R}^n) = \text{BMO}$;
(iii) $\dot{F}^{\alpha}_{p,r_1}(\mathbf{R}^n) \subset \dot{F}^{\alpha}_{p,r_2}(\mathbf{R}^n)$ if $r_1 < r_2$;
(iv) $\left(\dot{F}^{\alpha}_{p,r}(\mathbf{R}^n) \right)^* = \dot{F}^{-\alpha}_{p',r'}(\mathbf{R}^n)$.

In this work, we let $\mathcal{H}_d$ $(d \neq 0)$ to be the class of all smooth functions $\phi : (0, \infty) \to \mathbf{R}$ satisfying the following growth conditions:

$$|\phi(t)| \leq C_1 t^d, \qquad |\phi''(t)| \leq C_2 t^{d-2}, \qquad C_3 t^{d-1} \leq |\phi'(t)| \leq C_4 t^{d-1}$$

for $t \in (0, \infty)$, where the positive constants C_1, C_2, C_3, and C_4 are independent of the variable t.

It is worth mentioning that, when $d = 0$, the class $\mathcal{H}_d$ is empty. Some model examples for the class $\mathcal{H}_d$ are t^d with $d > 0$ and t^l with $l < 0$.

Here, and henceforth, we let p' denote the conjugate index of p defined by $1/p + 1/p' = 1$.

Our main results are formulated as follows:

Theorem 1. *Let $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$ satisfy the condition (1), and $h \in \Gamma_s(\mathbf{R}^+)$ for some $1 < s \leq 2$. Suppose that $\phi \in \mathcal{H}_d$ for some $d \neq 0$. Then, for any $f \in \dot{F}^{0}_{p,r}(\mathbf{R}^n)$, there exists a positive constant C_p (independent of Ω, ϕ, h, r, s, and q) such that*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p (q-1)^{-1} (s-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^{0}_{p,r}(\mathbf{R}^n)} \tag{3}$$

for $1 < p < r$; and

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p (q-1)^{-1/r} (s-1)^{-1/r} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^{0}_{p,r}(\mathbf{R}^n)} \tag{4}$$

for $r \leq p < \infty$.

Theorem 2. *Assume that ϕ and Ω are given as in Theorem 1. Suppose that $h \in \Gamma_s(\mathbf{R}^+)$ for some $s > 2$. Then, there is a constant $C_p > 0$ such that*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p (q-1)^{-1/r} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^{0}_{p,r}(\mathbf{R}^n)} \tag{5}$$

for $1 < p < r$ *with* $r \leq s'$ *and* $2 < s < \infty$*; and*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p (q-1)^{-1/r} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \tag{6}$$

for $s' < p < \infty$ *with* $r > s'$ *and* $2 < s \leq \infty$*.*

By the conclusions in Theorems 1 and 2 and the extrapolation arguments used in [18,22,23], we get the following results.

Theorem 3. *Assume that* $\phi \in \mathcal{H}_d$ *for some* $d \neq 0$ *and* Ω *satisfies (1).*
(i) If $\Omega \in B_q^{(0,\frac{1}{r}-1)}(\mathbf{S}^{n-1})$ *for some* $q > 1$ *and* $h \in \mathcal{N}^{1/r}(\mathbf{R}^+)$*, then*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \|\Omega\|_{B_q^{(0,\frac{1}{r}-1)}(\mathbf{S}^{n-1})} \right) (1 + N_{1/r}(h)) \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $r \leq p < \infty$*;*
(ii) If $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ *for some* $q > 1$ *and* $h \in \mathcal{N}^1(\mathbf{R}^+)$*, then*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \|\Omega\|_{B_q^{(0,0)}(\mathbf{S}^{n-1})} \right) (1 + N_1(h)) \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$*;*
(iii) If $\Omega \in L(\log L)^{1/r}(\mathbf{S}^{n-1})$ *and* $h \in \mathcal{N}^{1/r}(\mathbf{R}^+)$*, then*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \|\Omega\|_{L(\log L)^{1/r}(\mathbf{S}^{n-1})} \right) (1 + N_{1/r}(h)) \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $r \leq p < \infty$*;*
(iv) If $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ *and* $h \in \mathcal{N}^1(\mathbf{R}^+)$*, then*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \|\Omega\|_{L(\log L)(\mathbf{S}^{n-1})} \right) (1 + N_1(h)) \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$*, where* C_p *is a bounded positive constant independent of* h*,* Ω *and* ϕ*.*

Theorem 4. *Let* Ω *satisfy the condition (1),* $h \in \Gamma_s(\mathbf{R}^+)$ *for some* $s > 2$ *and* $\phi \in \mathcal{H}_d$ *for some* $d \neq 0$*.*
(i) If $\Omega \in B_q^{(0,\frac{1}{r}-1)}(\mathbf{S}^{n-1})$ *for some* $q > 1$*, then*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \|\Omega\|_{B_q^{(0,\frac{1}{r}-1)}(\mathbf{S}^{n-1})} \right) \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$ *with* $r \leq s'$ *and* $2 < s < \infty$*; and for* $s' < p < \infty$ *with* $r > s'$ *and* $2 < s \leq \infty$*.*
(ii) If $\Omega \in L(\log L)^{1/r}(\mathbf{S}^{n-1})$*, then*

$$\left\| \mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \|\Omega\|_{L(\log L)^{1/r}(\mathbf{S}^{n-1})} \right) \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)}$$

for $1 < p < r$ *with* $r \leq s'$ *and* $2 < s < \infty$*; and for* $s' < p < \infty$ *with* $r > s'$ *and* $2 < s \leq \infty$*.*

We point out that our results generalize what Al-Qassem found in [18]; and also extend and improve ([24], Theorems 1 and 2). Precisely, the results in [18] are acheived when we take $\phi(t) = t$ in our results. However, when we take $r = 2$, we directly obtain the results in [24].

2. Preparation

In this section, we establish some lemmas used in the proof of our results. Let us start this section by introducing some notations. Let $\theta \geq 2$. For a suitable mapping $\phi : \mathbf{R}^+ \to \mathbf{R}$, $\Omega : \mathbf{S}^{n-1} \to \mathbf{R}$ and a measurable function $h : \mathbf{R}^+ \to \mathbf{C}$; the family of measures $\{\sigma_{\Omega,\phi,h,t} : t \in \mathbf{R}^+\}$ and the corresponding maximal operators $\sigma^*_{\Omega,\phi,h}$ and $M_{\Omega,\phi,h,\theta}$ on $\mathbf{R}^n$ are defined by

$$\int_{\mathbf{R}^n} f d\sigma_{\Omega,\phi,h,t} = t^{-\rho} \int_{t/2\leq|u|\leq t} f(\phi(|u|)u')K_{\Omega,h}(u)du,$$

$$\sigma^*_{\Omega,\phi,h}(f) = \sup_{t\in\mathbf{R}^+} \left| |\sigma_{\Omega,\phi,h,t}| * f \right|,$$

and

$$M_{\Omega,\phi,h,\theta} f(u) = \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left| |\sigma_{\Omega,\phi,h,t}| * f(u) \right| \frac{dt}{t},$$

where $|\sigma_{\Omega,\phi,h,t}|$ is defined in the same way as $\sigma_{\Omega,\phi,h,t}$, but with replacing Ω by $|\Omega|$ and h by $|h|$. We write $r^{\pm\gamma} = \min\{r^\gamma, r^{-\gamma}\}$ and $\left\|\sigma_{\Omega,\phi,h,t}\right\|$ for the total variation of $\sigma_{\Omega,\phi,h,t}$.

We shall need the following lemma which can be derived by applying the same arguments (with only minor modifications) used in the proof of ([24], Lemma 4).

Lemma 1. *Let $\theta \geq 2$, $h \in \Gamma_s(\mathbf{R}^+)$ for some $s > 1$ and $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $q > 1$. Suppose that $\phi \in \mathcal{H}_d$ for some $d \neq 0$. Then, there exist constants C and a with $0 < 2aq' < 1$ such that, for all $k \in \mathbf{Z}$,*

$$\left\|\sigma_{\Omega,\phi,h,t}\right\| \leq C, \tag{7}$$

$$\int_{\theta^k}^{\theta^{k+1}} \left|\hat{\sigma}_{\Omega,\phi,h,t}(\zeta)\right|^2 \frac{dt}{t} \leq C(\ln\theta) \left|\zeta\theta^{kd}\right|^{\pm\frac{2a}{\ln\theta}} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \|h\|^2_{\Gamma_s(\mathbf{R}^+)}, \tag{8}$$

where the constant C is independent of ζ, k and ϕ.

By using ([9], Lemma 2.4) and following the same approaches employed in ([8], Lemmas 2.4 and 2.5) we immediately get the following lemma.

Lemma 2. *Let $\theta \geq 2$, $\phi \in \mathcal{H}_d$ for some $d \neq 0$, $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$, and $h \in \Gamma_s(\mathbf{R}^+)$ for some $s > 1$. Then, there is a constant C_p such that*

$$\|M_{\Omega,\phi,h,\theta}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p(\ln\theta) \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{L^p(\mathbf{R}^n)}, \tag{9}$$

$$\|\sigma^*_{\Omega,\phi,h}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{L^p(\mathbf{R}^n)} \tag{10}$$

for all $1 < p \leq \infty$ with $1 < s \leq 2$; and

$$\|\sigma^*_{\Omega,\phi,h}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{L^p(\mathbf{R}^n)} \tag{11}$$

for all $s' < p < \infty$ with $s \geq 2$.

By applying the same procedures (with only minor modifications) as those in [18], we obtain the following:

Lemma 3. *Let $\theta \geq 2$, $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$ and $h \in \Gamma_s(\mathbf{R}^+)$ for some $1 < s \leq 2$. Let $\phi \in \mathcal{H}_d$ for some $d \neq 0$ and $r > 1$ be a real number. Then, there is a positive constant C_p such that the inequalities*

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,\phi,h,t}*g_k\right|^r\frac{dt}{t}\right)^{\frac{1}{r}}\right\|_{L^p(\mathbf{R}^n)}\\ \leq\;& C_p(\ln\theta)^{1/r}\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}\left\|h\right\|_{\Gamma_s(\mathbf{R}^+)}\left\|\left(\sum_{k\in\mathbf{Z}}\left|g_k\right|^r\right)^{1/r}\right\|_{L^p(\mathbf{R}^n)}\quad for\ r\leq p<\infty \end{aligned}\tag{12}$$

and

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,\phi,h,t}*g_k\right|^r\frac{dt}{t}\right)^{\frac{1}{r}}\right\|_{L^p(\mathbf{R}^n)}\\ \leq\;& C_p(\ln\theta)\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}\left\|h\right\|_{\Gamma_s(\mathbf{R}^+)}\left\|\left(\sum_{k\in\mathbf{Z}}\left|g_k\right|^r\right)^{1/r}\right\|_{L^p(\mathbf{R}^n)}\quad for\ 1<p<r \end{aligned}\tag{13}$$

hold for arbitrary functions $\{g_k(\cdot), k \in \mathbf{Z}\}$ on $\mathbf{R}^n$.

Proof. Let us first prove the inequality (12). On one hand, if $p = r$, then Hölder's inequality and (9) lead us to

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,\phi,h,t}*g_k\right|^r\frac{dt}{t}\right)^{\frac{1}{r}}\right\|_{L^p(\mathbf{R}^n)}^r\leq C\left\|h\right\|_{\Gamma_1(\mathbf{R}^+)}^{(r/r')}\left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})}^{(r/r')}\\ \times\;&\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\theta^k}^{\theta^{k+1}}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}\left|g_k(x-\phi(l)u)\right|^r\left|\Omega(u)\right|\left|h(l)\right|d\sigma(u)\frac{dl}{l}\frac{dt}{t}dx\\ \leq\;& C(\ln\theta)\left\|h\right\|_{\Gamma_1(\mathbf{R}^+)}^{(r/r')+1}\left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})}^{(r/r')+1}\int_{\mathbf{R}^n}\left(\sum_{k\in\mathbf{Z}}\left|g_k(x)\right|^r dx\right)^{p/r}. \end{aligned}\tag{14}$$

Hence, (12) is true for the case $p = r$. On the other hand, if $p > r$, then, by duality, there exists a non-negative function $\Lambda \in L^{(p/r)'}(\mathbf{R}^n)$ with $\|\Lambda\|_{L^{(p/r)'}(\mathbf{R}^n)} \leq 1$ such that

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,\phi,h,t}*g_k\right|^r\frac{dt}{t}\right)^{1/r}\right\|_{L^p(\mathbf{R}^n)}^r=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,\phi,h,t}*g_k(x)\right|^r\frac{dt}{t}\Lambda(x)dx.\tag{15}$$

By Hölder's inequality, we obtain

$$\left|\sigma_{\Omega,\phi,h,t}*g_k(x)\right|^r\leq C\left\|h\right\|_{\Gamma_1(\mathbf{R}^+)}^{(r/r')}\left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})}^{(r/r')}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}\left|g_k(x-\phi(l)u)\right|^r\left|\Omega(u)\right|\left|h(l)\right|d\sigma(u)\frac{dl}{l}.$$

Thus, by a change of variable, Hölder's inequality and (9), we reach that

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * g_k\right|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)}^r \\
\leq\ & C \|h\|_{\Gamma_1(\mathbf{R}^+)}^{(r/r')} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r/r')} \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |g_k(x)|^r \right) M_{|\Omega|,\phi,|h|,\theta} \widetilde{\Lambda}(-x) dx \\
\leq\ & C \|h\|_{\Gamma_1(\mathbf{R}^+)}^{(r/r')} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r/r')} \left\| \sum_{k\in\mathbf{Z}} |g_k|^r \right\|_{L^{(p/r)}(\mathbf{R}^n)} \left\| M_{|\Omega|,\phi,|h|,\theta} \widetilde{\Lambda} \right\|_{L^{(p/r)'}(\mathbf{R}^n)} \\
\leq\ & C_p (\ln\theta) \|h\|_{\Gamma_s(\mathbf{R}^+)}^{(r/r')+1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{(r/r')+1} \left\| \sum_{k\in\mathbf{Z}} |g_k|^r \right\|_{L^{(p/r)}(\mathbf{R}^n)} \left\| \widetilde{\Lambda} \right\|_{L^{(p/r)'}(\mathbf{R}^n)},
\end{aligned}$$

where $\widetilde{\Lambda}(-x) = \Lambda(x)$. Therefore, (12) is satisfied.

Now, consider the case $1 < p < r$ which gives $r' < p'$. Again, by the duality, there exist functions $\zeta = \zeta_k(x,t)$ defined on $\mathbf{R}^n \times \mathbf{R}^+$ with $\left\| \left\| \|\zeta_k\|_{L^{r'}([\theta^k,\theta^{k+1}],\frac{dt}{t})} \right\|_{l^{r'}} \right\|_{L^{p'}(\mathbf{R}^n)} \leq 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} \left|\sigma_{\Omega,\phi,h,t} * g_k\right|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left(\sigma_{\Omega,\phi,h,t} * g_k(x) \right) \zeta_k(x,t) \frac{dt}{t} dx. \tag{16}$$

Let $\Upsilon(\zeta)$ be given by

$$\Upsilon(\zeta)(x) = \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * \zeta_k(x,t)\right|^{r'} \frac{dt}{t}.$$

As $(p'/r') > 1$, we conclude that there is a function $\vartheta \in L^{(p'/r')'}(\mathbf{R}^n)$ such that

$$\begin{aligned}
&\left\| (\Upsilon(\zeta))^{1/r'} \right\|_{L^{p'}(\mathbf{R}^n)}^{r'} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * \zeta_k(x,t)\right|^{r'} \frac{dt}{t} \vartheta(x) dx \\
\leq\ & C \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)} \|h\|_{\Gamma_s(\mathbf{R}^+)}^{(r'/r)} \left\| \sigma^*{}_{|\Omega|,\phi,|h|}(\vartheta) \right\|_{L^{(p'/r')'}(\mathbf{R}^n)} \left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\zeta_k(\cdot,t)|^{r'} \frac{dt}{t} \right) \right\|_{L^{(p'/r')}(\mathbf{R}^n)} \\
\leq\ & C_p (\ln\theta) \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{(r'/r)+1} \|h\|_{\Gamma_s(\mathbf{R}^+)}^{(r'/r)+1} \|\vartheta\|_{L^{(p'/r')'}(\mathbf{R}^n)}.
\end{aligned}$$

Hence, by Hölder's inequality and (16), we obtain that

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * g_k\right|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \\ \leq\ & C_p \ln(\theta)^{1/r} \left\| (\Upsilon(\zeta))^{1/r'} \right\|_{L^{p'}(\mathbf{R}^n)} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \\ \leq\ & C_p (\ln\theta) \|h\|_{\Gamma_s(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \end{aligned} \tag{17}$$

for all $1 < p < r$. Therefore, the proof of Lemma 3 is complete. □

In the same manner, we obtain the following:

Lemma 4. *Let $h \in \Gamma_s(\mathbf{R}^+)$ for some $2 \leq s < \infty$; and let Ω, θ, ϕ, and r be given as in Lemma 3. Then, a positive constant C_p exists such that*
(i) *If $r \leq s'$, we have*

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * g_k\right|^r \frac{dt}{t} \right)^{\frac{1}{r}} \right\|_{L^p(\mathbf{R}^n)} \\ \leq\ & C_p (\ln\theta)^{1/r} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \quad \mathit{for}\ 1 < p < r. \end{aligned} \tag{18}$$

(ii) *If $r > s'$, we have*

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * g_k\right|^r \frac{dt}{t} \right)^{\frac{1}{r}} \right\|_{L^p(\mathbf{R}^n)} \\ \leq\ & C_p (\ln\theta)^{1/r} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^r \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} \quad \mathit{for}\ s' < p < \infty, \end{aligned} \tag{19}$$

where $\{g_k(\cdot), k \in \mathbf{Z}\}$ are arbitrary functions on $\mathbf{R}^n$.

Proof. Let us first consider the case $1 < p < r$ with $r \leq s'$. As above, by the duality, there are functions $\psi = \psi_k(x,t)$ defined on $\mathbf{R}^n \times \mathbf{R}^+$ with $\left\| \left\| \|\psi_k\|_{L^{r'}([\theta^k,\theta^{k+1}],\frac{dt}{t})} \right\|_{l^{r'}} \right\|_{L^{p'}(\mathbf{R}^n)} \leq 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} \left|\sigma_{\Omega,\phi,h,t} * g_k\right|^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left(\sigma_{\Omega,\phi,h,t} * g_k(x)\right) \psi_k(x,t) \frac{dt}{t} dx$$

$$\leq \quad C_p \ln(\theta)^{1/r} \left\|(\Theta(\psi))^{1/r'}\right\|_{L^{p'}(\mathbf{R}^n)} \left\|\left(\sum_{k\in\mathbf{Z}} |g_k|^r\right)^{1/r}\right\|_{L^p(\mathbf{R}^n)}, \tag{20}$$

where

$$\Theta(\psi)(x) = \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^k} \left|\sigma_{\Omega,\phi,h,t} * \psi_k(x,t)\right|^{r'} \frac{dt}{t}.$$

As $r \leq s' \leq s$, then, by Hölder's inequality, we have that

$$\begin{aligned}\left|\sigma_{\Omega,\phi,h,t} * \psi_k(x,t)\right|^{r'} &\leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)} \|h\|_{\Gamma_r(\mathbf{R}^+)}^{r'} \int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \\ &\times \quad |\psi_k(x-\phi(l)\,u,t)|^{r'} \, d\sigma(u) \frac{dl}{l} \\ &\leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)} \|h\|_{\Gamma_s(\mathbf{R}^+)}^{r'} \int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \\ &\times \quad |\psi_k(x-\phi(l)\,u,t)|^{r'} \, d\sigma(u) \frac{dl}{l}.\end{aligned} \tag{21}$$

Again, since $(p'/r') > 1$, we deduce that there is a function $\nu \in L^{(p'/r')'}(\mathbf{R}^n)$ such that

$$\left\|(\Theta(\psi))^{1/r'}\right\|_{L^{p'}(\mathbf{R}^n)}^{r'} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{k+1}} \left|\sigma_{\Omega,\phi,h,t} * \psi_k(x,t)\right|^{r'} \frac{dt}{t} \nu(x) dx.$$

Hence, by a simple change of variables, Hölder's inequality, ([9], Lemma 2.5) and (21), we get that

$$\begin{aligned}\left\|(\Theta(\psi))^{1/r'}\right\|_{L^{p'}(\mathbf{R}^n)}^{r'} &\leq C\|h\|_{\Gamma_s(\mathbf{R}^+)}^{r'} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)} \left\|\sigma^*{}_{|\Omega|,\phi,1}(\nu)\right\|_{L^{(p'/r')'}(\mathbf{R}^n)} \\ &\times \left\|\left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\psi_k(\cdot,t)|^{r'} \frac{dt}{t}\right)\right\|_{L^{(p'/r')}(\mathbf{R}^n)} \\ &\leq C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(r'/r)+1} \|h\|_{\Gamma_s(\mathbf{R}^+)}^{r'} \|\nu\|_{L^{(p'/r')'}(\mathbf{R}^n)}.\end{aligned}$$

Therefore, by (20) and the last inequality, we reach (18) for any $1 < p < r$ with $r \leq s'$. Now, we consider the case $s' < p < \infty$ with $s' < r$. Thanks to (11), we get that

$$\begin{aligned}\left\|\sup_{k\in\mathbf{Z}} \sup_{t\in[1,\theta]} \left|\sigma_{\Omega,\phi,h,\theta^k t} * g_k\right|\right\|_{L^p(\mathbf{R}^n)} &\leq \left\|\sigma^*_{\Omega,\phi,h}\left(\sup_{k\in\mathbf{Z}} |g_k|\right)\right\|_{L^p(\mathbf{R}^n)} \\ &\leq C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \left\|\sup_{k\in\mathbf{Z}} |g_k|\right\|_{L^p(\mathbf{R}^n)}\end{aligned} \tag{22}$$

for all $s' < p < \infty$ and $s \geq 2$. This implies

$$\begin{aligned}\left\|\left\|\|\sigma_{\Omega,\phi,h,\theta^k t} * g_k\|_{L^\infty([1,\theta],\frac{dt}{t})}\right\|_{l^\infty(\mathbf{Z})}\right\|_{L^p(\mathbf{R}^n)} &\leq C_p \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \\ &\times \left\|\|g_k\|_{l^\infty(\mathbf{Z})}\right\|_{L^p(\mathbf{R}^n)}.\end{aligned} \tag{23}$$

Here, we follow the same above procedure; by Hölder's inequality, we get

$$\left|\sigma_{\Omega,\phi,h,\theta^k t} * g_k(x)\right|^{s'} \leq C\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(s'/s)}\|h\|_{\Gamma_s(\mathbf{R}^+)}^{s'}\int_{\theta^k t/2}^{\theta^k t}\int_{\mathbf{S}^{n-1}}|\Omega(u)| \times |g_k(x-\phi(l)\,u)|^{s'}\,d\sigma(u)\frac{dl}{l}.$$

By duality, there is a function $\varphi \in L^{(p/s')'}(\mathbf{R}^n)$ with $\|\varphi\|_{L^{(p/s')'}(\mathbf{R}^n)} \leq 1$ such that

$$\begin{aligned}
\left\|\left(\sum_{k\in\mathbf{Z}}\int_1^\theta\left|\sigma_{\Omega,\phi,h,\theta^k t} * g_k\right|^{s'}\frac{dt}{t}\right)^{\frac{1}{s'}}\right\|_{L^p(\mathbf{R}^n)}^{s'} &= \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_1^\theta\left|\sigma_{\Omega,\phi,h,\theta^k t} * g_k(x)\right|^{r'}\frac{dt}{t}\varphi(x)dx \\
&\leq C\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(s'/s)}\|h\|_{\Gamma_s(\mathbf{R}^+)}^{s'}\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}|g_k(x)|^{s'}\,\sigma^*_{\Omega,\phi,1}\overline{\varphi}(-x)dx \\
&\leq C\ln(\theta)\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(s'/s)}\|h\|_{\Gamma_s(\mathbf{R}^+)}^{s'}\left\|\sum_{k\in\mathbf{Z}}|g_k|^{s'}\right\|_{L^{(p/s')}(\mathbf{R}^n)}\left\|\sigma^*_{\Omega,\phi,1}\overline{\varphi}\right\|_{L^{(p/s')'}(\mathbf{R}^n)} \\
&\leq C\ln(\theta)\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{(s'/s)+1}\|h\|_{\Gamma_s(\mathbf{R}^+)}^{s'}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^{s'}\right)^{\frac{1}{s'}}\right\|_{L^p(\mathbf{R}^n)}^{s'}, \qquad (24)
\end{aligned}$$

where $\overline{\varphi}(x) = \varphi(-x)$. Thus, when we define the linear operator H on any function $\omega = g_k(x)$ by $H(g_k(x)) = \sigma_{\Omega,\phi,h,\theta^k t} * g_k(x)$, then, by interpolation (23) and (24), we directly obtain that

$$\begin{aligned}
\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,\phi,h,t} * g_k\right|^{r}\frac{dt}{t}\right)^{\frac{1}{r}}\right\|_{L^p(\mathbf{R}^n)} &\leq \left\|\left(\sum_{k\in\mathbf{Z}}\int_1^{\theta}\left|\sigma_{\Omega,\phi,h,\theta^k t} * g_k\right|^{r}\frac{dt}{t}\right)^{\frac{1}{r}}\right\|_{L^p(\mathbf{R}^n)} \\
&\leq C_p(\ln\theta)^{1/r}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Gamma_s(\mathbf{R}^+)}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^{r}\right)^{\frac{1}{r}}\right\|_{L^p(\mathbf{R}^n)}
\end{aligned}$$

for all $s' < p < \infty$ and $s \geq 2$. This ends the proof of Lemma 4. □

3. Proof of the Main Results

Proof of Theorem 1. The proof of this theorem depends on the arguments used in [9,18]. Let us first assume that $\phi \in \mathcal{H}_d$ for some $d \neq 0$, $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $q \in (1,2]$ and $h \in \Gamma_s(\mathbf{R}^+)$ for some $s \in (1,2]$. Thanks to Minkowski's inequality, we have that

$$\begin{aligned}
\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho}(f)(x) &\leq \sum_{k=0}^{\infty}\left(\int_0^\infty\left|t^{-\rho}\int_{2^{-k-1}t<|u|\leq 2^{-k}t}f(x-\phi(|u|)u')K_{\Omega,h}(u)du\right|^r\frac{dt}{t}\right)^{1/r} \\
&= \frac{2^\tau}{2^\tau-1}\left(\int_0^\infty\left|\sigma_{\Omega,\phi,h,t} * f(x)\right|^r\frac{dt}{t}\right)^{1/r}. \qquad (25)
\end{aligned}$$

Let $\theta = 2^{q's'}$. For $k \in \mathbf{Z}$, let $\{\Phi_k\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $\mathcal{I}_{k,\theta} = [\theta^{-kd-|d|}, \theta^{-kd+|d|}]$. In fact, we require the following:

$$0 \le \Phi_k \le 1, \ \sum_k \Phi_k(t) = 1,$$
$$\text{supp } \Phi_k \subseteq \mathcal{I}_{k,\theta}, \text{ and } \left| \frac{d^s \Phi_k(t)}{dt^s} \right| \le \frac{C_s}{t^s}.$$

Let $\widehat{\Psi_k}(\zeta) = \Phi_k(|\zeta|)$. Then, for $f \in \mathcal{S}(\mathbf{R}^n)$, one can deduce that

$$\mathcal{M}^{(r)}_{\Omega,\phi,h,\rho} f(x) \le \frac{2^\tau}{2^\tau - 1} \sum_{j \in \mathbf{Z}} \mathcal{G}^{(r)}_{\Omega,\phi,h,j}(f), \tag{26}$$

where

$$\mathcal{G}^{(r)}_{\Omega,\phi,h,j} f(x) = \left(\int_0^\infty \left| \mathcal{F}_{\Omega,\phi,h,j,\theta}(x,t) \right|^r \frac{dt}{t} \right)^{1/r},$$
$$\mathcal{F}_{\Omega,\phi,h,j,\theta}(x,t) = \sum_{k \in \mathbf{Z}} (\Psi_{k+j} * \sigma_{\Omega,\phi,h,t} * f)(x) \chi_{[\theta^k,\theta^{k+1})}(t).$$

Notice that, we prove Theorem 1 for the case $s \in (1,2]$ once we show that

$$\begin{aligned} &\left\| \mathcal{G}^{(r)}_{\Omega,\phi,h,j}(f) \right\|_{L^p(\mathbf{R}^n)} \\ \le\ & C 2^{-\varepsilon|j|} (q-1)^{-1/r} (s-1)^{-1/r} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \end{aligned} \tag{27}$$

for $r \le p < \infty$, and

$$\begin{aligned} &\left\| \mathcal{G}^{(r)}_{\Omega,\phi,h,j}(f) \right\|_{L^p(\mathbf{R}^n)} \\ \le\ & C 2^{-\varepsilon|j|} (q-1)^{-1} (s-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \end{aligned} \tag{28}$$

for $1 < p < r$ and for some $0 < \varepsilon < 1$.

Let us prove the inequality (27). First, we consider the case $p = r = 2$. In this case, we have $\|f\|_{\dot{F}^0_{2,2}(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}$. Thus, by Plancherel's theorem, (8), and the fact $\ln\theta \le C(s-1)^{-1}(q-1)^{-1}$ with $s, q \in (1,2]$, we get that

$$\begin{aligned} \left\| \mathcal{G}^{(2)}_{\Omega,\phi,h,j}(f) \right\|^2_{L^2(\mathbf{R}^n)} &\le \sum_{k \in \mathbf{Z}} \int_{\mathcal{B}_{k+j,\theta}} \left(\int_{\theta^k}^{\theta^{k+1}} \left| \hat{\sigma}_{\Omega,\phi,h,t}(\zeta) \right|^2 \frac{dt}{t} \right) \left| \hat{f}(\zeta) \right|^2 d\zeta \\ &\le C(\ln\theta) \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \|h\|^2_{\Gamma_s(\mathbf{R}^+)} \sum_{k \in \mathbf{Z}} \int_{\mathcal{B}_{k+j,\theta}} \left(\left| \theta^{kd} \zeta \right|^{\pm \frac{2a}{q's'}} \right) \left| \hat{f}(\zeta) \right|^2 d\zeta \\ &\le C(\ln\theta) \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \|h\|^2_{\Gamma_s(\mathbf{R}^+)} 2^{-\eta|j|} \sum_{k \in \mathbf{Z}} \int_{\mathcal{B}_{k+j,\theta}} \left| \hat{f}(\zeta) \right|^2 d\zeta \\ &\le C(s-1)^{-1}(q-1)^{-1} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \|h\|^2_{\Gamma_s(\mathbf{R}^+)} 2^{-\eta|j|} \|f\|^2_{L^2(\mathbf{R}^n)}, \end{aligned}$$

where $\mathcal{B}_{k,\theta} = \left\{ \zeta \in \mathbf{R}^n : |\zeta| \in \mathcal{I}_{k,\theta} \right\}$ and $0 < \eta < 1$. Therefore,

$$\begin{aligned} &\left\| \mathcal{G}^{(2)}_{\Omega,\phi,h,j}(f) \right\|_{L^2(\mathbf{R}^n)} \\ \le\ & C 2^{-\frac{\eta}{2}|j|} (s-1)^{-1/2} (q-1)^{-1/2} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Gamma_s(\mathbf{R}^+)} \|f\|_{\dot{F}^0_{2,2}(\mathbf{R}^n)}. \end{aligned} \tag{29}$$

On the other hand, by Lemma 3, we directly get that

$$\begin{aligned} & \left\| \mathcal{G}^{(r)}_{\Omega,\phi,h,j}(f) \right\|_{L^p(\mathbf{R}^n)} \\ \leq \quad & C\,(q-1)^{-1/r}\,(s-1)^{-1/r}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|h\|_{\Gamma_s(\mathbf{R}^+)}\,\|f\|_{\dot{F}^0_{p,r}(\mathbf{R}^n)} \end{aligned} \tag{30}$$

for $r \leq p < \infty$, and

$$\begin{aligned} & \left\| \mathcal{G}^{(r)}_{\Omega,\phi,h,j}(f) \right\|_{L^p(\mathbf{R}^n)} \\ \leq \quad & C\,(q-1)^{-1}\,(s-1)^{-1}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|h\|_{\Gamma_s(\mathbf{R}^+)}\,\|f\|_{\dot{F}^{0,r}_{p}(\mathbf{R}^n)} \end{aligned} \tag{31}$$

for $1 < p < r$. Consequently, interpolating (29) with (30) and (31), we achieve (27) and (28). □

Proof of Theorem 2. The proof of Theorem 2 can be obtained by applying the above approaches except we need to invoke $\theta = 2^{q'}$ instead of $\theta = 2^{q's'}$, and Lemma 4 instead of Lemma 3. □

Author Contributions: Formal analysis, investigation, and writing-original draft preparation M.A. and O.A.-R.

Funding: This research received no external funding.

Acknowledgments: The authors are grateful to the Editor for handling the full submission of the manuscript.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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