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On the Sign of the Curvature of a Contact Metric Manifold

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Abstract: In this expository article, we discuss the author's conjecture that an associated metric for a given contact form on a contact manifold of dimension ≥ 5 must have some positive curvature. In dimension 3, the standard contact structure on the 3-torus admits a flat associated metric; we also discuss a local example, due to Krouglov, where there exists a neighborhood of negative curvature on a particular 3-dimensional contact metric manifold. In the last section, we review some results on contact metric manifolds with negative sectional curvature for sections containing the Reeb vector field.

Keywords: contact manifolds; associated metrics; curvature

1. Introduction

During the 1974–75 academic year, the author was on sabbatical in Strasbourg, and at that time it was unknown if the 5-dimensional torus carried a contact form. Also, at that time, Georges Reeb commented that he felt that the existence of a contact form on a manifold "tightens up" the manifold. Today someone working in contact topology might think of this in terms of "tight" vs. "overtwisted" and someone working in contact geometry might ask if an associated metric should have some positive curvature. The author conjectures that there should be some positive curvature and this is one of the main themes of this article. In 1975 [1] the author proved that in dimensions greater than or equal to 5 there are no flat associated metrics. In 1979, R. Lutz [2] produced a very explicit contact form on the 5-torus; thus, the flat metric on the torus cannot be an associated metric. Today it is known that all odd-dimensional tori carry contact structures [3]. However, to date, the author knows of no specific associated metric on any torus of dimension ≥ 5; in principal, one can always construct one, but, so far, not in a naturally suggested way. The standard contact structure on the 3-torus does carry a flat associated metric.

In 1979, Z. Olszak [4] generalized the non-flatness result and showed that if a contact metric manifold of dimension ≥ 5 is of constant curvature, then it must be of constant curvature +1 and Sasakian. As mentioned above, the author conjectures that, at least in dimensions greater than or equal to 5, every contact metric manifold has positive sectional curvature for some section at some point on the manifold. In dimension 3, Kroglov [5] gave an example of a contact metric structure on $\mathbb{R}^3$, which is negatively curved in a neighborhood of the origin but is not globally negatively curved.

We first give a brief review of the rudiments of contact metric geometry and then discuss a number of curvature results. In the final section, we review some work on contact metric manifolds for which the sectional curvature in the direction of the Reeb vector field is negative.

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2. Review of Contact Metric Manifolds

A *contact manifold* is a C^∞ manifold M^{2n+1} carrying a 1-form η for which

$$\eta \wedge (d\eta)^n \neq 0.$$

Given such a structure there exists a unique vector field Z_η, called the *Reeb vector field*, satisfying $d\eta(X, Z_\eta) = 0$ and $\eta(Z_\eta) = 1$. Denote by $\mathcal{D}_\eta$ the *contact sub-bundle*, defined by

$$\{X \in T_m M : \eta(X) = 0\}.$$

Roughly speaking, the contact condition means that the contact sub-bundle is maximally far from being integrable; in fact, the maximum dimension of an integral submanifold of $\mathcal{D}_\eta$ is only n. Thinking about it another way, the contact sub-bundle rotates as one moves around on the manifold.

Many topologists think of a *contact structure*, sometimes referred to as a *contact structure in the wider sense*; the idea is that, first of all, one has a field of $2n$-planes $\mathcal{D}$ given locally by a contact form, and secondly, in the overlap of coordinate neighborhoods, $\mathcal{U} \cap \mathcal{U}'$, there is a function f, such that $\eta' = f\eta$ and thus $d\eta' = df \wedge \eta + f d\eta$, from which

$$\eta' \wedge (d\eta')^n = f^{n+1} \eta \wedge (d\eta)^n \neq 0.$$

Turning to contact metric geometry, an *associated metric* is a Riemannian metric g satisfying the following two conditions: First,

$$\eta(X) = g(X, Z_\eta)$$

and second, there exists a field of endomorphisms ϕ, such that

$$\phi^2 = -I + \eta \otimes Z_\eta \text{ and } d\eta(X, Y) = g(X, \phi Y).$$

The contact sub-bundle is orthogonal to the Reeb vector field, and the field of endomorphisms ϕ annihilates the Reeb vector field and acts as an almost complex structure on the contact sub-bundle. One also has that $\phi Z_\eta = 0$, $\eta \circ \phi = 0$. We refer to (ϕ, Z_η, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold*. All associated metrics have the same volume element, proportional to $\eta \wedge (d\eta)^n$. Associated metrics are far from unique, in fact, the space of all associated metrics for a given contact form is infinite dimensional, see, e.g., [6].

Associated metrics can be constructed much as in symplectic geometry. For any Riemannian metric $\bar{\bar{g}}$ on M^{2n+1}, define a metric $\bar{g}$ by

$$\bar{g}(X, Y) = \bar{\bar{g}}(-X + \eta(X) Z_\eta, -Y + \eta(Y) Z_\eta) + \eta(X)\eta(Y).$$

One immediately has $\bar{g}(X, Z_\eta) = \eta(X)$. Choosing a local $\bar{g}$-orthonormal basis $\{X_1, \ldots, X_{2n}\}$ of $\mathcal{D}_\eta$ and evaluating $d\eta$ on these vectors gives a non-singular $2n \times 2n$ matrix, $A_{ij} = d\eta(X_i, X_j)$. By polarization, A can be written as the product of an orthogonal matrix F and a positive definite symmetric matrix G. Now, define an associated metric g and almost complex structure ϕ on $\mathcal{D}_\eta$ by $g(X_i, X_j) = G_{ij}$ and $\phi X_i = F_i{}^j X_j$, then extend these to all tangent vectors by $g(X, Z_\eta) = \eta(X)$ and $\phi Z_\eta = 0$. Given another $\bar{g}$-orthonormal basis of $\mathcal{D}_\eta$, say $\{Y_1, \ldots, Y_{2n}\}$, there exists an orthogonal matrix P, such that

$$B_{ij} = d\eta(Y_i, Y_j) = d\eta(P^k{}_i X_k, P^l{}_j X_l) = (PAP^{-1})_{ij}.$$

If $B = \Phi\Gamma$ is the polar decomposition of B, then $\Phi\Gamma = PFP^{-1}PGP^{-1}$. By the uniqueness of the polar decompositions $\Phi = PFP^{-1}$ and $\Gamma = PGP^{-1}$, we have that g and ϕ are globally defined structure tensors. Next note that $A^T = GF^T = -FG$, and consequently $G = -FFF^T GF$. On the other hand, $F^T GF$ is positive definite symmetric and, again, by the uniqueness of the polar decomposition $F^2 = -I$

and $F = -F^T$. This construction dates from 1962 and is due to Y. Hatakeyama [7]; in the course of his work he proved the analyticity of the polar decomposition.

The product manifold, $M^{2n+1} \times \mathbb{R}$, carries a natural almost complex structure J, which is defined by

$$J(X,0) = \Big(\phi X, \eta(X)\frac{d}{dt}\Big) \quad J(0,\frac{d}{dt}) = \Big(-Z_\eta, 0\Big).$$

The structure on M^{2n+1} is said to be *normal* if J is integrable. The normality condition can be expressed as

$$[\phi,\phi](X,Y) + 2d\eta(X,Y)Z_\eta = 0$$

where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ.

Definition 1. *A Sasakian manifold is a normal contact metric manifold.*

Alternatively, a Sasakian manifold can be defined in the following way. Let (M^m, g) be a Riemannian manifold and consider the cone over M^m,

$$(\mathbb{R}_+ \times M^m, dr^2 + r^2 g).$$

The manifold becomes Sasakian if and only if the holonomy group of the cone reduces to a subgroup of $U(\frac{m+1}{2})$. Thus $(\mathbb{R}_+ \times M^m, dr^2 + r^2 g)$ is Kähler and $m = 2n+1$, $n \geq 1$.

In terms of the covariant derivative of ϕ, the Sasakian condition is

$$(\nabla_X \phi)Y = g(X,Y)Z_\eta - \eta(Y)X.$$

In view of this, a Sasakian manifold is often thought of as a contact metric analogue of a Kähler manifold where one has $\nabla J = 0$. Another common characterization of the Sasakian condition is given in terms of the curvature tensor by

$$R_{XY}Z_\eta = \eta(Y)X - \eta(X)Y.$$

Definition 2. *A K-contact manifold is contact metric manifold on which the Reeb vector field is a Killing vector field.*

It is well known that a Sasakian manifold is K-contact. In dimension 3, the K-contact condition is equivalent to the Sasakian condition, but in higher dimensions this is not true.

Another important structure tensor is the Lie derivative of ϕ with respect to the Reeb vector field; we set

$$h = \frac{1}{2}\pounds_{Z_\eta}\phi.$$

This operator is symmetric, it anticommutes with ϕ, and it annihilates the Reeb vector field. Moreover, h vanishes if and only if the Reeb vector field is a Killing vector field. A particularly important property of h is the following,

$$\nabla_X Z_\eta = -\phi X - \phi h X;$$

this exhibits the rotation of the Reeb vector field and by orthogonality, the rotation of the contact sub-bundle. From this formula one sees immediately that the integral curves of Z_η are geodesics. If λ is a non-zero eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue and ϕX a corresponding eigenvector.

For future use in this article we need a few more ideas. The first of these will be the notion of a *ϕ-basis* on a contact metric manifold. Consider a coordinate neighborhood $\mathcal{U}$ and any unit vector field X_1 on $\mathcal{U}$ orthogonal to Z_η. The vector field ϕX_1 is orthogonal to X_1 and Z_η. Choosing a unit vector field X_2 orthogonal to Z_η, X_1, and ϕX_1, it is easy to see that ϕX_2 is also a unit vector field orthogonal to Z_η, X_1, ϕX_1, and X_2. Continuing in this way we obtain a local orthonormal basis $\{X_i, \phi X_i, Z_\eta\}$, $i = 1, \ldots, n$.

Tanno introduced the idea of a *$\mathcal{D}$-homothetic deformation* [8]. For a given contact metric structure (ϕ, Z_η, η, g), the deformed structure

$$\bar{\eta} = a\eta, \quad \bar{Z}_{\bar{\eta}} = \frac{1}{a} Z_\eta, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant, is again a contact metric structure. A $\mathcal{D}$-homothetic deformation also preserves the states of being K-contact, and Sasakian. The name derives form the fact that, the metrics restricted to the contact sub-bundle $\mathcal{D}_\eta$ are homothetic.

Finally we mention ϕ-sectional curvature. A plane section in $T_m M^{2n+1}$ is a *ϕ-section* if there exists a vector $X \in \mathcal{D}_\eta$, such that $\{X, \phi X\}$ span the section. The sectional curvature $K(X, \phi X)$ is referred to as the *ϕ-sectional curvature* of the plane section. This idea in Sasakian geometry can be regarded as an analogue of holomorphic sectional curvature in Kähler geometry.

General references for the ideas of this section are [6,9].

3. The Sign of the Curvature of Contact Metric Manifolds

For the structure tensors—(ϕ, Z_η, η, g) together with the tensor field $h = \frac{1}{2}\pounds_{Z_\eta}\phi$ on a contact metric manifold—we noted above that

$$\nabla_X Z_\eta = -\phi X - \phi h X.$$

Differentiating this with respect to Z_η and computing $R_{Z_\eta X} Z_\eta$ we have the following formulas,

$$(\nabla_{Z_\eta} h)X = \phi X - h^2 \phi X - \phi R_{X Z_\eta} Z_\eta,$$

$$\frac{1}{2}(R_{X Z_\eta} Z_\eta - \phi R_{\phi X Z_\eta} Z_\eta) = -\phi^2 X - h^2 X.$$

Choosing a unit vector X orthogonal to Z_η, the inner product of X with the second formula gives the following formula for the sum of two sectional curvatures

$$K(Z_\eta, X) + K(Z_\eta, \phi X) = 2(1 - g(h^2 X, X)).$$

Thus, if $\{X_1, \ldots, X_n, \phi X_1, \ldots, \phi X_n, Z_\eta\}$ is a ϕ-basis, summing over $\{X_1, \ldots, X_n\}$ yields

$$Ric(Z_\eta) = 2n - \mathrm{tr} h^2. \tag{1}$$

An immediate consequence is that a contact metric manifold is K-contact if and only if

$$Ric(Z_\eta) = 2n.$$

We also note that a contact metric manifold is K-contact if and only if the sectional curvature of all plane sections containing Z_η are equal to +1. The sufficiency is clear from (1). Conversely, as $\nabla_X Z_\eta = -\phi X$ on a K-contact manifold and $\nabla_{Z\eta}\phi = 0$ on any contact metric manifold, we have by direct computation $R_{X Z_\eta} Z_\eta = X$.

As mentioned in the introduction, in dimensions ≥ 5 there are no flat associated metrics and one has the following theorem of Olszak [4].

Theorem 1. *A contact metric manifold of dimension ≥ 5 and of constant curvature is Sasakian and of constant curvature $+1$. In dimension 3, constant curvature exists only in the flat case or the Sasakian case of constant curvature $+1$.*

For our conjecture on positive curvature, we discuss a substantial supporting result. This is a result of Rukimbira [10], utilizing a result of Zeghib [11] on geodesic plane fields. A k-dimensional plane field on an n-dimensional Riemannian manifold is said to be *geodesic* if any geodesic tangent to the plane field at one point is tangent to it at every point. Zeghib's result is the following.

Theorem 2. *A compact Riemannian manifold of strictly negative curvature cannot carry a C^1 geodesic plane field.*

As we have seen, for any contact metric structure the integral curves of Z_η are geodesics; therefore, Z_η determines a geodesic line field. Rukimbira applied Zeghib's result to compact contact manifolds giving the following result.

Theorem 3. *On a compact contact manifold, there is no associated metric of strictly negative curvature.*

Without compactness, in dimension 3, this is false, as was shown by an example of Krouglov [5] of a contact metric structure on $\mathbb{R}^3$, which is negatively curved on some neighborhood of the origin. However, this metric is not globally of non-positive curvature. We present this example here. Starting with the standard Darboux form $\eta = \frac{1}{2}(dz - y\,dx)$ and its Reeb vector field $Z_\eta = 2\frac{\partial}{\partial z}$, an associated metric of the type studied by Krouglov is

$$g = \frac{1}{4}\begin{pmatrix} \sqrt{2}\,e^{-z} + y^2 & -1 & -y \\ -1 & \sqrt{2}\,e^{z} & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

The vector fields

$$X = \frac{\sqrt{2}\,e^{z/2}}{\sqrt{\sqrt{2}-1}}\begin{pmatrix} 1 \\ e^{-z} \\ y \end{pmatrix}, \quad Y = \frac{\sqrt{2}\,e^{z/2}}{\sqrt{\sqrt{2}+1}}\begin{pmatrix} 1 \\ -e^{-z} \\ y \end{pmatrix}$$

belong to the sub-bundle $\mathcal{D}_\eta$, and together with Z_η form an orthonormal basis. Now let

$$U = \alpha X + \beta Y + \gamma Z_\eta, \quad V = \lambda X + \mu Y + \nu Z_\eta$$

be two independent vector fields. Computing $R(U,V,V,U)$, we have

$$R(U,V,V,U) = -(\alpha\mu - \beta\lambda)^2(1 + 2\sqrt{2}\,y^2 e^z) - (\alpha\nu - \gamma\lambda)^2 - (\beta\nu - \gamma\mu)^2$$
$$-4\sqrt{1+\sqrt{2}}\,(\alpha\mu - \beta\lambda)(\alpha\nu - \gamma\lambda)ye^{z/2} + \frac{4}{\sqrt{1+\sqrt{2}}}(\alpha\mu - \beta\lambda)(\beta\nu - \gamma\mu)ye^{z/2};$$

this is negative at the origin in $\mathbb{R}^3$, and therefore negative in a neighborhood of the origin giving a contact metric manifold of negative curvature. This metric is not of negative curvature everywhere; indeed,

$$R(X, Y + Z_\eta, Y + Z_\eta, X) = -(1 + 2\sqrt{2}\,y^2 e^z) - 1 - 4\sqrt{1+\sqrt{2}}\,ye^{z/2},$$

which is positive for x arbitrary, $y = -1$, $z = 0$. Note that $\{X, Y + Z_\eta\}$ is not a plane section in the contact sub-bundle. It is interesting that, for this metric, we do have the following everywhere, negative sectional curvatures

$$K(X, Z_\eta) = -1, \quad K(Y, Z_\eta) = -1, \quad K(X,Y) = -1 - 2\sqrt{2}y^2 e^z.$$

The non-completeness of the neighborhood may be important and the conjecture in dimension 3 may be true, aside from the flat case, when the contact metric manifold is complete.

Krouglov [5] does however give a proof of the conjecture in dimension 3 for contact structures which are sufficiently non-trivial as fibrations; his result is the following.

Theorem 4. *Let M^3 be a compact contact metric manifold, for which the contact sub-bundle cannot be decomposed as a sum of two 1-dimensional fibrations. Then, M^3 admits some positive curvature.*

Turning to the homogeneous case, i.e., contact manifolds admitting a transitive Lie group of diffeomorphisms preserving the contact form, a contact metric manifold is said to be *homogeneous* if it admits a transitive Lie group of diffeomorphisms preserving the structure tensors (ϕ, Z_η, η, g). This case was studied by A. Lotta [12], and he proved the following two theorems.

Theorem 5. *In dimensions ≥ 5 there are no homogeneous, simply connected contact manifolds which admit a Riemannian metric of non-positive curvature and for which the Reeb vector field is orthogonal to the contact sub-bundle.*

Theorem 6. *Let M^{2n+1} be a homogeneous, simply connected contact metric manifold of non-positive curvature. Then, the manifold is 3-dimensional, flat, and equivalent to the universal cover of $E(2)$ with a left invariant contact metric structure.*

We have seen that Sasakian manifolds always have some positive curvature; in particular, the Z_η-sectional curvatures are $+1$. A more general class of contact metric manifolds, which includes the Sasakian manifolds, called (κ, μ)-manifolds, has become of more interest than one might have expected. We begin with the following theorem from [13].

Theorem 7. *A contact metric manifold M^{2n+1} satisfying $R_{XY}Z_\eta = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

The contact metric structure on $E^{n+1} \times S^n(4)$ is the standard one on the tangent sphere bundle of Euclidean space (see, e.g., [6], Section 9.2).

The condition $R_{XY}Z_\eta = 0$ is not a $\mathcal{D}$-homothetic invariant as was observed by Themis Koufogiorgos, unlike a number of other conditions we mentioned above; instead, it takes the form

$$R_{XY}Z_\eta = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2}$$

for constants κ and μ. However, the form of (2) is $\mathcal{D}$-homothetic invariant, in particular, for the deformed metric $\bar{g} = ag + a(a-1)\eta \otimes \eta$, we have this form for $\bar{R}_{XY}\bar{Z}_\eta$ with

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

A contact metric manifold for which (2) is satisfied is called a (κ, μ)-*manifold*, and was introduced and developed as an interesting class of contact metric manifolds in [14] by Koufogiorgos, Papatoniou, and the author. E. Boeckx [15] gave an important classification of (κ, μ)-manifolds; here, we simply note that (κ, μ)-manifolds exist for all values of $\kappa \leq 1$ and μ.

We state the following theorem and give a few remarks.

Theorem 8. *On a (κ, μ)-manifold we have $\kappa \leq 1$. If $\kappa = 1$, the structure is Sasakian and if $\kappa < 1$, the (κ, μ) condition determines the curvature of M^{2n+1} completely.*

In the equality case, the proof shows that $h = 0$ and consequently μ is indeterminant. For $\kappa < 1$, the non-zero eigenvalues of h are $\pm\sqrt{1-\kappa}$ each with multiplicity n. Recall that the non-zero eigenvalues come in pairs, λ and $-\lambda$. The (κ, μ)-manifold then admits three mutually orthogonal sub-bundles, $\mathcal{D}(0)$, $\mathcal{D}(\lambda)$, and $\mathcal{D}(-\lambda)$, all of which are integrable.

In Boeckx [16], the curvature tensor of a non-Sasakian (κ, μ)-manifold was computed. Here, we just note that for a unit vector $X \in [\lambda]$, $\phi X \in [-\lambda]$, we have the following.

$$K(X, Z_\eta) = \kappa + \lambda\mu, \quad K(\phi X, Z_\eta) = \kappa - \lambda\mu, \quad K(X, \phi X) = -(\kappa + \mu).$$

Thus, concerning the sign of the curvature question, we have that if a (κ, μ)-manifold were to be of negative curvature, then at least one eigenvalue would be > 1 by (1) and $\kappa \pm \lambda\mu < 0$, giving $\lambda^2 - 1 > \lambda|\mu|$. Then, $K(X, \phi X) = -(\kappa + \mu) < 0$ would give $\lambda|\mu| < \lambda^2 - 1 < \mu$, a contradiction.

We now briefly discuss an implication of some control on the Ricci curvature. In 1941 S. B. Myers [17] proved a famous implication of the sign of the Ricci curvature, viz. that a complete Riemannian manifold for which the Ricci curvature is bounded below by a positive constant is compact and has finite fundamental group. In 1981 Hasegawa and Seino [18] proved a stronger result for K-contact manifolds.

Theorem 9. *A complete K-contact manifold for which Ric $\geq \delta > -2$ is compact and has finite fundamental group.*

Hasegawa and Seino state their result for Sasakian manifolds, but their proof uses only the K-contact property. Note also that in the example of Krouglov above, the Ricci curvature in the direction of the Reeb vector field, Z_η, is equal to -2.

We close this section with the following remark and include a brief sketch of its proof. If a contact metric manifold of non-positive curvature exists, then a $\mathcal{D}$-homothetic deformation can be made to the contact metric structure to yield one with some positive curvature. Suppose that $M(\phi, Z_\eta, \eta, g)$ is a contact metric manifold of non-positive curvature, and let

$$S = \sup g(R_{XZ_\eta} Z_\eta, X) \leq 0$$

on M with $hX = \lambda X$, $\lambda \geq 0$, and X a g-unit eigenvector. Under a $\mathcal{D}$-homothetic deformation

$$\bar{g}(\bar{R}_{X\bar{Z}_\eta} \bar{Z}_\eta, X) = \frac{1}{a^2} \bar{g}(R_{XZ_\eta} Z_\eta + 2(a-1)(X + hX) + (a-1)^2 X, X).$$

Now, for some $\epsilon > 0$, there exists a point $p \in M$ such that $g(R_{XZ_\eta} Z_\eta, X) = S - \epsilon < 0$. At the point p, $\bar{g}(\bar{R}_{X\bar{Z}_\eta} \bar{Z}_\eta, X) = \frac{1}{a}(a^2 + 2\lambda a + S - \epsilon - 2\lambda - 1)$, which is positive for some $a > 0$.

4. Negative Z_η-Sectional Curvature

Although positive Z_η-sectional curvature is immediate in the case of K-contact manifolds, we indicate in this section that positive Z_η-sectional curvature is not necessarily abundant for contact metric manifolds in general. We begin this section with the following theorem of H. Chen and the author [19]. Denote by Q the Ricci operator, i.e., the Ricci tensor of type (1,1).

Theorem 10. *Let M^3 be a contact metric manifold on which $Q\phi = \phi Q$. Then, M^3 is either Sasakian, flat, or locally isometric to a left invariant metric on the Lie group $SU(2)$ or $SL(2, \mathbb{R})$. In the Lie group cases, M^3 has constant Z_η-sectional curvature $k < 1$ and constant ϕ-sectional curvature $-k$; such structures can occur with $k > 0$ for $SU(2)$ and $k < 0$ for $SL(2, \mathbb{R})$.*

Note the positive curvature in this theorem on $SL(2, \mathbb{R})$. The contact metric structure on the Lie group $SL(2, \mathbb{R})$ is given explicitly in [19] and in [20] as well as Section 11.2 of [6].

We next recall the formula

$$\nabla_X Z_\eta = -\phi X - \phi h X$$

and that it reflects the rotation of the contact structure and its Reeb vector field. In particular, if $hX = \lambda X$, one has $\nabla_X Z_\eta = -(1+\lambda)\phi X$. Thus, we can view Z_η as "turning" or "falling" toward $-\phi X$ for $\lambda > -1$, or toward ϕX for $\lambda < -1$. We also recall that if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is an eigenvalue with eigenvector ϕX.

We now ask if there exist directions orthogonal to Z_η, along which Z_η "falls" forward or backward as one moves in such a direction.

Theorem 11. *Let M be a contact metric manifold and suppose the tensor field h admits an eigenvalue $\lambda > 1$ at a point p, then there exists a vector Y orthogonal to Z_η at p, such that $\nabla_Y Z_\eta$ is collinear with Y. In particular, if M has negative Z_η-sectional curvature, such directions, Y, exist.*

Suppose now that $\lambda > 1$ is a positive eigenvalue of h with eigenvector X, then a and b may be chosen, such that $Y = aX + b\phi X$ is such a direction where $a^2 + b^2 = 1$ and $\nabla_Y Z_\eta = -(\sqrt{\lambda^2 - 1})Y$. Moreover if $Z = aX - b\phi X$, then $\nabla_Z Z_\eta = (\sqrt{\lambda^2 - 1})Z$. Thus, we can think of Z_η as "falling backward" as we move in the direction Y and "falling forward" as we move in the direction Z. For the inner product one has $g(Y, Z) = -\frac{1}{\lambda}$ and hence the directions Y and Z cannot be orthogonal. If λ has multiplicity $m \geq 1$, then there exist m-dimensional sub-bundles $\mathcal{Y}$ and $\mathcal{Z}$ such that $\nabla_Y Z_\eta$ is collinear with Y for any $Y \in \mathcal{Y}$ and $\nabla_Z Z_\eta$ is collinear with Z for any $Z \in \mathcal{Z}$. Directions along which the covariant derivative of Z_η is collinear with the direction are called *special directions*.

Classically an Anosov flow [21] is defined as follows. Consider a compact differentiable manifold M admitting a nonvanishing vector field Z and let $\{\psi_t\}$ denote its 1-parameter group of (C^k) diffeomorphisms. The flow $\{\psi_t\}$ and the vector field Z are said to be *Anosov* if there exist invariant sub-bundles E^s and E^u, such that $TM = E^s \oplus E^u \oplus \{Z\}$ and there exists a Riemannian metric, such that

$$|\psi_{t*}Y| \leq ae^{-ct}|Y| \text{ for } t \geq 0 \text{ and } Y \in E^s_p,$$

$$|\psi_{t*}Y| \leq ae^{ct}|Y| \text{ for } t \leq 0 \text{ and } Y \in E^u_p$$

for some positive constants a and c independent of $p \in M$ and Y in E^s_p or E^u_p. The sub-bundles E^s and E^u are called the *stable* and *unstable* sub-bundles. The sub-bundles E^s and E^u are integrable with C^k integral submanifolds.

When the manifold M is compact this idea is independent of the Riemannian metric. For noncompact manifolds, the notion is metric-dependent; for example, one can have a metric on $\mathbb{R}^3$ with respect to which one of the coordinate fields is an Anosov vector field, even though a coordinate field is not Anosov with respect to the Euclidean metric on $\mathbb{R}^3$, see Section 11.2 of [6]. Here, of course, we are dealing with Riemannian metrics associated to a contact form, thus when we speak of the "Reeb vector field being Anosov", we will mean that it is an Anosov vector field with respect to an associated metric of the contact form.

The most notable example of a contact manifold for which the Reeb vector field is Anosov is the tangent sphere bundle of a negatively curved manifold; here the Reeb vector field is (twice) the geodesic flow, see, e.g., [6], Section 9.2. The tangent sphere bundle of a surface is closely related to the structure on $SL(2,\mathbb{R})$, from both the topological and Anosov points of view. Setting $Z_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$, $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/Z_2$ is homeomorphic to the tangent sphere bundle of the hyperbolic plane. For a compact surface of constant negative curvature, the geodesic flow may be realized on $PSL(2,\mathbb{R})/\Gamma$ by $\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}$, where Γ is a discrete subgroup of $SL(2,\mathbb{R})$, for which $SL(2,\mathbb{R})/\Gamma$ is compact, (see, e.g.,, Auslander, Green, and Hahn [22] pp. 26–27). From the Riemannian point of view, however, these examples are quite different as we next observe.

Theorem 12. *With respect to the standard contact metric structure on the tangent sphere bundle of a negatively curved surface, the Reeb vector field is Anosov, but the special directions never agree with the stable and unstable directions.*

In contrast, if one assumes that the special directions do agree with the Anosov directions, we have the following result.

Theorem 13. *Consider a 3-dimensional contact metric manifold M^3 with negative Z_η-sectional curvature. If the Reeb vector field Z_η generates an Anosov flow with respect to the associated metric and the special directions agree with the stable and unstable directions, then the contact metric structure satisfies $\nabla_{Z_\eta} h = 0$. Moreover, if M^3 is compact, it is a compact quotient of $\widetilde{SL}(2, \mathbb{R})$.*

For proofs of the preceding theorems see Section 11.2 of [6].

Y. Mitsumatsu [23] and Eliashberg and Thurston [24] introduced a generalization of Anosov flows as follows. A flow ψ_t and its corresponding vector field are said to be *conformally Anosov*, Eliashberg and Thurston (*projectively Anosov*, Mitsumatsu), if there is a continuous Riemannian metric and a continuous, invariant splitting, $TM = E^s \oplus E^u \oplus \{Z_\eta\}$, as in the Anosov case, such that for $Z \in E^u$ and $Y \in E^s$,

$$\frac{|\psi_{t*}Z|}{|\psi_{t*}Y|} \geq e^{ct}\frac{|Z|}{|Y|}$$

for some constant $c > 0$ and all $t \geq 0$.

A contact form determines an orientation on the underlying manifold; even for a contact structure in the wider sense, this is true in dimension 3 , as the sign of $\eta \wedge d\eta$ is independent of the choice of the local contact form η.

The main result from Mitsumatsu [23] and Eliashberg and Thurston [24] for our purpose is the following.

Theorem 14. *If two contact structures (in the wider sense) on a compact* 3*-dimensional contact manifold M^3 induce opposition orientations, then the vector field directing the intersection of the two contact sub-bundles is a conformally Anosov flow. Conversely, given a conformally Anosov flow on M^3, there exist two contact structures giving opposite orientations on M^3 whose contact sub-bundles intersect tangent to the flow.*

Now, we might expect certain curvature hypotheses on a compact contact metric 3-manifold to imply that the Reeb vector field is conformally Anosov. In particular, negative Z_η-sectional curvature is such a hypothesis, and we state a result of D. Perrone and the author [25].

Theorem 15. *Consider a compact* 3*-dimensional contact metric manifold with nowhere vanishing h. Let $\{e_1, e_2(= \phi e_1), Z_\eta\}$ be an orthonormal eigenvector basis of h with $he_1 = \lambda e_1$ and λ the positive eigenvalue. If $K(Z_\eta, e_1) < (1+\lambda)^2$ and $K(Z_\eta, e_2) < (1-\lambda)^2$, then the Reeb vector field Z_η is conformally Anosov.*

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