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Convergence Theorem of Two Sequences for Solving the Modified Generalized System of Variational Inequalities and Numerical Analysis

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Abstract: The purpose of this paper is to introduce an iterative algorithm of two sequences which depend on each other by using the intermixed method. Then, we prove a strong convergence theorem for solving fixed-point problems of nonlinear mappings and we treat two variational inequality problems which form an approximate modified generalized system of variational inequalities (MGSV). By using our main theorem, we obtain the additional results involving the split feasibility problem and the constrained convex minimization problem. In support of our main result, a numerical example is also presented.

Keywords: the intermixed algorithm; strong convergence theorem; variational inequality; fixed point; modified generalized system of variational inequalities (MGSV)

MSC: 47H09; 47H10; 90C33

Let $\mathcal{H}$ be a real Hilbert space. Let C be a nonempty closed convex subset of $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ respectively and let $\mathcal{T}$ be a self-mapping of C. We use $F(\mathcal{T})$ to denote the set of fixed points of $\mathcal{T}$ (i.e., $F(\mathcal{T}) = \{x \in C : \mathcal{T}x = x\}$).

Recall that $\mathcal{T}$ is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|\mathcal{T}x - \mathcal{T}y\|^2 \leq \|x - y\|^2 + \kappa\|(I - \mathcal{T})x - (I - \mathcal{T})y\|^2, \ \ \forall x, y \in C. \tag{1}$$

Please note that the class of κ-strict pseudo-contractions strictly includes the class of *nonexpansive* mappings which are self-mappings $\mathcal{T}$ on C such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|, \ \ \forall x, y \in C. \tag{2}$$

In particular, $\mathcal{T}$ is nonexpansive mapping if and only if $\mathcal{T}$ is a 0-strict pseudo-contraction.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of weak and strong convergence theorem, see for example [1–3] and the references therein.

Over recent decades, many authors have constructed various types of iterative methods to approximate fixed points. The first one is the Mann iteration introduced by Mann [4] in 1953 which is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\mathcal{T}x_n, \ \ n \geq 0, \tag{3}$$

where $x_0 \in C$ is chosen arbitrarily and $\alpha_n \in [0, 1]$, $\mathcal{T} : C \to C$ is a mapping. If $\mathcal{T}$ is a nonexpansive mapping, the sequence $\{x_n\}$ be generated by (3) converges weakly to an element of $F(\mathcal{T})$.

It is well known that in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm [4] is only weakly convergent.

It is clear that strict pseudo-contractions are more general than nonexpansive mappings, and therefore they have a wider range of applications. Therefore, it is important to develop the theory of iterative methods for strict pseudo-contractions. Indeed, Browder and Petryshyn [5] proved that if the sequence $\{x_n\}$ is generated by (3) with a constant control parameter $\alpha_n \equiv \alpha$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges weakly to a fixed point of the strict pseudo-contraction T. Moreover, many mathematicians proposed iterative algorithms and proved the strong convergence theorems for a nonexpansive mapping and a κ-strictly pseudo-contractive mapping in Hilbert space to find their fixed points, see for example [6–9].

To prove the strong convergence of iterations determined by nonexpansive mapping, Moudafi [1] established a theorem for finding fixed points of nonexpansive mappings. More precisely, he established the following result, known as the *viscosity approximation method*.

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_1 \in C \text{ is arbitrarily chosen}, \\ x_{n+1} = \frac{1}{1+\varepsilon_n} Sx_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n), \forall n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\varepsilon_n\}$ is a sequence of positive real numbers having to go to zero. Then the sequence $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)}f(z)$ and $P_{F(S)}$ is a metric projection of $\mathcal{H}$ onto $F(S)$.

The Moudafi viscosity approximation method can be applied to elliptic differential equations, linear programming, convex optimization and monotone inclusions, it has been widely studied in the literature (see [10–12]).

To construct an iterative algorithm such that it converges strongly to the fixed points of a finite family of strict pseudo-contractions by using the concept of the viscosity approximation method (4) and Manns iteration (3), Yao et al. [13] proposed the intermixed algorithm for two strict pseudo-contractions as follows:

Algorithm 1. *For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n], \ n \geq 0, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n], \ n \geq 0, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real number in (0,1), $T, S : C \to C$ are a strict λ-pseudo-contractions, $f : C \to \mathcal{H}$ is a ρ_1-contraction and $g : C \to \mathcal{H}$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant.

Then they proved the strong convergence theorem of the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (5) as follows.

Theorem 2. *Suppose that $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Assume the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\beta_n \in [\xi_1, \xi_2] \subset (0,1)$ *for all* $n \geq 0$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (5) converge strongly to $P_{fix(T)}f(y^*)$ and $P_{fix(S)}g(x^*)$, respectively.

If putting $C = \mathcal{H}$ and $\beta_n = 1$ in (5), we have

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n, \ n \geq 0, \\ y_{n+1} = \alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n, \ n \geq 0, \end{cases} \tag{6}$$

which is a modified version of viscosity approximation method. Observe that the sequence $\{x_n\}$ and $\{y_n\}$ are mutually dependent on each other.

Let $B : C \to \mathcal{H}$. The *variational inequality problem* is to find a point $u^* \in C$ such that

$$\langle Bu^*, v - u^* \rangle \geq 0, \tag{7}$$

for all $v \in C$. The set of solutions of (7) is denoted by $VI(C, B)$. It is known that the variational inequality, as a strong and important tool, has already been studied for a wide class of optimization problems in economics, and equilibrium problems arising in physics and several other branches of pure and applied sciences, see for example [14–17].

Recently, in 2018, Siriyan and Kangtunyakarn [18] introduced the following *modified generalized system of variational inequalities (MGSV)*, which involves finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \ \ \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \ \ \forall x \in C. \end{cases} \tag{8}$$

where $D_1, D_2, D_3 : C \to \mathcal{H}$, $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0, 1]$.

If putting $a = 0$, in (8), we have

$$\begin{aligned} &\langle x^* - (I - \lambda_1 D_1)y^*, x - x^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle y^* - (I - \lambda_2 D_2)z^*, x - y^* \rangle \geq 0, \ \ \forall x \in C, \\ &\langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \ \ \forall x \in C. \end{aligned} \tag{9}$$

which is generalized system of variational inequalities modified by Ceng et al. [19],

To find an element of the set of solutions of modified generalized system of variational inequalities problem (8), Siriyan and Kangtunyakarn [18] introduced the following iterative scheme:

$$\begin{aligned} &x_{n+1} = \beta_n^1 x_n + \beta_n^2 T x_n + \beta_n^3 P_C(I - \lambda D) y_n, \\ &y_n = \alpha_n \gamma f(x_n) + \left(I - \alpha_n \overline{A}\right) G x_n, \end{aligned} \tag{10}$$

where $D, D_1, D_2, D_3 : C \to \mathcal{H}$ be d, d_1, d_2, d_3-inverse strongly monotone mappings, respectively, $G : C \to C$ is defined by

$$G(x) = P_C(I - \lambda_1 D_1)\left(ax + (1-a)P_C(I - \lambda_2 D_2)\left(ax + (1-a)P_C(I - \lambda_3 D_3)x\right)\right),$$

and $a \in [0, 1)$. Under some suitable conditions, see more details [18], they proved that the sequence $\{x_n\}$ converges strongly to $x_0 = P_\Omega(I - \overline{A} + \gamma f)x_0$ and (x_0, y_0, z_0) is a solution of (10) where $y_0 = P_C(I - \lambda_2 D_2)\left(ax_0 + (1-a)z_0\right)$ and $z_0 = P_C(I - \lambda_3 D_3)x_0$.

Moreover, they proved Lemma 3 in the next section which involving MGSV and the set of solution of fixed point of nonlinear equation related to a metric projection onto C. This lemma is very important to prove our main result in Section 2.

By using the concept of (5), we introduce a new iterative method for solving a modified generalized system of variational inequalities as follows:

Algorithm 2. *Starting with $x_1, w_1 \in C$, let the sequences $\{x_n\}$ and $\{w_n\}$ be defined by*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \lambda_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)G_C^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \lambda_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)G_C^2 w_n). \end{cases}$$

By putting $B_1 = B_2 = 0$, we get

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)G_C^1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)G_C^2 w_n). \end{cases}$$

which is a modified version of (5).

Under some extra conditions in Theorem 3, we prove a strong convergence theorem for solving fixed-point problems of nonlinear mappings and two variational inequality problems by using Algorithm 2 which is an approximate MGSV. Moreover, using our main result, we obtain additional results involving the split feasibility problem (SFP) and the constrained convex minimization problem. Finally, we give a numerical example for the main theorem.

1. Preliminaries

We denote the weak convergence and the strong convergence by "$\rightharpoonup$" and "$\rightarrow$", respectively. For every $x \in \mathcal{H}$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of $\mathcal{H}$ onto C.

Definition 1. *A mapping $f : C \to C$ is called contractive if there exists a constant $\xi \in (0,1)$ such that*

$$\|f(u) - f(v)\| \leq \xi \|u - v\|,$$

for all $u, v \in C$.

A mapping $f : C \to \mathcal{H}$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle fu - fv, u - v \rangle \geq \alpha \|fu - fv\|^2,$$

for all $u, v \in C$.

The following lemmas are needed to prove the main theorem.

Lemma 1 ([20]). *Each Hilbert space $\mathcal{H}$ satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset \mathcal{H}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in \mathcal{H}$ with $y \neq x$.

Lemma 2. *Let $\mathcal{H}$ be a real Hilbert space. Then, for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

(i) $\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2,$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \ \ \forall x, y \in \mathcal{H}.$

Lemma 3 ([18]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $D_1, D_2, D_3 : C \to \mathcal{H}$ are three mappings. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0,1]$. The following statements are equivalent* *(i)* $(x^*, y^*, z^*) \in C \times C \times C$ *is a solution of problem (8)*

(ii) x^* *is a fixed point of the mapping* G*, i.e.,* $x^* \in F(G)$*, defined the mapping* $G : C \to C$ *by* $G(x) = P_C(I - \lambda_1 D_1)\left(ax + (1-a)P_C(I - \lambda_2 D_2)\left(ax + (1-a)P_C(I - \lambda_3 D_3)x\right)\right), \forall x \in C$*, where* $y^* = P_C(I - \lambda_2 D_2)\left(ax^* + (1-a)z^*\right)$ *and* $z^* = P_C(I - \lambda_3 D_3)x^*$.

Lemma 4 ([21])**.** *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \alpha_n)s_n + \delta_n, \quad \forall n \ge 0,$$

where α_n *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim sup_{n\to\infty} \dfrac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 5 ([22])**.** *For a given* $z \in \mathcal{H}$ *and* $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \ge 0, \forall v \in C$.
Furthermore, P_C *is a firmly nonexpansive mapping of* $\mathcal{H}$ *onto* C*, i.e.,* $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \forall x, y \in \mathcal{H}$.

Lemma 6 ([23])**.** *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and let* $T : C \to C$ *be a* κ*-strictly pseudo-contractive mapping with* $F(T) \ne \emptyset$*. Then, the following statements hold:*

(i) $F(T) = VI(C, I - T)$,

(ii) For every $u \in C$ *and* $v \in F(T)$,

$$\|P_C(I - \lambda(I - T))u - v\| \le \|u - v\|,$$

for $u \in C$ *and* $v \in F(T)$ *and* $\lambda \in (0, 1 - \kappa)$.

2. Main Result

In this section, we introduce a strong convergence theorem for solving fixed-point problems of nonlinear mappings and two variational inequality problems by using Algorithm 2.

Theorem 3. *Let* C *be nonempty closed convex subset of a real Hilbert* $\mathcal{H}$*. For* $i = 1, 2$*, let* $B_i : C \to \mathcal{H}$ *be* α_i*-inverse strongly monotone mapping with* $\alpha = \min\{\alpha_1, \alpha_2\}$ *and let* $f, g : \mathcal{H} \to \mathcal{H}$ *be* a_f *and* a_g*-contraction mappings with* $a = \max\{a_f, a_g\}$*. For* $i = 1, 2$ *and* $j = 1, 2, 3$ *let* $D_j^i : C \to \mathcal{H}$ *be* d_j^i*-inverse strongly monotone, where* $\lambda_j^i \in (0, 2\omega_i)$ *with* $\omega_i = \min\limits_{j=1,2,3}\{d_j^i\}$*. For* $i = 1, 2$*, define* $G_i : C \to C$ *by* $G_i(x) = P_C(I - \lambda_1^i D_1^i)(ax + (1-a)P_C(I - \lambda_2^i D_2^i)(ax + (1-a)P_C(I - \lambda_3^i D_3^i)x)), \forall x \in C$*. Let the sequences* $\{x_n\}$ *and* $\{w_n\}$ *be generated by* $x_1, w_1 \in C$ *and by*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)G_1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 B_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)G_2 w_n) \end{cases} \tag{11}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \eta_n + \mu_n = 1$ *and* $\gamma \in (0, 2\alpha)$ *with* $\gamma = \min\{\gamma_1, \gamma_2\}$*. Assume the following conditions hold:*

(i) $\mathcal{F}_i = F(G_i) \cap VI(C, B_i) \ne \emptyset$ *for* $i = 1, 2$,

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim\limits_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \le \delta_n, \eta_n, \mu_n \le \theta$ *for all* $n \in N$ *and for some* $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then $\{x_n\}$ *converges strongly to* $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, *where* $y_1^* = P_C(I - \lambda_2^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ *and* $z_1^* = P_C(I - \lambda_3^1 D_3^1)x_1^*$ *and* $\{w_n\}$ *converges strongly to* $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, *where* $y_2^* = P_C(I - \lambda_2^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ *and* $z_1^* = P_C(I - \lambda_3^2 D_3^2)x_1^*$.

Proof. The proof of this theorem will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

First, we will prove that $I - \gamma B_i$ is nonexpansive with $\gamma = \min\{\gamma_1, \gamma_2\}$, for $i = 1, 2$ we get

$$\begin{aligned} \|(I - \gamma B_i)x - (I - \gamma B_i)w\|^2 &= \|x - w - \gamma(B_i x - B_i w)\|^2 \\ &= \|x - w\|^2 - 2\gamma\langle x - w, B_i x - B_i w\rangle \\ &\quad + \gamma^2 \|B_i x - B_i w\|^2 \\ &\le \|x - w\|^2 - 2\alpha\gamma \|B_i x - B_i w\|^2 \\ &\quad + \gamma^2 \|B_i x - B_i w\|^2 \\ &= \|x - w\|^2 - \gamma(2\alpha - \gamma)\|B_i x - B_i w\|^2 \\ &\le \|x - w\|^2. \end{aligned}$$

Thus, $I - \gamma B_i$ is a nonexpansive mapping, for $i = 1$ and $i = 2$.
Let $\tilde{x} \in \mathcal{F}_1$ and $\tilde{w} \in \mathcal{F}_2$. Then we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}\| &= \|\delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)G_1 x_n) \\ &\quad - (\delta_n + \eta_n + \mu_n)\tilde{x}\| \\ &\le \delta_n \|x_n - \tilde{x}\| + \eta_n \|P_C(I - \gamma_1 B_1)x_n - \tilde{x}\| + \mu_n \|P_C(\alpha_n f(w_n) \\ &\quad + (1-\alpha_n)G_1 x_n) - \tilde{x}\| \\ &\le (1-\mu_n)\|x_n - \tilde{x}\| + \mu_n \|\alpha_n(f(w_n) - \tilde{x}) + (1-\alpha_n)(G_1 x_n - \tilde{x})\| \\ &\le (1-\mu_n)\|x_n - \tilde{x}\| + \mu_n\alpha_n \|f(w_n) - \tilde{x}\| + \mu_n(1-\alpha_n)\|x_n - \tilde{x}\| \\ &\le (1-\mu_n)\|x_n - \tilde{x}\| + \mu_n\alpha_n a\|w_n - \tilde{w}\| + \mu_n\alpha_n \|f(\tilde{w}) - \tilde{x}\| \\ &\quad + \mu_n(1-\alpha_n)\|x_n - \tilde{x}\| \\ &= (1-\mu_n\alpha_n)\|x_n - \tilde{x}\| + \mu_n\alpha_n a\|w_n - \tilde{w}\| + \mu_n\alpha_n\|f(\tilde{w}) - \tilde{x}\|. \end{aligned} \tag{12}$$

Similarly, we get

$$\|w_{n+1} - \tilde{w}\| \le (1-\mu_n\alpha_n)\|w_n - \tilde{w}\| + \mu_n\alpha_n a\|x_n - \tilde{x}\| + \mu_n\alpha_n\|g(\tilde{x}) - \tilde{w}\|. \tag{13}$$

Combining (12) and (13), we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}\| + \|w_{n+1} - \tilde{w}\| &\le (1-\mu_n\alpha_n)\left[\|x_n - \tilde{x}\| + \|w_n - \tilde{w}\|\right] \\ &\quad + \mu_n\alpha_n a\left[\|x_n - \tilde{x}\| + \|w_n - \tilde{w}\|\right] \\ &\quad + \mu_n\alpha_n\left[\|g(\tilde{x}) - \tilde{w}\| + \|f(\tilde{w}) - \tilde{x}\|\right] \\ &= (1 - \mu_n\alpha_n(1-a))\left[\|x_n - \tilde{x}\| + \|w_n - \tilde{w}\|\right] \\ &\quad + \mu_n\alpha_n\left[\|g(\tilde{x}) - \tilde{w}\| + \|f(\tilde{w}) - \tilde{x}\|\right]. \end{aligned}$$

By induction, we can derive that

$$\|x_n - \tilde{x}\| + \|w_n - \tilde{w}\| \leq \max\left\{\|x_1 - \tilde{x}\| + \|w_1 - \tilde{w}\|, \frac{\|g(\tilde{x}) - \tilde{w}\| + \|f(\tilde{w}) - \tilde{x}\|}{1-a}\right\},$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{w_n\}$ are bounded.

Step 2. Claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|w_{n+1} - w_n\| = 0$.

First, we let $U_n = P_C(\alpha_n f(w_n) + (1-\alpha_n)G_1 x_n)$ and $V_n = P_C(\alpha_n g(x_n) + (1-\alpha_n)G_2 w_n)$. Then, observe that

$$\begin{aligned}
\|U_n - U_{u-1}\| &= \|P_C(\alpha_n f(w_n) + (1-\alpha_n)G_1 x_n) - P_C(\alpha_{n-1} f(w_{n-1}) \\
&\quad + (1-\alpha_{n-1})G_1 x_{n-1})\| \\
&\leq \alpha_n \|f(w_n) - f(w_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(w_{n-1})\| \\
&\quad + (1-\alpha_n)\|G_1 x_n - G_1 x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|G_1 x_{n-1}\| \\
&\leq \alpha_n a\|w_n - w_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left[\|f(w_{n-1})\| + \|G_1 x_{n-1}\|\right] \\
&\quad + (1-\alpha_n)\|x_n - x_{n-1}\|.
\end{aligned} \tag{14}$$

By the definition of x_n and (14) we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\delta_n x_n + \eta_n P_C(I - \gamma_1 B_1)x_n + \mu_n U_n - \delta_{n-1} x_{n-1} \\
&\quad - \eta_{n-1} P_C(I - \gamma_1 B_1)x_{n-1} - \mu_{n-1} U_{n-1}\| \\
&\leq \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + \eta_n \|P_C(I - \gamma_1 B_1)x_n \\
&\quad - P_C(I - \gamma_1 B_1)x_{n-1}\| + |\eta_n - \eta_{n-1}|\|P_C(I - \gamma_1 B_1)x_{n-1}\| \\
&\quad + \mu_n \|U_n - U_{n-1}\| + |\mu_n - \mu_{n-1}|\|U_{n-1}\| \\
&= (1-\mu_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
&\quad + |\eta_n - \eta_{n-1}|\|P_C(I - \gamma_1 B_1)x_{n-1}\| \\
&\quad + \mu_n |\alpha_n - \alpha_{n-1}| \left[\|f(w_{n-1})\| + \|G_1 x_{n-1}\|\right] \\
&\quad + \mu_n (1-\alpha_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|U_{n-1}\| \\
&\quad + \mu_n \alpha_n a\|w_n - w_{n-1}\|.
\end{aligned} \tag{15}$$

Using the same method as derived in (15), we have

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq (1-\mu_n)\|w_n - w_{n-1}\| + |\delta_n - \delta_{n-1}|\|w_{n-1}\| \\
&\quad + |\eta_n - \eta_{n-1}|\|P_C(I - \gamma_2 B_2)w_{n-1}\| \\
&\quad + \mu_n |\alpha_n - \alpha_{n-1}| \left[\|g(x_{n-1})\| + \|G_2 w_{n-1}\|\right] \\
&\quad + \mu_n (1-\alpha_n)\|w_n - w_{n-1}\| + |\mu_n - \mu_{n-1}|\|V_{n-1}\| \\
&\quad + \mu_n \alpha_n a\|x_n - x_{n-1}\|.
\end{aligned} \tag{16}$$

From (15) and (16), then we get

$$\begin{aligned}
\|x_{n+1} - x_n\| + \|w_{n+1} - w_n\| &\leq (1-\mu_n) \left[\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\|\right] \\
&\quad + |\delta_n - \delta_{n-1}| \left[\|x_{n-1}\| + \|w_{n-1}\|\right] \\
&\quad + |\eta_n - \eta_{n-1}|\left[\|P_C(I - \gamma_1 B_1)x_{n-1}\| \right. \\
&\quad \left. + \|P_C(I - \gamma_2 B_2)w_{n-1}\|\right]
\end{aligned}$$

$$
\begin{aligned}
&+ |\mu_n - \mu_{n-1}| \, [\|U_{n-1}\| + \|V_{n-1}\|] \\
&+ \mu_n \alpha_n a \, [\|w_n - w_{n-1}\| + \|x_n - x_{n-1}\|] \\
&+ \mu_n |\alpha_n - \alpha_{n-1}| [\|f(w_{n-1})\| + \|G_1 x_{n-1}\| \\
&+ \|g(x_{n-1})\| + \|G_2 w_{n-1}\|] \\
&+ \mu_n (1 - \alpha_n) \, [\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\|] \\
\leq\ & (1 - \alpha_n \bar{\theta}(1 - a)) \, [\|x_n - x_{n-1}\| + \|w_n - w_{n-1}\|] \\
&+ |\delta_n - \delta_{n-1}| \, [\|x_{n-1}\| + \|w_{n-1}\|] \\
&+ |\eta_n - \eta_{n-1}| [\|P_C(I - \gamma_1 B_1) x_{n-1}\| \\
&+ \|P_C(I - \gamma_2 B_2) w_{n-1}\|] \\
&+ |\mu_n - \mu_{n-1}| \, [\|U_{n-1}\| + \|V_{n-1}\|] \\
&+ \theta |\alpha_n - \alpha_{n-1}| [\|f(w_{n-1})\| + \|G_1 x_{n-1}\| \\
&+ \|g(x_{n-1})\| + \|G_2 w_{n-1}\|].
\end{aligned}
$$

Applying Lemma 4 and the condition (ii), (iii) and (iv) we can conclude that

$$
\|x_{n+1} - x_n\| \to 0 \text{ and } \|w_{n+1} - w_n\| \to 0 \ \ as \ \ n \to \infty. \tag{17}
$$

Step 3. Prove that $\lim_{n\to\infty} \|U_n - P_C(I - \gamma_1 B_1) U_n\| = \lim_{n\to\infty} \|U_n - G_1 U_n\| = 0$.

To show this, take $\tilde{u}_n = \alpha_n f(w_n) + (1 - \alpha_n) G_1 x_n, \ \forall n \in \mathbb{N}$. Then we derive that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|\delta_n (x_n - \tilde{x}) + \eta_n (P_C(I - \gamma_1 B_1) x_n - \tilde{x}) + \mu_n (U_n - \tilde{x})\|^2 \\
&\leq \delta_n \|x_n - \tilde{x}\|^2 + \eta_n \|P_C(I - \gamma_1 B_1) x_n - \tilde{x}\|^2 \\
&\quad - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 + \mu_n \|\tilde{u}_n - \tilde{x}\|^2 \\
&\leq (1 - \mu_n) \|x_n - \tilde{x}\|^2 - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 \\
&\quad + \mu_n \|\alpha_n (f(w_n) - G_1 x_n) + (G_1 x_n - \tilde{x})\|^2 \\
&\leq (1 - \mu_n) \|x_n - \tilde{x}\|^2 - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 A_1) x_n\|^2 \\
&\quad + \mu_n \left[\|G_1 x_n - \tilde{x}\|^2 + 2\alpha_n \langle f(w_n) - G_1 x_n, \tilde{u}_n - \tilde{x} \rangle \right] \\
&\leq \|x_n - \tilde{x}\|^2 - \delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 \\
&\quad + 2\mu_n \alpha_n \|f(w_n) - G_1 x_n\| \|\tilde{u}_n - \tilde{x}\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\delta_n \eta_n \|x_n - P_C(I - \gamma_1 B_1) x_n\|^2 &\leq \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\
&\quad + 2\mu_n \alpha_n \|f(w_n) - G_1 x_n\| \|\tilde{u}_n - \tilde{x}\| \\
&\leq \|x_n - x_{n+1}\| \, [\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|] \\
&\quad + 2\mu_n \alpha_n \|f(w_n) - G_1 x_n\| \|\tilde{u}_n - \tilde{x}\|.
\end{aligned}
$$

Then, we have

$$
\|x_n - P_C(I - \gamma_1 B_1) x_n\| \to 0 \ \ as \ \ n \to \infty. \tag{18}
$$

Observe that

$$
x_{n+1} - x_n = \eta_n (P_C(I - \gamma_1 B_1) x_n - x_n) + \mu_n (U_n - x_n).
$$

This follows that

$$\mu_n\|U_n - x_n\| \leq \eta_n\|P_C(I-\gamma_1 B_1)x_n - x_n\| + \|x_{n+1} - x_n\|.$$

From (17) and (18), we obtain

$$\|U_n - x_n\| \to 0 \ \ as \ \ n \to \infty. \tag{19}$$

Observe that

$$\begin{aligned}\|U_n - P_C(I-\gamma_1 B_1)U_n\| &\leq \|U_n - x_n\| + \|x_n - P_C(I-\gamma_1 B_1)x_n\| \\ &\quad + \|P_C(I-\gamma_1 B_1)x_n - P_C(I-\gamma_1 B_1)U_n\| \\ &\leq \|U_n - x_n\| + \|x_n - P_C(I-\gamma_1 B_1)x_n\| + \|x_n - U_n\| \\ &= 2\|U_n - x_n\| + \|x_n - P_C(I-\gamma_1 B_1)x_n\|,\end{aligned}$$

by (18) and (19), we obtain

$$\|U_n - P_C(I-\gamma_1 B_1)U_n\| \to 0 \ \ as \ \ n \to \infty. \tag{20}$$

Applying the same arguments as for deriving (20), we also obtain

$$\|V_n - P_C(I-\gamma_2 B_2)V_n\| \to 0 \ \ as \ \ n \to \infty.$$

Consider

$$\|x_{n+1} - U_n\| \leq \|x_{n+1} - x_n\| + \|x_n - U_n\|.$$

From (17) and (19), we have

$$\|x_{n+1} - U_n\| \to 0 \ \ as \ \ n \to \infty. \tag{21}$$

Since

$$\begin{aligned}\|x_n - G_1 x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|U_n - G_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \|\tilde{u}_n - G_1 x_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| \\ &\quad + \|\alpha_n f(w_n) + (1-\alpha_n)G_1 x_n - G_1 x_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - U_n\| + \alpha_n\|f(w_n) - G_1 x_n\|.\end{aligned}$$

From (17), (21) and condition (ii), we get

$$\|x_n - G_1 x_n\| \to 0 \ \ as \ \ n \to \infty. \tag{22}$$

Consider

$$\begin{aligned}\|U_n - G_1 U_n\| &\leq \|U_n - x_n\| + \|x_n - G_1 x_n\| + \|G_1 x_n - G_1 U_n\| \\ &\leq \|U_n - x_n\| + \|x_n - G_1 x_n\| + \|x_n - U_n\| \\ &\leq 2\|U_n - x_n\| + \|x_n - G_1 x_n\|.\end{aligned}$$

From (19) and (22), we have

$$\|U_n - G_1 U_n\| \to 0 \ \ as \ \ n \to \infty. \tag{23}$$

Applying the same method as (22), we also have

$$\|V_n - G_2 V_n\| \to 0 \ \ as \ \ n \to \infty.$$

Step 4. Claim that $\limsup_{n\to\infty} \langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle \le 0$, where $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$.

First, take a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle = \lim_{k\to\infty} \langle f(x_2^*) - x_1^*, U_{n_k} - x_1^* \rangle. \tag{24}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$ as $k \to \infty$. From (19), we obtain $U_{n_k} \rightharpoonup \hat{x} \ \ as \ \ k \to \infty$.

Next, we need to show that $\hat{x} \in \mathcal{F}_1 = F(G_1) \cap VI(C, B_1)$. Assume $\hat{x} \notin F(G_1)$. Then, we have $\hat{x} \ne G_1 \hat{x}$. By the Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|U_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \|U_{n_k} - G_1 \hat{x}\| \\
&\le \liminf_{k\to\infty} \|U_{n_k} - G_1 U_{n_k}\| + \liminf_{k\to\infty} \|G_1 U_{n_k} - G_1 \hat{x}\| \\
&\le \liminf_{k\to\infty} \|U_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction.

Therefore

$$\hat{x} \in F(G_1). \tag{25}$$

Assume $\hat{x} \notin VI(C, B_1)$, then we get $\hat{x} \ne P_C(I - \lambda_1 B_1)\hat{x}$.

From the Opial's condition and (20), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|U_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \|U_{n_k} - P_C(I - \gamma_1 B_1)\hat{x}\| \\
&\le \liminf_{k\to\infty} \|U_{n_k} - P_C(I - \gamma_1 B_1) U_{n_k}\| \\
&\quad + \liminf_{k\to\infty} \|P_C(I - \gamma_1 B_1) U_{n_k} - P_C(I - \gamma_1 B_1)\hat{x}\| \\
&\le \liminf_{k\to\infty} \|U_{n_k} - \hat{x}\|.
\end{aligned}$$

This is a contradiction.

Therefore

$$\hat{x} \in VI(C, B_1). \tag{26}$$

By (25) and (26), this yields that

$$\hat{x} \in \mathcal{F}_1 = F(G_1) \cap VI(C, B_1). \tag{27}$$

Since $U_{n_k} \rightharpoonup \hat{x} \ \ as \ \ k \to \infty$, (27) and Lemma 5, we can derive that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(x_2^*) - x_1^*, U_n - x_1^* \rangle &= \lim_{k\to\infty} \langle f(x_2^*) - x_1^*, U_{n_k} - x_1^* \rangle \\
&= \langle f(x_2^*) - x_1^*, \hat{x} - x_1^* \rangle \\
&\le 0.
\end{aligned} \tag{28}$$

Following the same method as for (28), we obtain that

$$\limsup_{n\to\infty}\langle g(x_1^*) - x_2^*, V_n - x_2^*\rangle \leq 0. \tag{29}$$

Step 5. Finally, prove that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$ and $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, respectively.

By firm nonexpansiveness of P_C, we derive that

$$\begin{aligned}
\|U_n - x_1^*\|^2 &= \|P_C \tilde{u}_n - x_1^*\|^2 \\
&\leq \langle \tilde{u}_n - x_1^*, U_n - x_1^*\rangle \\
&= \langle \alpha_n (f(w_n) - x_1^*) + (1-\alpha_n)(G_1 x_n - x_1^*), U_n - x_1^*\rangle \\
&= \alpha_n \langle f(w_n) - x_1^*, U_n - x_1^*\rangle + (1-\alpha_n)\langle G_1 x_n - x_1^*, U_n - x_1^*\rangle \\
&= \alpha_n \langle f(w_n) - f(x_2^*), U_n - x_1^*\rangle + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\quad + (1-\alpha_n)\|G_1 x_n - x_1^*\|\|U_n - x_1^*\| \\
&\leq \alpha_n a \|w_n - x_2^*\|\|U_n - x_1^*\| + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\quad + (1-\alpha_n)\|x_n - x_1^*\|\|U_n - x_1^*\| \\
&\leq \frac{\alpha_n a}{2}\left\{\|w_n - x_2^*\|^2 + \|U_n - x_1^*\|^2\right\} + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\quad + \frac{(1-\alpha_n)}{2}\left\{\|x_n - x_1^*\|^2 + \|U_n - x_1^*\|^2\right\} \\
&= \frac{\alpha_n a}{2}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{2}\|x_n - x_1^*\|^2 \\
&\quad + \left(\frac{\alpha_n a}{2} + \frac{(1-\alpha_n)}{2}\right)\|U_n - x_1^*\|^2 \\
&\quad + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&= \frac{\alpha_n a}{2}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{2}\|x_n - x_1^*\|^2 \\
&\quad + \left(\frac{1-\alpha_n(1-a)}{2}\right)\|U_n - x_1^*\|^2 \\
&\quad + \alpha_n \langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle,
\end{aligned}$$

which yields

$$\begin{aligned}
\|U_n - x_1^*\|^2 \leq{}& \frac{\alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 + \frac{(1-\alpha_n)}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \\
&+ \frac{\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle.
\end{aligned}\tag{30}$$

From the definition of x_n and (30), we get

$$\begin{aligned}
\|x_{n+1} - x_1^*\|^2 &\leq \delta_n \|x_n - x_1^*\|^2 + \eta_n \|P_C(I - \gamma_1 B_1)x_n - x_1^*\|^2 + \mu_n \|U_n - x_1^*\|^2 \\
&\leq (1-\mu_n)\|x_n - x_1^*\|^2 + \frac{\mu_n \alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2 \\
&\quad + \frac{\mu_n \alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*) - x_1^*, U_n - x_1^*\rangle \\
&\quad + \frac{\mu_n(1-\alpha_n)}{1+\alpha_n(1-a)}\|x_n - x_1^*\|^2 \\
&= \left(1 - \frac{\mu_n \alpha_n (2-a)}{1+\alpha_n(1-a)}\right)\|x_n - x_1^*\|^2 + \frac{\mu_n \alpha_n a}{1+\alpha_n(1-a)}\|w_n - x_2^*\|^2
\end{aligned}$$

$$+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\langle f(x_2^*)-x_1^*,U_n-x_1^*\rangle. \tag{31}$$

Similarly, as derived above, we also have

$$\|w_{n+1}-x_2^*\|^2 \le \left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\|w_n-x_2^*\|^2+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\|x_n-x_1^*\|^2 + \frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\langle g(x_1^*)-x_2^*,V_n-x_2^*\rangle. \tag{32}$$

From (31) and (32), we deduce that

$$\begin{aligned}
&\|x_{n+1}-x_1^*\|^2+\|w_{n+1}-x_2^*\|^2\\
&\le \left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}\right)\left(\|x_n-x_1^*\|^2+\|w_n-x_2^*\|^2\right)\\
&\quad+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\left(\|x_n-x_1^*\|^2+\|w_n-x_2^*\|^2\right)\\
&\quad+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\langle f(x_2^*)-x_1^*,U_n-x_1^*\rangle+\langle g(x_1^*)-x_2^*,V_n-x_2^*\rangle\right)\\
&=\left(1-\frac{\mu_n\alpha_n(2-a)}{1+\alpha_n(1-a)}+\frac{\mu_n\alpha_n a}{1+\alpha_n(1-a)}\right)\left(\|x_n-x_1^*\|^2+\|w_n-x_2^*\|^2\right)\\
&\quad+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\langle f(x_2^*)-x_1^*,U_n-x_1^*\rangle+\langle g(x_1^*)-x_2^*,V_n-x_2^*\rangle\right)\\
&=\left(1-\frac{2\mu_n\alpha_n(1-a)}{1+\alpha_n(1-a)}\right)\left\{\|x_n-x_1^*\|^2+\|w_n-x_2^*\|^2\right\}\\
&\quad+\frac{\mu_n\alpha_n}{1+\alpha_n(1-a)}\left(\langle f(x_2^*)-x_1^*,U_n-x_1^*\rangle+\langle g(x_1^*)-x_2^*,V_n-x_2^*\rangle\right).
\end{aligned}$$

Applying the condition (ii), (28), (29), and Lemma 4, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $x_1^*=P_{\mathcal{F}_1}f(x_2^*)$ and $x_2^*=P_{\mathcal{F}_2}g(x_1^*)$, respectively. This completes the proof. □

Corollary 1. *Let C be nonempty closed convex subset of a real Hilbert $\mathcal{H}$. For $i=1,2$, let $T_i: C\to C$ be a κ_i-strictly pseudo-contractive mapping with $F(T_i)\neq\emptyset$ and let $f,g:\mathcal{H}\to\mathcal{H}$ be a_f and a_g-contraction mappings with $a=\max\{a_f,a_g\}$. For $i=1,2$ and $j=1,2,3$ let $D_j^i: C\to H$ be d_j^i-inverse strongly monotone, where $\lambda_j^i\in(0,2\omega_i)$ with $\omega_i=\min_{j=1,2,3}\{d_j^i\}$. For $i=1,2$, define $G_i: C\to C$ by $G_i(x)=P_C(I-\lambda_1^iD_1^i)(ax+(1-a)P_C(I-\lambda_2^iD_2^i)(ax+(1-a)P_C(I-\lambda_3^iD_3^i)x)),\forall x\in C$. Let the sequence $\{x_n\}$ and $\{w_n\}$ be generated by $x_1,w_1\in C$ and by*

$$\begin{cases} x_{n+1}=\delta_nx_n+\eta_nP_C(I-\gamma_1(I-T_1))x_n+\mu_nP_C(\alpha_nf(w_n)+(1-\alpha_n)G_1x_n)\\ w_{n+1}=\delta_nw_n+\eta_nP_C(I-\gamma_2(I-T_2))w_n+\mu_nP_C(\alpha_ng(x_n)+(1-\alpha_n)G_2w_n)\end{cases} \tag{33}$$

where $\{\delta_n\},\{\eta_n\},\{\mu_n\},\{\alpha_n\}\subseteq[0,1]$ with $\delta_n+\eta_n+\mu_n=1$, $\gamma\in(0,2\alpha)$ with $\alpha=\min\{\frac{1-\kappa_1}{2},\frac{1-\kappa_2}{2}\}$ and $\gamma=\min\{\gamma_1,\gamma_2\}$. Assume the following conditions hold:

(i) $\mathcal{F}_i=F(G_i)\cap F(T_i)\neq\emptyset$ for $i=1,2$,

(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\alpha_n=0$,

(iii) $0<\bar{\theta}\le\delta_n,\eta_n,\mu_n\le\theta$ for $n\in N$ and for some $\bar{\theta},\theta>0$,

(iv) $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|\eta_{n+1}-\eta_n|<\infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$.

Then $\{x_n\}$ *converges strongly to* $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, *where* $y_1^* = P_C(I - \lambda_2^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ *and* $z_1^* = P_C(I - \lambda_3^1 D_3^1)x_1^*$ *and* $\{w_n\}$ *converges strongly to* $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, *where* $y_2^* = P_C(I - \lambda_2^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ *and* $z_1^* = P_C(I - \lambda_3^2 D_3^2)x_1^*$.

Proof. From Theorems 3 and 6, we have the desired conclusion. □

3. Application

In this section, we obtain Theorems 4 and 5 which solve the split feasibility problem and the constrained convex minimization problem. To prove these theorems, the following definition and lemmas are needed.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces and let C, Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $A_1, A_2 : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operator with A_1^*, A_2^* are adjoint of A_1 and A_2, respectively.

3.1. The Split Feasibility Problem

The split feasibility problem (SFP) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving [24]. The set of all solution (SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. The split feasibility problem was studied extensively as an extremely powerful tool in various fields such as medical image reconstruction, signal processing, intensity-modulated radiation therapy problems and computer tomograph; see [25–27] and the references therein.

In 2012, Ceng [28] introduced the following lemma to solve SFP;

Lemma 7. *Given* $x^* \in \mathcal{H}_1$, *the following statements are equivalent:*

(i) $x^* \in \Gamma$;

(ii) $x^* = P_C(I - \lambda A^*(I - P_Q)A)x^*$, *where* A^* *is adjoint of* A;

(iii) x^* *solves the variational inequality problem (VIP) of finding* $x^* \in C$ *such that* $\langle y - x^*, \nabla g(x^*)\rangle \geq 0$, *for all* $y \in C$ *and* $\nabla g = A^*(I - P_Q)A$.

By using these results, we obtain the following theorem

Theorem 4. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be a real Hilbert spaces and let* C, Q *be a nonempty closed convex subsets of a real Hilbert space* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively. Let* $B_1, B_2 : \mathcal{H}_1 \to \mathcal{H}_2$ *be bounded linear operator with* B_1^*, B_2^* *are adjoint of* B_1 *and* B_2, *respectively and* L_1, L_2 *are spectral radius of* $B_1^*B_1$ *and* $B_2^*B_2$, *respectively with* $L = \max\{L_1, L_2\}$. *Let* $f, g : \mathcal{H} \to \mathcal{H}$ *be* a_f *and* a_g*-contraction mappings with* $a = \max\{a_f, a_g\}$. *For* $i = 1,2$ *and* $j = 1,2,3$ *let* $D_j^i : C \to \mathcal{H}$ *be* d_j^i*-inverse strongly monotone, where* $\lambda_j^i \in (0, 2\omega_i)$ *with* $\omega_i = \min_{j=1,2,3}\{d_j^i\}$. *For* $i = 1,2$, *define* $G_i : C \to C$ *by* $G_i(x) = P_C(I - \lambda_1^i D_1^i)(ax + (1-a)P_C(I - \lambda_2^i D_2^i)(ax + (1-a)P_C(I - \lambda_3^i D_3^i)x)), \forall x \in C$. *Let the sequences* $\{x_n\}$ *and* $\{w_n\}$ *be generated by* $x_1, w_1 \in C$ *and by*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 \nabla\Im_1)x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n)G_1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 \nabla\Im_2)w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)G_2 w_n) \end{cases} \quad (34)$$

where $\nabla\Im_1 = B_1^*(I - P_Q)B_1 x$, $\nabla\Im_2 = B_2^*(I - P_Q)B_2 x$ *for all* $x \in \mathcal{H}_1$, $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ *with* $\delta_n + \eta_n + \mu_n = 1$ *and* $\gamma \in (0, \frac{2}{L})$ *with* $\gamma = \min\{\gamma_1, \gamma_2\}$. *Assume the following conditions hold:*

(i) $\mathcal{F}_i = F(G_i) \cap \Gamma_i \neq \emptyset$ *where* $\Gamma_i\{x \in C; B_i x \in Q\}$ *for* $i = 1,2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ *for all* $n \in N$ *and for some* $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $x_1^* = P_{\mathcal{F}_1} f(x_2^*)$, *where* $y_1^* = P_C(I - \lambda_2^1 D_2^1)(ax_1^* + (1-a)z_1^*)$ *and* $z_1^* = P_C(I - \lambda_3^1 D_3^1)x_1^*$ *and* $\{w_n\}$ *converges strongly to* $x_2^* = P_{\mathcal{F}_2} g(x_1^*)$, *where* $y_2^* = P_C(I - \lambda_2^2 D_2^2)(ax_2^* + (1-a)z_2^*)$ *and* $z_1^* = P_C(I - \lambda_3^2 D_3^2)x_1^*$.

Proof. Let $x, y \in \mathcal{H}_1$.

First, we will show that $\nabla \Im_1$ is $\frac{1}{L_1}$-inverse strongly monotone.
Consider,

$$\begin{aligned}
\|\nabla \Im_1(x) - \nabla \Im_1(y)\|^2 &= \|B_1^*(I - P_Q)B_1 x - B_1^*(I - P_Q)B_1 y\|^2 \\
&= \langle B_1^*(I - P_Q)B_1 x - B_1^*(I - P_Q)B_1 y, B_1^*(I - P_Q)B_1 x \\
&\quad - B_1^*(I - P_Q)B_1 y\rangle \\
&= \langle (I - P_Q)B_1 x - (I - P_Q)B_1 y, B_1 B_1^*(I - P_Q)B_1 x \\
&\quad - B_1 B_1^*(I - P_Q)B_1 y\rangle \\
&\leq L\|(I - P_Q)B_1 x - (I - P_Q)B_1 y\|^2.
\end{aligned}$$

From the property of P_C, we have

$$\begin{aligned}
\|(I - P_Q)B_1 x - (I - P_Q)B_1 y\|^2 &= \langle (I - P_Q)B_1 x - (I - P_Q)B_1 y, (I - P_Q)B_1 x \\
&\quad - (I - P_Q)B_1 y\rangle \\
&= \langle (I - P_Q)B_1 x - (I - P_Q)B_1 y, B_1 x - B_1 y\rangle \\
&\quad - \langle (I - P_Q)B_1 x - (I - P_Q)B_1 y, P_Q B_1 x \\
&\quad - P_Q B_1 y\rangle \\
&= \langle B_1^*(I - P_Q)B_1 x - B_1^*(I - P_Q)B_1 y, x - y\rangle \\
&\quad - \langle (I - P_Q)B_1 x - (I - P_Q)B_1 y, P_Q B_1 x \\
&\quad - P_Q B_1 y\rangle \\
&= \langle B_1^*(I - P_Q)B_1 x - B_1^*(I - P_Q)B_1 y, x_1 - y\rangle \\
&\quad - \langle (I - P_Q)B_1 x, P_Q B_1 x - P_Q B_1 y\rangle \\
&\quad + \langle (I - P_Q)B_1 y, P_Q B_1 x - P_Q B_1 y\rangle \\
&\leq \langle A_1^*(I - P_Q)B_1 x - B_1^*(I - P_Q)B_1 y, x - y\rangle.
\end{aligned}$$

Since $\nabla \Im_1(x) = B_1^*(I - P_Q)B_1 x$, we get

$$\langle \nabla \Im_1(x) - \nabla \Im_1(y), x - y\rangle \geq \frac{1}{L_1}\|\nabla \Im_1(x) - \nabla \Im_1(y)\|^2. \tag{35}$$

Then $\nabla \Im_1 = B_1^*(I - P_Q)B_1 x$ is $\frac{1}{L_1}$-inverse strongly monotone.
Using the same method as (35), we have

$$\langle \nabla \Im_2(x) - \nabla \Im_2(y), x - y\rangle \geq \frac{1}{L_2}\|\nabla \Im_2(x) - \nabla \Im_2(y)\|^2.$$

Then $\nabla \Im_2 = B_2^*(I - P_Q)B_2 x$ is $\frac{1}{L_2}$-inverse strongly monotone.
By using Theorems 3 and 7, we obtain the conclusion. □

3.2. The Constrained Convex Minimization Problem

Let C be closed convex subset of $\mathcal{H}$. The constrained convex minimization problem is to find $u^* \in C$ such that

$$\Im(u^*) = \min_{u \in C} \Im(u), \tag{36}$$

where $\Im : \mathcal{H} \to \mathbb{R}$ is a continuous differentiable function. The set of all solution of (36) is denoted by $\Gamma_{\Im}$.

It is known that the gradient-projection algorithm is one of the powerful methods for solving the minimization problem (36), see [29–31].

Before we prove the theorem, we need the following lemma.

Lemma 8 ([32])**.** *A necessary condition of optimality for a point $u^* \in C$ to be a solution of the minimization problem (36) is that u^* solves the variational inequality*

$$\langle \nabla \Im(u^*), x - u^* \rangle \geq 0, \forall x \in C. \tag{37}$$

Equivalently, $u^* \in C$ solves the fixed-point equation

$$u^* = P_C(u^* - \lambda \nabla \Im(u^*)),$$

for every constant $\lambda > 0$. If, in addition, $\Im$ is convex, then the optimality condition (37) is also sufficient.

By using these results, we obtain the following theorem.

Theorem 5. *Let C be nonempty closed convex subset of a real Hilbert $\mathcal{H}$. For $i = 1$ and $i = 2$ let $\Im_i : \mathcal{H} \to \mathbb{R}$ be continuous differentiable function with $\nabla \Im_i$ is $\frac{1}{L_i}$-inverse strongly monotone with $L = \max\{L_1, L_2\}$. Let $f, g : \mathcal{H} \to \mathcal{H}$ be a_f and a_g-contraction mappings with $a = \max\{a_f, a_g\}$. For $i = 1, 2$ and $j = 1, 2, 3$ let $D^i_j : C \to \mathcal{H}$ be d^i_j-inverse strongly monotone, where $\lambda^i_j \in (0, 2\omega_i)$ with $\omega_i = \min_{j=1,2,3}\{d^i_j\}$. For $i = 1, 2$, define $G_i : C \to C$ by $G_i(x) = P_C(I - \lambda^i_1 D^i_1)(ax + (1-a)P_C(I - \lambda^i_2 D^i_2)(ax + (1-a)P_C(I - \lambda^i_3 D^i_3)x))$, $\forall x \in C$. Let the sequences $\{x_n\}$ and $\{w_n\}$ be recursively defined by $x_1, w_1 \in C$ and by*

$$\begin{cases} x_{n+1} = \delta_n x_n + \eta_n P_C(I - \gamma_1 \nabla \Im_1) x_n + \mu_n P_C(\alpha_n f(w_n) + (1-\alpha_n) G_1 x_n) \\ w_{n+1} = \delta_n w_n + \eta_n P_C(I - \gamma_2 \nabla \Im_2) w_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n) G_2 w_n) \end{cases} \tag{38}$$

where $\{\delta_n\}, \{\eta_n\}, \{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$ with $\delta_n + \eta_n + \mu_n = 1$, $\gamma \in (0, 2\alpha)$ with $\alpha = \min\{\frac{1}{L_1}, \frac{1}{L_2}\}$ and $\gamma = \min\{\gamma_1, \gamma_2\}$. Assume that the following conditions are satisfied:

(i) $\mathcal{F}_i = F(G_i) \cap \Gamma_{\Im_i} \neq \emptyset$ for $i = 1, 2$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \alpha_n = 0$,

(iii) $0 < \bar{\theta} \leq \delta_n, \eta_n, \mu_n \leq \theta$ for all $n \in N$ and for some $\bar{\theta}, \theta > 0$,

(iv) $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $x^*_1 = P_{\mathcal{F}_1} f(x^*_2)$, where $y^*_1 = P_C(I - \lambda^1_2 D^1_2)(ax^*_1 + (1-a)z^*_1)$ and $z^*_1 = P_C(I - \lambda^1_3 D^1_3)x^*_1$ and $\{w_n\}$ converges strongly to $x^*_2 = P_{\mathcal{F}_2} g(x^*_1)$, where $y^*_2 = P_C(I - \lambda^2_2 D^2_2)(ax^*_2 + (1-a)z^*_2)$ and $z^*_1 = P_C(I - \lambda^2_3 D^2_3)x^*_1$.

Proof. By using Theorems 3 and 8, we obtain the conclusion. □

4. A Numerical Example

In this section, we give an example to support our main theorem.

Example 1. *Let $\mathbb{R}$ be the set of real numbers, $C = [-50, 50] \times [-50, 50]$, $\mathcal{H} = \mathbb{R}^2$. Let $T_1, T_2 : C \to C$ be defined by $T_1 x = \{\max\{0, 12 - x_1\}, \max\{0, 12 - x_2\}\}$, and $T_2 x = \{\max\{\frac{18-x_1}{2}, 0\}, \max\{\frac{18-x_1}{2}, 0\}\}$ for every $x = (x_1, x_2) \in C$. For every $i = 1, 2$ let $B_i : C \to \mathcal{H}$ be defined by $B_i(x) = x - T_i x$, for every $x = (x_1, x_2) \in C$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = (\frac{x_1}{2}, \frac{x_2}{2})$ and $g(x) = (\frac{x_1}{3}, \frac{x_2}{3})$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. For every $i = 1, 2, j = 1, 2, 3$ let $D^i_j : C \to \mathcal{H}$ be defined by $D^1_1(x) = (\frac{x_1-6}{3}, 0), D^1_2(x) = (\frac{x_1-6}{5}, 0), D^1_3(x) = (\frac{x_1-6}{7}, 0), D^2_1(x) = (0, \frac{x_2-6}{2}), D^1_1(x) = (0, \frac{x_2-6}{3}), D^1_1(x) = (0, \frac{x_2-6}{4})$, define $G_1, G_2 : C \to C$ by $G_1(x) = P_C(I - \frac{1}{2}D^1_1)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{1}{3}D^1_2)(\frac{1}{2}x + \frac{1}{2}P_C(I - \frac{1}{4}D^1_3)x))$ and $G_2(x) = P_C(I - 0.75D^2_1)(\frac{1}{2}x + \frac{1}{2}P_C(I - 0.25D^2_2)(\frac{1}{2}x + \frac{1}{2}P_C(I - 0.3D^2_3)x))$.*

Let the sequences $x_n = (x^1_n, x^2_n)$ and $w_n = (w^1_n, w^2_n)$ be generated by $x_1, w_1 \in C$ and by

$$\begin{cases} x_{n+1} = \frac{n}{5n+2}x_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - 0.5B_1)x_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\frac{1}{8n}f(w_n) + (1 - \frac{1}{8n})G_1 x_n), \\ w_{n+1} = \frac{n}{5n+2}w_n + \frac{2n+\frac{1}{2}}{5n+2}P_C(I - 0.7B_2)w_n + \frac{2n+\frac{3}{2}}{5n+2}P_C(\frac{1}{8n}g(x_n) + (1 - \frac{1}{8n})G_2 w_n), \end{cases} \tag{39}$$

for all $n \in \mathbb{N}$. Then the sequence $x_n = (x^1_n, x^2_n)$ converges strongly to $(6, 6)$ and $w_n = (w^1_n, w^2_n)$ converges strongly to $(6, 6)$.

Solution. By the definition of $T_i, B_i, f, g, D^i_j, G_i$ for every $i = 1, 2, j = 1, 2, 3$ we have $(6, 6) \in F(G_i) \cap VI(C, B_i)$. From Theorem 3, we can conclude that the sequences $\{x_n\}$ and $\{w_n\}$ converge strongly to $(6, 6)$.

The following Table 1 and Figure 1 show the numerical results of the sequences $\{x_n\}$ and $\{w_n\}$ where $x_1 = (20, 20)$, $w_1 = (20, 20)$ and $n = N = 30$.

Table 1. The values of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (20, 20)$, $w_1 = (20, 20)$ and $n = N = 30$.

n	$x_n = (x^1_n, x^2_n)$	$w_n = (w^1_n, w^2_n)$
1	(20.000000, 20.000000)	(20.000000, 20.000000)
2	(14.570064, 15.803571)	(14.166667, 11.644491)
3	(10.882109, 12.554478)	(10.857685, 9.291785)
⋮	⋮	⋮
15	(5.976595, 5.983309)	(5.970351, 5.980201)
⋮	⋮	⋮
28	(5.987723, 5.984952)	(5.984472, 5.986233)
29	(5.988190, 5.985531)	(5.985060, 5.986751)
30	(5.988622, 5.986069)	(5.985605, 5.987233)

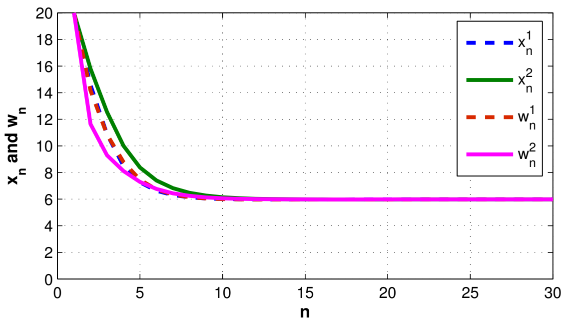

Figure 1. The convergence of $\{x_n\}$ and $\{w_n\}$ with initial values $x_1 = (20, 20)$, $w_1 = (20, 20)$ and $n = N = 30$.

5. Conclusions

From the above numerical results, we can conclude that Table 1 and Figure 1 show that the sequences $\{x_n\}$ and $\{w_n\}$ converge to $(6,6)$ and the convergence of $\{x_n\}$ and $\{w_n\}$ can be guaranteed by Theorem 3.

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