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Article

On Mann Viscosity Subgradient Extragradient Algorithms for Fixed Point Problems of Finitely Many Strict Pseudocontractions and Variational Inequalities

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Abstract: In a real Hilbert space, we denote CFPP and VIP as common fixed point problem of finitely many strict pseudocontractions and a variational inequality problem for Lipschitzian, pseudomonotone operator, respectively. This paper is devoted to explore how to find a common solution of the CFPP and VIP. To this end, we propose Mann viscosity algorithms with line-search process by virtue of subgradient extragradient techniques. The designed algorithms fully assimilate Mann approximation approach, viscosity iteration algorithm and inertial subgradient extragradient technique with line-search process. Under suitable assumptions, it is proven that the sequences generated by the designed algorithms converge strongly to a common solution of the CFPP and VIP, which is the unique solution to a hierarchical variational inequality (HVI).

Keywords: method with line-search process; pseudomonotone variational inequality; strictly pseudocontractive mappings; common fixed point; sequentially weak continuity

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1. Introduction and Preliminaries

Throughout this article, we suppose that the real vector space H is a Hilbert one and the nonempty subset C of H is a convex and closed one. An operator $S : C \to H$ is called:

(i) L-Lipschitzian if there exists $L > 0$ such that $\|Su - Sv\| \le L\|u - v\|\ \forall u, v \in C$;

(ii) sequentially weakly continuous if for any $\{w_n\} \subset C$, the following implication holds: $w_n \rightharpoonup w \Rightarrow Sw_n \rightharpoonup Sw$;

(iii) pseudomonotone if $\langle Su, u - v\rangle \le 0 \Rightarrow \langle Sv, u - v\rangle \le 0\ \forall u, v \in C$;

(iv) monotone if $\langle Su - Sv, v - u\rangle \le 0\ \forall u, v \in C$;

(v) γ-strongly monotone if $\exists \gamma > 0$ s.t. $\langle Su - Sw, u - w\rangle \ge \gamma\|u - w\|^2\ \forall u, w \in C$.

It is not difficult to observe that monotonicity ensures the pseudomonotonicity. A self-mapping $S : C \to C$ is called a η-strict pseudocontraction if the relation holds: $\langle Su - Sv, u - v\rangle \le \|u - v\|^2 - \frac{1-\eta}{2}\|(I - S)u - (I - S)v\|^2\ \forall u, v \in C$ for some $\eta \in [0, 1)$. By [1] we know that, in the case where S is η-strictly pseudocontractive, S is Lipschitzian, i.e., $\|Su - Sv\| \le \frac{1+\eta}{1-\eta}\|u - v\|\ \forall u, v \in C$. It is clear that the class of strict pseudocontractions includes the class of nonexpansive operators, i.e., $\|Su - Sv\| \le \|u - v\|\ \forall u, v \in C$. Both classes of nonlinear operators received much attention and many numerical algorithms were designed for calculating their fixed points in Hilbert or Banach spaces; see e.g., [2–11].

Let A be a self-mapping on H. The classical variational inequality problem (VIP) is to find $z \in C$ such that $\langle Az, y-z\rangle \geq 0 \ \forall y \in C$. The solution set of such a VIP is indicated by VI(C, A). To the best of our knowledge, one of the most effective methods for solving the VIP is the gradient-projection method. Recently, many authors numerically investigated the VIP in finite dimensional spaces, Hilbert spaces or Banach spaces; see e.g., [12–20].

In 2014, Kraikaew and Saejung [21] suggested a Halpern-type gradient-like algorithm to deal with the VIP

$$\begin{cases} v_k = P_C(u_k - \ell A u_k), \\ C_k = \{v \in H : \langle u_k - \ell A u_k - v_k, v_k - v\rangle \geq 0\}, \\ w_k = P_{C_k}(u_n - \ell A v_k), \\ u_{k+1} = \varrho_k u_0 + (1-\varrho_k) w_k \quad \forall k \geq 0, \end{cases}$$

where $\ell \in (0, \frac{1}{L})$, $\{\varrho_k\} \subset (0,1)$, $\lim_{k\to\infty} \varrho_k = 0$, $\sum_{k=1}^{\infty} \varrho_k = +\infty$, and established strong convergence theorems for approximation solutions in Hilbert spaces. Later, Thong and Hieu [22] designed an inertial algorithm, i.e., for arbitrarily given $u_0, u_1 \in H$, the sequence $\{u_k\}$ is constructed by

$$\begin{cases} z_k = u_k + \varrho_k(u_k - u_{k-1}), \\ v_k = P_C(z_k - \ell A z_k), \\ C_k = \{v \in H : \langle z_k - \ell A z_k - v_k, v_k - v\rangle \geq 0\}, \\ u_{k+1} = P_{C_k}(z_n - \ell A v_k) \quad \forall k \geq 1, \end{cases}$$

with $\ell \in (0, \frac{1}{L})$. Under mild assumptions, they proved that $\{u_k\}$ converge weakly to a point of VI(C, A). Very recently, Thong and Hieu [23] suggested two inertial algorithms with linear-search process, to solve the VIP for Lipschitzian, monotone operator A and the FPP for a quasi-nonexpansive operator S satisfying a demiclosedness property in H. Under appropriate assumptions, they proved that the sequences constructed by the suggested algorithms converge weakly to a point of Fix$(S) \cap$ VI(C, A). Further research on common solutions problems, we refer the readers to [24–38].

In this paper, we first introduce Mann viscosity algorithms via subgradient extragradient techniques, and then establish some strong convergence theorems in Hilbert spaces. It is remarkable that our algorithms involve line-search process.

The following lemmas are useful for the convergence analysis of our algorithms in the sequel.

Lemma 1. *[39] Let the operator A be pseudomonotone and continuous on C. Given a point $w \in C$. Then the relation holds: $\langle Aw, w-y\rangle \leq 0 \ \forall y \in C \Leftrightarrow \langle Ay, w-y\rangle \leq 0 \ \forall y \in C$.*

Lemma 2. *[40] Suppose that $\{s_k\}$ is a sequence in $[0, +\infty)$ such that $s_{k+1} \leq t_k b_k + (1-t_k)s_k \ \forall k \geq 1$, where $\{t_k\}$ and $\{b_k\}$ lie in real line $\mathbf{R} := (-\infty, \infty)$, such that:*

(a) $\{t_k\} \subset [0,1]$ and $\sum_{k=1}^{\infty} t_k = \infty$;

(b) $\limsup_{k\to\infty} b_k \leq 0$ or $\sum_{k=1}^{\infty} |t_k b_k| < \infty$. Then $s_k \to 0$ as $k \to \infty$.

From Ceng et al. [2] it is not difficult to find that the following lemmas hold.

Lemma 3. *Let Γ be an η-strictly pseudocontractive self-mapping on C. Then $I - \Gamma$ is demiclosed at zero.*

Lemma 4. *For $l = 1, ..., N$, let Γ_l be an η_l-strictly pseudocontractive self-mapping on C. Then for $l = 1, ..., N$, the mapping Γ_l is an η-strict pseudocontraction with $\eta = \max\{\eta_l : 1 \leq l \leq N\}$, such that*

$$\|\Gamma_l u - \Gamma_l v\| \leq \frac{1+\eta}{1-\eta}\|u - v\| \quad \forall u, v \in C.$$

Lemma 5. *Let Γ be an η-strictly pseudocontractive self-mapping on C. Given two reals $\gamma, \beta \in [0, +\infty)$. If $(\gamma+\beta)\eta \leq \gamma$, then $\|\gamma(u-v) + \beta(\Gamma u - \Gamma v)\| \leq (\gamma+\beta)\|u-v\| \ \forall u, v \in C$.*

2. Main Results

Our first algorithm is specified below.

Algorithm 1

Initial Step: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$.
Iteration Steps: Compute x_{n+1} below:
Step 1. Put $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $u_n = P_C(v_n - \ell_n A v_n)$, where ℓ_n is picked to be the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ s.t.

$$\ell \|Av_n - Au_n\| \le \mu \|v_n - u_n\|. \tag{1}$$

Step 2. Calculate $z_n = (1-\alpha_n)P_{C_n}(v_n - \ell_n A u_n) + \alpha_n f(x_n)$ with $C_n := \{v \in H : \langle v_n - \ell_n A v_n - u_n, u_n - v\rangle \ge 0\}$.
Step 3. Calculate

$$x_{n+1} = \gamma_n P_{C_n}(v_n - \ell_n A u_n) + \delta_n T_n z_n + \beta_n x_n. \tag{2}$$

Update $n := n+1$ and return to Step 1.
In this section, we always suppose that the following hypotheses hold:
T_k is a ζ_k-strictly pseudocontractive self-mapping on H for $k = 1, ..., N$ s.t. $\zeta \in [0,1)$ with $\zeta = \max\{\zeta_k : 1 \le k \le N\}$.
A is L-Lipschitzian, pseudomonotone self-mapping on H, and sequentially weakly continuous on C, such that $\Omega := \cap_{k=1}^N \mathrm{Fix}(T_k) \cap \mathrm{VI}(C, A) \ne \emptyset$.
$f : H \to C$ is a δ-contraction with $\delta \in [0, \frac{1}{2})$.
$\{\sigma_n\} \subset [0,1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ are such that:
(i) $\beta_n + \gamma_n + \delta_n = 1$ and $\sup_{n\ge 1} \frac{\sigma_n}{\alpha_n} < \infty$;
(ii) $(1-2\delta)\delta_n > \gamma_n \ge (\gamma_n + \delta_n)\zeta\ \forall n \ge 1$ and $\liminf_{n\to\infty}((1-2\delta)\delta_n - \gamma_n) > 0$;
(iii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
(iv) $\liminf_{n\to\infty} \beta_n > 0$, $\liminf_{n\to\infty} \delta_n > 0$ and $\limsup_{n\to\infty} \beta_n < 1$.
Following Xu and Kim [40], we denote $T_n := T_{n \bmod N}$, $\forall n \ge 1$, where the mod function takes values in $\{1, 2, ..., N\}$, i.e., whenever $n = jN + q$ for some $j \ge 0$ and $0 \le q < N$, we obtain that $T_n = T_N$ in the case of $q = 0$ and $T_n = T_q$ in the case of $0 < q < N$.

Lemma 6. *The Armijo-like search rule (1) is well defined, and* $\min\{\gamma, \frac{\mu l}{L}\} \le \ell_n \le \gamma$.

Proof. Obviously, (1) holds for all $\gamma l^m \le \frac{\mu}{L}$. So, ℓ_n is well defined and $\ell_n \le \gamma$. In the case of $\ell_n = \gamma$, the inequality is true. In the case of $\ell_n < \gamma$, (1) ensures $\|Av_n - AP_C(v_n - \frac{\ell_n}{l}Av_n)\| > \frac{\mu}{\frac{\ell_n}{l}}\|v_n - P_C(v_n - \frac{\ell_n}{l}Av_n)\|$. The L-Lipschitzian property of A yields $\ell_n > \frac{\mu l}{L}$. □

Lemma 7. *Let* $\{v_n\}, \{u_n\}$ *and* $\{z_n\}$ *be the sequences constructed by Algorithm 1. Then*

$$\begin{aligned}\|z_n - \omega\|^2 &\le (1-\alpha_n)\|v_n - \omega\|^2 + \alpha_n\delta\|x_n - \omega\|^2 - (1-\alpha_n)(1-\mu)[\|v_n - u_n\|^2 \\ &\quad + \|h_n - u_n\|^2] + 2\alpha_n\langle f\omega - \omega, z_n - \omega\rangle \quad \forall \omega \in \Omega,\end{aligned} \tag{3}$$

where $h_n := P_{C_n}(v_n - \ell_n A u_n)\ \forall n \ge 1$.

Proof. First, taking an arbitrary $p \in \Omega \subset C \subset C_n$, we observe that

$$\begin{aligned}2\|h_n - p\|^2 &\le 2\langle h_n - p, v_n - \ell_n A u_n - p\rangle \\ &= \|h_n - p\|^2 + \|v_n - p\|^2 - \|h_n - v_n\|^2 - 2\langle \ell_n A u_n, h_n - p\rangle.\end{aligned}$$

So, it follows that $\|v_n - p\|^2 - 2\langle h_n - p, \ell_n A u_n\rangle - \|h_n - v_n\|^2 \ge \|h_n - p\|^2$, which together with (1), we deduce that $0 \ge \langle p - u_n, Au_n\rangle$ and

$$\begin{aligned}\|h_n - p\|^2 &\le \|v_n - p\|^2 - \|h_n - v_n\|^2 + 2\ell_n(\langle Au_n, p - u_n\rangle + \langle Au_n, u_n - h_n\rangle) \\ &\le \|v_n - p\|^2 - \|u_n - h_n\|^2 - \|v_n - u_n\|^2 + 2\langle u_n - v_n + \ell_n A u_n, u_n - h_n\rangle.\end{aligned} \tag{4}$$

Since $h_n = P_{C_n}(v_n - \ell_n A u_n)$ with $C_n := \{v \in H : \langle u_n - v_n + \ell_n A v_n, u_n - v\rangle \leq 0\}$, we have $\langle u_n - v_n + \ell_n A v_n, u_n - h_n\rangle \leq 0$, which together with (1), implies that

$$\begin{aligned}2\langle u_n - v_n + \ell_n A u_n, u_n - h_n\rangle &= 2\langle u_n - v_n + \ell_n A v_n, u_n - h_n\rangle + 2\ell_n\langle A v_n - A u_n, h_n - u_n\rangle \\ &\leq 2\mu\|u_n - v_n\|\|u_n - h_n\| \leq \mu(\|v_n - u_n\|^2 + \|h_n - u_n\|^2).\end{aligned}$$

Therefore, substituting the last inequality for (4), we infer that

$$\|h_n - p\|^2 \leq \|v_n - p\|^2 - (1-\mu)\|v_n - u_n\|^2 - (1-\mu)\|h_n - u_n\|^2 \quad \forall p \in \Omega. \tag{5}$$

In addition, we have

$$z_n - p = (1-\alpha_n)(h_n - p) + \alpha_n(f - I)p + \alpha_n(f(x_n) - f(p)).$$

Using the convexity of the function $h(t) = t^2 \ \forall t \in \mathbf{R}$, from (5) we get

$$\begin{aligned}\|z_n - p\|^2 &\leq [\alpha_n\delta\|x_n - p\| + (1-\alpha_n)\|h_n - p\|]^2 + 2\alpha_n\langle (f-I)p, z_n - p\rangle \\ &\leq \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n)\|h_n - p\|^2 + 2\alpha_n\langle (f-I)p, z_n - p\rangle \\ &\leq \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 - (1-\alpha_n)(1-\mu)[\|v_n - u_n\|^2 \\ &\quad + \|h_n - u_n\|^2] + 2\alpha_n\langle (f-I)p, z_n - p\rangle.\end{aligned}$$

□

Lemma 8. *Let $\{x_n\}, \{u_n\}$, and $\{z_n\}$ be bounded sequences constructed by Algorithm 1. If $x_n - x_{n+1} \to 0$, $v_n - u_n \to 0$, $v_n - z_n \to 0$ and $\exists \{v_{n_i}\} \subset \{v_n\}$ s.t. $v_{n_i} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. According to Algorithm 1, we get $\sigma_n(x_n - x_{n-1}) = v_n - x_n \ \forall n \geq 1$, and hence $\|x_n - x_{n-1}\| \geq \|v_n - x_n\|$. Using the assumption $x_n - x_{n+1} \to 0$, we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{6}$$

So,

$$\|z_n - x_n\| \leq \|v_n - z_n\| + \|v_n - x_n\| \to 0.$$

Since $\{x_n\}$ is bounded, from $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ we know that $\{v_n\}$ is a bounded vector sequence. According to (5), we obtain that $h_n := P_{C_n}(v_n - \ell_n A u_n)$ is a bounded vector sequence. Also, by Algorithm 1 we get $\alpha_n f(x_n) + h_n - x_n - \alpha_n h_n = z_n - x_n$. So, the boundedness of $\{x_n\}, \{h_n\}$ guarantees that as $n \to \infty$,

$$\|h_n - x_n\| = \|z_n - x_n - \alpha_n f(x_n) + \alpha_n h_n\| \leq \|z_n - x_n\| + \alpha_n(\|f(x_n)\| + \|h_n\|) \to 0.$$

It follows that

$$x_{n+1} - z_n = \gamma_n(h_n - x_n) + \delta_n(T_n z_n - z_n) + (1-\delta_n)(x_n - z_n),$$

which immediately yields

$$\begin{aligned}\delta_n\|T_n z_n - z_n\| &= \|x_{n+1} - x_n + x_n - z_n - (1-\delta_n)(x_n - z_n) - \gamma_n(h_n - x_n)\| \\ &= \|x_{n+1} - x_n + \delta_n(x_n - z_n) - \gamma_n(h_n - x_n)\| \\ &\leq \|x_{n+1} - x_n\| + \|x_n - z_n\| + \|h_n - x_n\|.\end{aligned}$$

Since $x_n - x_{n+1} \to 0$, $z_n - x_n \to 0$, $h_n - x_n \to 0$ and $\liminf_{n\to\infty} \delta_n > 0$, we obtain $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$. This further implies that

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - z_n\| + \|z_n - T_n z_n\| + \tfrac{1+\zeta}{1-\zeta}\|z_n - x_n\| \\ &\leq \tfrac{2}{1-\zeta}\|x_n - z_n\| + \|z_n - T_n z_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{7}$$

We have $\langle v_n - \ell_n A v_n - u_n, v - u_n\rangle \leq 0\ \forall v \in C$, and

$$\langle v_n - u_n, v - u_n\rangle + \ell_n\langle A v_n, u_n - v_n\rangle \leq \ell_n\langle A v_n, v - v_n\rangle \quad \forall v \in C. \tag{8}$$

Note that $\ell_n \geq \min\{\gamma, \frac{\mu l}{L}\}$. So, $\liminf_{i\to\infty}\langle A v_{n_i}, v - v_{n_i}\rangle \geq 0\ \forall v \in C$. This yields $\liminf_{i\to\infty}\langle A u_{n_i}, v - u_{n_i}\rangle \geq 0\ \forall v \in C$. Since $v_n - x_n \to 0$ and $v_{n_i} \rightharpoonup z$, we get $x_{n_i} \rightharpoonup z$. We may assume $k = n_i \mathrm{mod} N$ for all i. By the assumption $x_n - x_{n+k} \to 0$, we have $x_{n_i+j} \rightharpoonup z$ for all $j \geq 1$. Hence, $\|x_{n_i+j} - T_{k+j}x_{n_i+j}\| = \|x_{n_i+j} - T_{n_i+j}x_{n_i+j}\| \to 0$. Then the demiclosedness principle implies that $z \in \mathrm{Fix}(T_{k+j})$ for all j. This ensures that

$$z \in \bigcap_{k=1}^{N} \mathrm{Fix}(T_k). \tag{9}$$

We now take a sequence $\{\varsigma_i\} \subset (0,1)$ satisfying $\varsigma_i \downarrow 0$ as $i \to \infty$. For all $i \geq 1$, we denote by m_i the smallest natural number satisfying

$$\langle A u_{n_j}, v - u_{n_j}\rangle + \varsigma_i \geq 0 \quad \forall j \geq m_i. \tag{10}$$

Since $\{\varsigma_i\}$ is decreasing, it is clear that $\{m_i\}$ is increasing. Noticing that $\{u_{m_i}\} \subset C$ ensures $A u_{m_i} \neq 0\ \forall i \geq 1$, we set $e_{m_i} = \frac{A u_{m_i}}{\|A u_{m_i}\|^2}$, we get $\langle A u_{m_i}, e_{m_i}\rangle = 1\ \forall i \geq 1$. So, from (10) we get $\langle A u_{m_i}, v + \varsigma_i e_{m_i} - u_{m_i}\rangle \geq 0\ \forall i \geq 1$. Also, the pseudomonotonicity of A implies $\langle A(v + \varsigma_i e_{m_i}), v + \varsigma_i e_{m_i} - u_{m_i}\rangle \geq 0\ \forall i \geq 1$. This immediately leads to

$$\langle A v - A(v + \varsigma_i h_{m_i}), v + \varsigma_i e_{m_i} - u_{m_i}\rangle - \varsigma_i\langle A v, h_{m_i}\rangle \leq \langle A v, v - u_{m_i}\rangle \quad \forall i \geq 1. \tag{11}$$

We claim $\lim_{i\to\infty} \varsigma_i e_{m_i} = 0$. Indeed, from $v_{n_i} \rightharpoonup z$ and $v_n - u_n \to 0$, we obtain $u_{n_i} \rightharpoonup z$. So, $\{u_n\} \subset C$ ensures $z \in C$. Also, the sequentially weak continuity of A guarantees that $A u_{n_i} \rightharpoonup Az$. Thus, we have $Az \neq 0$ (otherwise, z is a solution). Moreover, the sequentially weak lower semicontinuity of $\|\cdot\|$ ensures $0 < \|Az\| \leq \liminf_{i\to\infty}\|A u_{n_i}\|$. Since $\{u_{m_i}\} \subset \{u_{n_i}\}$ and $\varsigma_i \downarrow 0$ as $i \to \infty$, we deduce that $0 \leq \limsup_{i\to\infty}\|\varsigma_i e_{m_i}\| = \limsup_{i\to\infty}\frac{\varsigma_i}{\|A u_{m_i}\|} \leq \frac{\limsup_{i\to\infty}\varsigma_i}{\liminf_{i\to\infty}\|A u_{n_i}\|} = 0$. Hence we get $\varsigma_i e_{m_i} \to 0$.

Finally we claim $z \in \Omega$. In fact, letting $i \to \infty$, we conclude that the right hand side of (11) tends to zero by the Lipschitzian property of A, the boundedness of $\{u_{m_i}\}$, $\{h_{m_i}\}$ and the limit $\lim_{i\to\infty}\varsigma_i e_{m_i} = 0$. Thus, we get $\langle A v, v - z\rangle = \liminf_{i\to\infty}\langle A v, v - u_{m_i}\rangle \geq 0\ \forall v \in C$. So, $z \in \mathrm{VI}(C, A)$. Therefore, from (9) we have $z \in \cap_{k=1}^{N}\mathrm{Fix}(T_k) \cap \mathrm{VI}(C, A) = \Omega$. □

Theorem 1. *Assume $A(C)$ is bounded. Let $\{x_n\}$ be constructed by Algorithm 1. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ \sup_{n\geq 1}\|x_n - f x_n\| < \infty \end{cases}$$

where $x^* \in \Omega$ is the unique solution to the hierarchical variational inequality (HVI): $\langle (I - f)x^*, x^* - \omega\rangle \leq 0,\ \forall \omega \in \Omega$.

Proof. Taking into account condition (iv) on $\{\gamma_n\}$, we may suppose that $\{\beta_n\} \subset [a,b] \subset (0,1)$. Applying Banach's Contraction Principle, we obtain existence and uniqueness of a fixed point $x^* \in H$ for the mapping $P_\Omega \circ f$, which means that $x^* = P_\Omega f(x^*)$. Hence, the HVI

$$\langle (I-f)x^*, x^* - \omega\rangle \leq 0, \quad \forall \omega \in \Omega \tag{12}$$

has a unique solution $x^* \in \Omega := \cap_{k=1}^N \mathrm{Fix}(T_k) \cap \mathrm{VI}(C,A)$

It is now obvious that the necessity of the theorem is true. In fact, if $x_n \to x^* \in \Omega$, then we get $\sup_{n\geq 1} \|x_n - f(x_n)\| \leq \sup_{n\geq 1}(\|x_n - x^*\| + \|x^* - f(x^*)\| + \|f(x^*) - f(x_n)\|) < \infty$ and

$$\|x_n - x_{n+1}\| \leq \|x_n - x^*\| + \|x_{n+1} - x^*\| \to 0 \quad (n \to \infty).$$

For the sufficient condition, let us suppose $x_n - x_{n+1} \to 0$ and $\sup_{n\geq 1} \|(I-f)x_n\| < \infty$. The sufficiency of our conclusion is proved in the following steps. □

Step 1. We show the boundedness of $\{x_n\}$. In fact, let p be an arbitrary point in Ω. Then $T_n p = p\ \forall n \geq 1$, and

$$\|v_n - p\|^2 - (1-\mu)\|h_n - u_n\|^2 - (1-\mu)\|v_n - u_n\|^2 \geq \|h_n - p\|^2, \tag{13}$$

which hence leads to

$$\|v_n - p\| \geq \|h_n - p\| \quad \forall n \geq 1. \tag{14}$$

By the definition of v_n, we have

$$\|v_n - p\| \leq \|x_n - p\| + \sigma_n\|x_n - x_{n-1}\| \leq \|x_n - p\| + \alpha_n \cdot \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|. \tag{15}$$

Noticing $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\sup_{n\geq 1} \|x_n - x_{n-1}\| < \infty$, we obtain that $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| < \infty$. This ensures that $\exists M_1 > 0$ s.t.

$$\frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1 \quad \forall n \geq 1. \tag{16}$$

Combining (14)–(16), we get

$$\|h_n - p\| \leq \|v_n - p\| \leq \|x_n - p\| + \alpha_n M_1 \quad \forall n \geq 1. \tag{17}$$

Note that $A(C)$ is bounded, $u_n = P_C(v_n - \ell_n A v_n)$, $f(H) \subset C \subset C_n$ and $h_n = P_{C_n}(v_n - \ell_n A u_n)$. Hence we know that $\{Au_n\}$ is bounded. So, from $\sup_{n\geq 1} \|(I-f)x_n\| < \infty$, it follows that

$$\begin{aligned} \|h_n - f(x_n)\| &\leq \|v_n - \ell_n A u_n - f(x_n)\| \\ &\leq \|x_n - x_{n-1}\| + \|x_n - f(x_n)\| + \gamma\|Au_n\| \leq M_0, \end{aligned}$$

where $\exists M_0 > 0$ s.t. $M_0 \geq \sup_{n\geq 1}(\|x_n - x_{n-1}\| + \|x_n - f(x_n)\| + \gamma\|Au_n\|)$ (due to the assumption $x_n - x_{n+1} \to 0$). Consequently,

$$\begin{aligned} \|z_n - p\| &\leq \alpha_n\delta\|x_n - p\| + (1-\alpha_n)\|h_n - p\| + \alpha_n\|(f-I)p\| \\ &\leq (1 - \alpha_n(1-\delta))\|x_n - p\| + \alpha_n(M_1 + \|(f-I)p\|), \end{aligned}$$

which together with $(\gamma_n + \delta_n)\zeta \leq \gamma_n$, yields

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(T_n z_n - p)]\| + \gamma_n\|h_n - z_n\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)[(1 - \alpha_n(1-\delta))\|x_n - p\| + \alpha_n(M_0 + M_1 + \|(f-I)p\|)] \\ &= [1 - \alpha_n(1-\beta_n)(1-\delta)]\|x_n - p\| + \alpha_n(1-\beta_n)(1-\delta)\tfrac{M_0 + M_1 + \|(f-I)p\|}{1-\delta}. \end{aligned}$$

This shows that $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{M_0+M_1+\|(I-f)p\|}{1-\delta}\}\ \forall n \ge 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{h_n\}, \{v_n\}, \{u_n\}, \{z_n\}, \{T_n z_n\}$.

Step 2. We show that $\exists M_4 > 0$ s.t.

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.$$

In fact, using Lemma 7 and the convexity of $\|\cdot\|^2$, we get

$$\begin{array}{ll} \|x_{n+1} - p\|^2 & \le \|\beta_n(x_n - p) + \gamma_n(z_n - p) + \delta_n(T_n z_n - p)\|^2 + 2\gamma_n\alpha_n\langle h_n - f(x_n), x_{n+1} - p\rangle \\ & \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + 2(1-\beta_n)\alpha_n\|h_n - f(x_n)\|\|x_{n+1} - p\| \\ & \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ & \quad - (1-\alpha_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] + \alpha_n M_2\}, \end{array} \tag{18}$$

where $\exists M_2 > 0$ s.t. $M_2 \ge \sup_{n\ge1} 2(\|(f-I)p\|\|z_n - p\| + \|u_n - f(x_n)\|\|x_{n+1} - p\|)$. Also,

$$\begin{array}{ll} \|v_n - p\|^2 & \le \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\ & \le \|x_n - p\|^2 + \alpha_n M_3, \end{array} \tag{19}$$

where $\exists M_3 > 0$ s.t. $M_3 \ge \sup_{n\ge1}(2M_1\|x_n - p\| + \beta_n M_1^2)$. Substituting (19) for (18), we have

$$\begin{array}{ll} \|x_{n+1} - p\|^2 & \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\{(1-\alpha_n(1-\delta))\|x_n - p\|^2 + (1-\alpha_n)\alpha_n M_3 \\ & \quad - (1-\alpha_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] + \alpha_n M_2\} \\ & \le \|x_n - p\|^2 - (1-\alpha_n)(1-\beta_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] + \alpha_n M_4, \end{array} \tag{20}$$

where $M_4 := M_2 + M_3$. This immediately implies that

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4. \tag{21}$$

Step 3. We show that $\exists M > 0$ s.t.

$$\begin{array}{l} \|x_{n+1} - p\|^2 \\ \le [1 - \frac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n - p\|^2 + \frac{[(1-2\delta)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n} \cdot \{\frac{2\gamma_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ \quad + \frac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_n\| + \frac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\langle f(p) - p, x_n - p\rangle \\ \quad + \frac{\gamma_n + \delta_n}{(1-2\delta)\delta_n - \gamma_n} \cdot \frac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M\}. \end{array}$$

In fact, we get

$$\begin{array}{ll} \|v_n - p\|^2 & \le \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|) \\ & \le \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|3M, \end{array} \tag{22}$$

where $\exists M > 0$ s.t. $M \ge \sup_{n\ge1}\{\|x_n - p\|, \sigma_n\|x_n - x_{n-1}\|\}$. By Algorithm 1 and the convexity of $\|\cdot\|^2$, we have

$$\begin{array}{ll} \|x_{n+1} - p\|^2 & \le \|\beta_n(x_n - p) + \gamma_n(z_n - p) + \delta_n(T_n z_n - p)\|^2 + 2\gamma_n\alpha_n\langle h_n - f(x_n), x_{n+1} - p\rangle \\ & \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(T_n z_n - p)]\|^2 \\ & \quad + 2\gamma_n\alpha_n\langle h_n - p, x_{n+1} - p\rangle + 2\gamma_n\alpha_n\langle p - f(x_n), x_{n+1} - p\rangle, \end{array}$$

which leads to

$$\begin{array}{ll} \|x_{n+1} - p\|^2 & \le \beta_n\|x_n - p\|^2 + (1-\beta_n)[(1-\alpha_n)\|h_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, z_n - p\rangle] \\ & \quad + \gamma_n\alpha_n(\|h_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\gamma_n\alpha_n\langle p - f(x_n), x_{n+1} - p\rangle. \end{array}$$

Using (17) and (22) we obtain that $\|h_n - p\|^2 \le \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|3M$. Hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le [1 - \alpha_n(1-\beta_n)]\|x_n - p\|^2 + (1-\beta_n)(1-\alpha_n)\sigma_n\|x_n - x_{n-1}\|3M \\ &\quad + 2\alpha_n\delta_n\langle f(x_n) - p, z_n - p\rangle + \gamma_n\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\ &\quad + (1-\beta_n)\alpha_n\sigma_n\|x_n - x_{n-1}\|3M + 2\gamma_n\alpha_n\langle f(x_n) - p, z_n - x_{n+1}\rangle \\ &\le [1 - \alpha_n(1-\beta_n)]\|x_n - p\|^2 + 2\gamma_n\alpha_n\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ &\quad + 2\alpha_n\delta_n\langle f(x_n) - p, x_n - p\rangle + 2\alpha_n\delta_n\langle f(x_n) - p, z_n - x_n\rangle \\ &\quad + \gamma_n\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + (1-\beta_n)\sigma_n\|x_n - x_{n-1}\|3M \\ &\le [1 - \alpha_n(1-\beta_n)]\|x_n - p\|^2 + 2\gamma_n\alpha_n\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ &\quad + 2\alpha_n\delta_n\delta\|x_n - p\|^2 + 2\alpha_n\delta_n\langle f(p) - p, x_n - p\rangle + 2\alpha_n\delta_n\|f(x_n) - p\|\|z_n - x_n\| \\ &\quad + \gamma_n\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + (1-\beta_n)\sigma_n\|x_n - x_{n-1}\|3M,\end{aligned}$$

which immediately yields

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\le [1 - \tfrac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n - p\|^2 + \tfrac{[(1-2\delta)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ &\quad + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_n\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\langle f(p) - p, x_n - p\rangle \\ &\quad + \tfrac{\gamma_n + \delta_n}{(1-2\delta)\delta_n - \gamma_n}\cdot\tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M\}.\end{aligned} \tag{23}$$

Step 4. We show that $x_n \to x^* \in \Omega$, where x^* is the unique solution of (12). Indeed, putting $p = x^*$, we infer from (23) that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le [1 - \tfrac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n - x^*\|^2 + \tfrac{[(1-2\delta)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - x^*\|\|z_n - x_{n+1}\| \\ &\quad + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - x^*\|\|z_n - x_n\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\langle f(x^*) - x^*, x_n - x^*\rangle \\ &\quad + \tfrac{\gamma_n + \delta_n}{(1-2\delta)\delta_n - \gamma_n}\cdot\tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M\}.\end{aligned} \tag{24}$$

It is sufficient to show that $\limsup_{n\to\infty}\langle (f - I)x^*, x_n - x^*\rangle \le 0$. From (21), $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$ and $\{\beta_n\} \subset [a, b] \subset (0, 1)$, we get

$$\begin{aligned}&\limsup_{n\to\infty}(1-\alpha_n)(1-b)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] \\ &\le \limsup_{n\to\infty}[(\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n M_4] = 0.\end{aligned}$$

This ensures that

$$\lim_{n\to\infty}\|v_n - u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|h_n - u_n\| = 0. \tag{25}$$

Consequently,

$$\|x_n - u_n\| \le \|x_n - v_n\| + \|v_n - u_n\| \to 0 \quad (n \to \infty).$$

Since $z_n = \alpha_n f(x_n) + (1-\alpha_n)h_n$ with $h_n := P_{C_n}(v_n - \ell_n A u_n)$, we get

$$\begin{aligned}\|z_n - u_n\| &= \|\alpha_n f(x_n) - \alpha_n h_n + h_n - u_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|h_n\|) + \|h_n - u_n\| \to 0 \quad (n \to \infty),\end{aligned} \tag{26}$$

and hence

$$\|z_n - x_n\| \le \|z_n - u_n\| + \|u_n - x_n\| \to 0 \quad (n \to \infty). \tag{27}$$

Obviously, combining (25) and (26), guarantees that

$$\|v_n - z_n\| \le \|v_n - u_n\| + \|u_n - z_n\| \to 0 \quad (n \to \infty).$$

From the boundedness of $\{x_n\}$, it follows that $\exists\{x_{n_i}\} \subset \{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle (f-I)x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle (f-I)x^*, x_{n_i} - x^*\rangle. \tag{28}$$

Since $\{x_n\}$ is bounded, we may suppose that $x_{n_i} \rightharpoonup \tilde{x}$. Hence from (28) we get

$$\limsup_{n\to\infty}\langle (f-I)x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle (f-I)x^*, x_{n_i} - x^*\rangle = \langle (f-I)x^*, \tilde{x} - x^*\rangle. \tag{29}$$

It is easy to see from $v_n - x_n \to 0$ and $x_{n_i} \rightharpoonup \tilde{x}$ that $v_{n_i} \rightharpoonup \tilde{x}$. Since $x_n - x_{n+1} \to 0$, $v_n - u_n \to 0$, $v_n - z_n \to 0$ and $v_{n_i} \rightharpoonup \tilde{x}$, by Lemma 8 we infer that $\tilde{x} \in \Omega$. Therefore, from (12) and (29) we conclude that

$$\limsup_{n\to\infty}\langle (f-I)x^*, x_n - x^*\rangle = \langle (f-I)x^*, \tilde{x} - x^*\rangle \le 0. \tag{30}$$

Note that $\liminf_{n\to\infty} \frac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n} > 0$. It follows that $\sum_{n=0}^{\infty} \frac{(1-2\delta)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}\alpha_n = \infty$. It is clear that

$$\begin{aligned}\limsup_{n\to\infty}\{&\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-x^*\|\|z_n - x_{n+1}\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-x^*\|\|z_n-x_n\| \\ &+ \tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(x^*)-x^*, x_n - x^*\rangle + \tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}\cdot\tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M\} \le 0.\end{aligned} \tag{31}$$

Therefore, by Lemma 2 we immediately deduce that $x_n \to x^*$.

Next, we introduce another Mann viscosity algorithm with line-search process by the subgradient extragradient technique.

Algorithm 2

Initial Step: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$.
Iteration Steps: Compute x_{n+1} below:
Step 1. Put $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $u_n = P_C(v_n - \ell_n A v_n)$, where ℓ_n is picked to be the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ s.t.

$$\ell\|Av_n - Au_n\| \le \mu\|v_n - u_n\|. \tag{32}$$

Step 2. Calculate $z_n = (1-\alpha_n)P_{C_n}(v_n - \ell_n A u_n) + \alpha_n f(x_n)$ with $C_n := \{v \in H : \langle v_n - \ell_n A v_n - u_n, u_n - v\rangle \ge 0\}$.
Step 3. Calculate

$$x_{n+1} = \gamma_n P_{C_n}(v_n - \ell_n A u_n) + \delta_n T_n z_n + \beta_n v_n. \tag{33}$$

Update $n := n+1$ and return to Step 1.

It is remarkable that Lemmas 6, 7 and 8 remain true for Algorithm 2.

Theorem 2. *Assume $A(C)$ is bounded. Let $\{x_n\}$ be constructed by Algorithm 2. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ \sup_{n\ge 1}\|(I-f)x_n\| < \infty \end{cases}$$

where $x^* \in \Omega$ is the unique solution of the HVI: $\langle (I-f)x^*, x^* - \omega\rangle \le 0,\ \forall \omega \in \Omega$.

Proof. For the necessity of our proof, we can observe that, by a similar approach to that in the proof of Theorem 1, we obtain that there is a unique solution $x^* \in \Omega$ of (12).

We show the sufficiency below. To this aim, we suppose $x_n - x_{n+1} \to 0$ and $\sup_{n\ge1}\|(I-f)x_n\| < \infty$, and prove the sufficiency by the following steps. □

Step 1. We show the boundedness of $\{x_n\}$. In fact, by the similar inference to that in Step 1 for the proof of Theorem 1, we obtain that (13)–(17) hold. So, using Algorithm 2 and (17) we obtain

$$\|z_n - p\| \quad \le (1-\alpha_n(1-\delta))\|x_n - p\| + \alpha_n(M_1 + \|(f-I)p\|),$$

which together with $(\gamma_n + \delta_n)\zeta \le \gamma_n$, yields

$$\begin{aligned}\|x_{n+1} - p\| &\le \beta_n\|v_n - p\| + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(T_n z_n - p)]\| + \gamma_n\|h_n - z_n\| \\ &\le \beta_n(\|x_n - p\| + \alpha_n M_1) + (1-\beta_n)[(1-\alpha_n(1-\delta))\|x_n - p\| \\ &\quad + \alpha_n(M_0 + M_1 + \|(f-I)p\|)] \\ &= [1-\alpha_n(1-\beta_n)(1-\delta)]\|x_n - p\| + \alpha_n(1-\beta_n)(1-\delta)\frac{M_0 + \frac{1}{1-\beta_n}M_1 + \|(f-I)p\|}{1-\delta}.\end{aligned}$$

Therefore, we get the boundedness of $\{x_n\}$ and hence the one of sequences $\{h_n\}, \{v_n\}, \{u_n\}, \{z_n\}, \{T_n z_n\}$.

Step 2. We show that $\exists M_4 > 0$ s.t.

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.$$

In fact, by Lemma 7 and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|\beta_n(v_n - p) + \gamma_n(z_n - p) + \delta_n(T_n z_n - p)\|^2 + 2\gamma_n\alpha_n\langle h_n - f(x_n), x_{n+1} - p\rangle \\ &\le \beta_n\|v_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + 2(1-\beta_n)\alpha_n\|h_n - f(x_n)\|\|x_{n+1} - p\| \\ &\le \beta_n\|v_n - p\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &\quad - (1-\alpha_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] + \alpha_n M_2\},\end{aligned} \tag{34}$$

where $\exists M_2 > 0$ s.t. $M_2 \ge \sup_{n\ge1} 2(\|(f-I)p\|\|z_n - p\| + \|u_n - f(x_n)\|\|x_{n+1} - p\|)$. Also,

$$\begin{aligned}\|v_n - p\|^2 &\le \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\ &\le \|x_n - p\|^2 + \alpha_n M_3,\end{aligned} \tag{35}$$

where $\exists M_3 > 0$ s.t. $M_3 \ge \sup_{n\ge1}(2M_1\|x_n - p\| + \beta_n M_1^2)$. Substituting (35) for (34), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\{(1-\alpha_n(1-\delta))\|x_n - p\|^2 + (1-\alpha_n)\alpha_n M_3 \\ &\quad - (1-\alpha_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] + \alpha_n M_2\} + \beta_n\alpha_n M_3 \\ &= \|x_n - p\|^2 - (1-\alpha_n)(1-\beta_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] + \alpha_n M_4,\end{aligned} \tag{36}$$

where $M_4 := M_2 + M_3$. This ensures that

$$(1-\alpha_n)(1-\beta_n)(1-\mu)[\|v_n - u_n\|^2 + \|h_n - u_n\|^2] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4. \tag{37}$$

Step 3. We show that $\exists M > 0$ s.t.

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\le [1 - \tfrac{(1-2\delta)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n - p\|^2 + \tfrac{[(1-2\delta)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_{n+1}\| \\ &\quad + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\|f(x_n) - p\|\|z_n - x_n\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n}\langle f(p) - p, x_n - p\rangle \\ &\quad + \tfrac{1}{(1-2\delta)\delta_n - \gamma_n}\cdot\tfrac{\sigma_n}{\alpha_n}\|x_n - x_{n-1}\|3M\}.\end{aligned}$$

In fact, we get

$$\begin{aligned}\|v_n - p\|^2 &\le \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|) \\ &\le \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|3M,\end{aligned} \tag{38}$$

where $\exists M > 0$ s.t. $M \geq \sup_{n\geq 1}\{\|x_n - p\|, \sigma_n\|x_n - x_{n-1}\|\}$. Using Algorithm 1 and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|\beta_n(v_n-p)+\gamma_n(z_n-p)+\delta_n(T_nz_n-p)\|^2+2\gamma_n\alpha_n\langle h_n-f(x_n),x_{n+1}-p\rangle\\ &\leq \beta_n\|v_n-p\|^2+(1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n-p)+\delta_n(T_nz_n-p)]\|^2\\ &\quad+2\gamma_n\alpha_n\langle h_n-p,x_{n+1}-p\rangle+2\gamma_n\alpha_n\langle p-f(x_n),x_{n+1}-p\rangle,\end{aligned}$$

which leads to

$$\begin{aligned}\|x_{n+1}-p\|^2 \quad &\beta_n\|v_n-p\|^2+(1-\beta_n)[(1-\alpha_n)\|h_n-p\|^2+2\alpha_n\langle f(x_n)-p,z_n-p\rangle]\\ &+\gamma_n\alpha_n(\|h_n-p\|^2+\|x_{n+1}-p\|^2)+2\gamma_n\alpha_n\langle p-f(x_n),x_{n+1}-p\rangle.\end{aligned}$$

Using (17) and (38) we deduce that $\|h_n-p\|^2 \leq \|v_n-p\|^2 \leq \|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|3M$. Hence,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\alpha_n(1-\beta_n)]\|x_n-p\|^2+[1-\alpha_n(1-\beta_n)]\sigma_n\|x_n-x_{n-1}\|3M\\ &\quad+2\alpha_n\delta_n\langle f(x_n)-p,z_n-p\rangle+\gamma_n\alpha_n(\|x_n-p\|^2+\|x_{n+1}-p\|^2)\\ &\quad+(1-\beta_n)\alpha_n\sigma_n\|x_n-x_{n-1}\|3M+2\gamma_n\alpha_n\langle f(x_n)-p,z_n-x_{n+1}\rangle\\ &\leq [1-\alpha_n(1-\beta_n)]\|x_n-p\|^2+2\gamma_n\alpha_n\|f(x_n)-p\|\|z_n-x_{n+1}\|\\ &\quad+2\alpha_n\delta_n\langle f(x_n)-p,x_n-p\rangle+2\alpha_n\delta_n\langle f(x_n)-p,z_n-x_n\rangle\\ &\quad+\gamma_n\alpha_n(\|x_n-p\|^2+\|x_{n+1}-p\|^2)+\sigma_n\|x_n-x_{n-1}\|3M\\ &\leq [1-\alpha_n(1-\beta_n)]\|x_n-p\|^2+2\gamma_n\alpha_n\|f(x_n)-p\|\|z_n-x_{n+1}\|\\ &\quad+2\alpha_n\delta_n\delta\|x_n-p\|^2+2\alpha_n\delta_n\langle f(p)-p,x_n-p\rangle+2\alpha_n\delta_n\|f(x_n)-p\|\|z_n-x_n\|\\ &\quad+\gamma_n\alpha_n(\|x_n-p\|^2+\|x_{n+1}-p\|^2)+\sigma_n\|x_n-x_{n-1}\|3M,\end{aligned}$$

which immediately yields

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq [1-\tfrac{(1-2\delta)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}\alpha_n]\|x_n-p\|^2+\tfrac{[(1-2\delta)\delta_n-\gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_{n+1}\|\\ &\quad+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_n\|+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(p)-p,x_n-p\rangle\\ &\quad+\tfrac{1}{(1-2\delta)\delta_n-\gamma_n}\cdot\tfrac{\sigma_n}{\alpha_n}\|x_n-x_{n-1}\|3M\}.\end{aligned} \tag{39}$$

Step 4. In order to show that $x_n \to x^* \in \Omega$, which is the unique solution of (12), we can follow a similar method to that in Step 4 for the proof of Theorem 1.

Finally, we apply our main results to solve the VIP and common fixed point problem (CFPP) in the following illustrating example.

The starting point $x_0 = x_1$ is randomly picked in the real line. Put $f(u) = \frac{1}{8}\sin u$, $\gamma = l = \mu = \frac{1}{2}$, $\sigma_n = \alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{6}$ and $\delta_n = \frac{1}{2}$.

We first provide an example of Lipschitzian, pseudomonotone self-mapping A satisfying the boundedness of $A(C)$ and strictly pseudocontractive self-mapping T_1 with $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $C = [-1, 2]$ and H be the real line with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Then f is a δ-contractive map with $\delta = \frac{1}{8} \in [0, \frac{1}{2})$ and $f(H) \subset C$ because $\|f(u) - f(v)\| = \frac{1}{8}\|\sin u - \sin v\| \leq \frac{1}{8}\|u - v\|$ for all $u, v \in H$.

Let $A : H \to H$ and $T_1 : H \to H$ be defined as $Au := \frac{1}{1+|\sin u|} - \frac{1}{1+|u|}$, and $T_1u := \frac{1}{2}u - \frac{3}{8}\sin u$ for all $u \in H$. Now, we first show that A is L-Lipschitzian, pseudomonotone operator with $L = 2$, such that $A(C)$ is bounded. In fact, for all $u, v \in H$ we get

$$\begin{aligned}\|Au-Av\| &\leq |\tfrac{1}{1+\|u\|}-\tfrac{1}{1+\|v\|}|+|\tfrac{1}{1+\|\sin u\|}-\tfrac{1}{1+\|\sin v\|}|\\ &= |\tfrac{\|v\|-\|u\|}{(1+\|u\|)(1+\|v\|)}|+|\tfrac{\|\sin v\|-\|\sin u\|}{(1+\|\sin u\|)(1+\|\sin v\|)}|\\ &\leq \tfrac{\|u-v\|}{(1+\|u\|)(1+\|v\|)}+\tfrac{\|\sin u-\sin v\|}{(1+\|\sin u\|)(1+\|\sin v\|)}\\ &\leq 2\|u-v\|.\end{aligned}$$

This implies that A is 2-Lipschitzian. Next, we show that A is pseudomonotone. For any given $u, v \in H$, it is clear that the relation holds:

$$\langle Au, u-v\rangle = (\frac{1}{1+|\sin u|} - \frac{1}{1+|u|})(u-v) \le 0 \Rightarrow \langle Av, u-v\rangle = (\frac{1}{1+|\sin v|} - \frac{1}{1+|v|})(u-v) \le 0.$$

Furthermore, it is easy to see that T_1 is strictly pseudocontractive with constant $\zeta_1 = \frac{1}{4}$. In fact, we observe that for all $u, v \in H$,

$$\|T_1u - T_1v\| \le \frac{1}{2}\|u-v\| + \frac{3}{8}\|\sin u - \sin v\| \le \|u-v\| + \frac{1}{4}\|(I-T_1)u - (I-T_1)v\|.$$

It is clear that $(\gamma_n + \delta_n)\zeta_1 = (\frac{1}{6} + \frac{1}{2}) \cdot \frac{1}{4} \le \frac{1}{6} = \gamma_n < (1-2\delta)\delta_n = (1 - 2 \cdot \frac{1}{8})\frac{1}{2} = \frac{3}{8}$ for all $n \ge 1$. In addition, it is clear that $\text{Fix}(T_1) = \{0\}$ and $A0 = 0$ because the derivative $d(T_1u)/du = \frac{1}{2} - \frac{3}{8}\cos u > 0$. Therefore, $\Omega = \{0\} \ne \emptyset$. In this case, Algorithm 1 can be rewritten below:

$$\begin{cases} v_n = x_n - \frac{1}{n+1}(x_{n-1} - x_n), \\ u_n = P_C(v_n - \ell_n A v_n), \\ z_n = \frac{1}{n+1} f(x_n) + \frac{n}{n+1} P_{C_n}(v_n - \ell_n A u_n), \\ x_{n+1} = \frac{1}{3}x_n + \frac{1}{6}P_{C_n}(v_n - \ell_n A u_n) + \frac{1}{2}T_1 z_n \quad \forall n \ge 1, \end{cases}$$

with $\{C_n\}$ and $\{\ell_n\}$, selected as in Algorithm 1. Then, by Theorem 1, we know that $x_n \to 0 \in \Omega$ iff $x_n - x_{n+1} \to 0$ $(n \to \infty)$ and $\sup_{n\ge1} |x_n - \frac{1}{8}\sin x_n| < \infty$.

On the other hand, Algorithm 2 can be rewritten below:

$$\begin{cases} v_n = x_n - \frac{1}{n+1}(x_{n-1} - x_n), \\ u_n = P_C(v_n - \ell_n A v_n), \\ z_n = \frac{1}{n+1} f(x_n) + \frac{n}{n+1} P_{C_n}(v_n - \ell_n A u_n), \\ x_{n+1} = \frac{1}{3}v_n + \frac{1}{6}P_{C_n}(v_n - \ell_n A u_n) + \frac{1}{2}T_1 z_n \quad \forall n \ge 1, \end{cases}$$

with $\{C_n\}$ and $\{\ell_n\}$, selected as in Algorithm 2. Then, by Theorem 2 , we know that $x_n \to 0 \in \Omega$ iff $x_n - x_{n+1} \to 0$ $(n \to \infty)$ and $\sup_{n\ge1} |x_n - \frac{1}{8}\sin x_n| < \infty$.

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