mathematics

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Gradient Methods with Selection Technique for the Multiple-Sets Split Equality Problem

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Received: 17 August 2019; Accepted: 26 September 2019; Published: 8 October 2019

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Abstract: The inverse problem is one of the four major problems in computational mathematics. There is an inverse problem in medical image reconstruction and radiotherapy that is called the multiple-sets split equality problem. The multiple-sets split equality problem is a unified form of the split feasibility problem, split equality problem, and split common fixed point problem. In this paper, we present two iterative algorithms for solving it. The suggested algorithms are based on the gradient method with a selection technique. Based on this technique, we only need to calculate one projection in each iteration.

Keywords: multiple-sets split equality problem; split equality problem; selection technique

1. Introduction

The inverse problem is one of the four major problems in computational mathematics. The rapid development of the inverse problem has been a feature of recent decades; it can be found in computer vision, machine learning, statistics, geography, medical imaging, remote sensing, ocean acoustics, tomography, aviation, physics, and other fields. There is an inverse problem in medical image reconstruction and radiotherapy that can be expressed as a split feasibility problem [1–9], split equality problem [10–13], and split common fixed point problem [14–19].

In this paper, we focus on a unified form of the split feasibility problem, split equality problem, and split common fixed point problem that is called the multiple-sets split equality problem.

Let H_1, H_2, H_3 be three real Hilbert spaces. r, t are positive integers, $\{C_i\}_{i=1}^r$ and $\{Q_j\}_{j=1}^t$ are two families of closed and convex subsets of H_1 and H_2, respectively. $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded and linear operators. Then the multiple-sets split equality problem (MSSEP for short) can be formulated as

$$finding \quad x \in \bigcap_{i=1}^{r} C_i, \quad y \in \bigcap_{j=1}^{t} Q_j \quad such \quad that \quad Ax = By. \tag{1}$$

It reduces to the split equality problem if $r = t = 1$; moreover, it is the split feasibility problem if $H_2 = H_3$ and B is the identity operator on H_2. In addition, it is the split common fixed point problem if we take $x \in C_i$ to $x = P_{C_i}x$, $y \in Q_j$ to $y \in P_{Q_j}$ where P_{C_i}, P_{Q_j} are the metric projections on C_i, Q_j.

In the problem (1), without loss of generality, we may assume that $t \geq r$ and let $C_{r+1} = C_{r+2} = \cdots = C_t = H_1$. Then the problem (1) can be described equivalently as:

$$finding \quad x \in \bigcap_{i=1}^{t} C_i, \quad y \in \bigcap_{j=1}^{t} Q_j \quad such \quad that \quad Ax = By. \tag{2}$$

Let $S_i = C_i \times Q_i \subseteq H = H_1 \times H_2, i \in \Lambda = \{1, 2, \cdots, t\}, G = [A, -B] : H \to H_3, G^*$ be the adjoint operator of G where $H_1 \times H_2$ is the Cartesian product of H_1 and H_2. Then the original problem (1) can be modified as

$$finding \quad w = (x, y) \in \bigcap_{i=1}^{t} S_i \quad such \quad that \quad Gw = 0. \tag{3}$$

Assume the problem (3) is consistent and let Ω denote its solution set, that is, Ω is not empty. We consider the proximity function

$$f(w) = \frac{1}{2}\Sigma_{i=1}^{t} \alpha_i \|w - P_{S_i} w\|^2 + \frac{1}{2}\|Gw\|^2,$$

where $\alpha_i, i = 1, \cdots, t$ are positive real numbers and $P_{S_i}, i = 1, \cdots, t$ are metric projections from H onto S_i. Since C_i and Q_i are closed convex, so are S_i, and then P_{S_i} are well defined. Then problem (3) can be transformed into the minimization problem

$$\min_{w \in \bigcap_{i=1}^{t} S_i} f(w). \tag{4}$$

Note that the proximity function $f(w)$ is convex and differentiable with gradient

$$\nabla f(w) = \Sigma_{i=1}^{t} \alpha_i (I - P_{S_i}) w + G^* G w,$$

where I is the identity operator on H. The gradient function $\nabla f(w)$ is L-Lipschitz continuous with Lipschitz constant [20]

$$L = \Sigma_{i=1}^{t} \alpha_i + \|G\|^2.$$

To solve the minimization problem (4), a classical method is the gradient algorithm, which takes the iterative issue

$$w_{n+1} = w_n - \gamma_n \nabla f(w_n), \tag{5}$$

where γ_n is the iterative step size in step n.

Note that in iteration (5), we need to calculate projections for t times in each step. On the other hand, notice that $w^* \in \Omega$ if and only if $g(w^*) = 0$, where

$$g(w) = \frac{1}{2}\|w - P_{S_{i(n)}} w\|^2 + \frac{1}{2}\|Gw\|^2,$$

in which

$$i(n) \in \{i | \max_{1 \le i \le t} \|w - P_{S_i} w\|\}.$$

Then we consider the iterative issue

$$w_{n+1} = w_n - \gamma_n \nabla g(w_n). \tag{6}$$

In iteration (6), we only need to implement a projection once in each step. Motivated by this point, we present Algorithms 1 and 2 in Section 3 to solve problem (3).

The general structure of this paper is as follows. In the next section, we go over some preliminaries. In Section 3, we present the main algorithms and their convergence. In Section 4, several numerical results are shown to confirm the effectiveness of the suggested algorithm. In the last section, there are some conclusions.

2. Preliminaries

For convenience, note that H is a real Hilbert space and I denotes the identity operator on H. By $x_n \to x^*$ and $x_n \rightharpoonup x^*$, the strong and weak convergence of sequence $\{x_n\}$ to a point x^*, respectively, and $\omega_w(x_n)$ denotes the set of weak cluster points of the sequence $\{x_n\}$. P_S is the projection from H onto its closed and convex subset S.

Lemma 1 ([21]). *Let S be a closed, convex, and nonempty subset of H, then for any $x, y \in H$ and $z \in S$,*
(i) $\langle x - P_S x, z - P_S x\rangle \leq 0$;
(ii) $\|P_S x - P_S y\|^2 \leq \langle P_S x - P_S y, x - y\rangle$;
(iii) $\langle x - P_S x, x - z\rangle \geq \|x - P_S x\|^2$.

Lemma 2 ([22]). *Let $\{a_n\}$, $\{\alpha_n\}$, $\{u_n\}$ be sequences of non-negative real numbers with*

$$\{\alpha_n\} \subset [0,1], \quad \Sigma_{n=1}^{\infty}\alpha_n = \infty, \quad \Sigma_{n=1}^{\infty}u_n < \infty.$$

Let $\{t_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} t_n \leq 0$ and

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n t_n + u_n.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 3 ([23]). *Let S be a closed and convex subset of H, and $T : S \to S$ be non-expansive, and $\{x_n\} \subseteq S$. If $x_n \rightharpoonup x$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $Tx = x$.*

Lemma 4 ([24]). *Let S be a closed, convex, and nonempty subset of H and $\{x_n\}$ be a sequence in H. If*
(i) $\lim_{n\to\infty}\|x_n - x\|$ *exists for each* $x \in S$;
(ii) $\omega_w(x_n) \subseteq S$;
then $\{x_n\}$ converges weakly to a point in S.

3. Main Results

Assume that the problem (3) is consistent and let Ω denote its solution set. That is, Ω is not empty and $\Omega := \{w \in H : w \in \bigcap_{i=1}^{t} S_i, Gw = 0\}$.

Remark 1. *w_n is a solution of the problem (3) if and only if the equality (8) holds.*

On the one hand, if $\|w_n + q_n - z_n\| = 0$, then take $z \in \Omega$. We have

$$\begin{aligned} 0 &= \langle w_n + q_n - z_n, w_n - z\rangle \\ &= \langle w_n + G^*Gw_n - P_{S_{i(n)}}w_n, w_n - z\rangle \\ &= \langle w_n - P_{S_{i(n)}}w_n, w_n - z\rangle + \langle Gw_n, Gw_n - Gz\rangle \\ &\geq \|w_n - P_{S_{i(n)}}w_n\|^2 + \|Gw_n\|^2. \end{aligned}$$

The first equality is from $\|w_n + q_n - z_n\| = 0$, the second one is from the definition of q_n and z_n, and the last inequality is from Lemma 1(iii) and $Gz = 0$. Then

$$\|w_n - P_{S_{i(n)}}w_n\| = 0 \quad and \quad \|Gw_n\| = 0,$$

which implies that

$$\|w_n - P_{S_i}w_n\| = 0, i \in \Lambda \quad and \quad \|Gw_n\| = 0.$$

Hence $w_n \in \bigcap_{i=1}^{t} S_i$ and $Gw_n = 0$. Namely, $w_n \in \Omega$.

Conversely, if w_n is a solution of the problem (3), that is $w_n \in \bigcap_{i=1}^{t} S_i$ and $Gw_n = 0$, then $q_n = G^*Gw_n = 0$ and $z_n = P_{S_{i(n)}}w_n = w_n$, so $w_n + q_n - z_n = 0$. That is, $\|w_n + q_n - z_n\| = 0$.

Next we discuss the convergence of the iterative sequence $\{w_n\}$ generated by Algorithm 1 if it does not terminate in finite steps.

Algorithm 1: Gradient method 1

Take $w_0 \in H$ arbitrarily and compute

$$z_n = P_{S_{i(n)}} w_n, \quad q_n = G^* G w_n, \tag{7}$$

where $n \geq 0$ and

$$i(n) \in \{i | \max_{i \in \Lambda} \|w_n - P_{S_i} w_n\|, \Lambda = \{1, 2, \cdots, t\}\}.$$

If

$$\|w_n + q_n - z_n\| = 0, \tag{8}$$

then stop. w_n is the solution (based on Remark 1). Otherwise, calculate

$$w_{n+1} = w_n - \tau_n (w_n + q_n - z_n), \tag{9}$$

where

$$\tau_n = \lambda_n \frac{\|w_n - z_n\|^2 + \|G w_n\|^2}{2\|w_n + q_n - z_n\|^2},$$

in which $\lambda_n \in (0, 4)$.

Theorem 1. *If $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 4$, taking initial point $w_0 \in H$ arbitrarily, then the sequence $\{w_n\}$ generated by Algorithm 1 converges weakly to a solution of the problem (3).*

Proof. First, we show the boundedness of $\{w_n\}$. Take $z \in \Omega$. Based on the inequality in the process of Remark 1, we get

$$\begin{aligned}
\|w_{n+1} - z\|^2 &= \|w_n - z - \tau_n (w_n + q_n - z_n)\|^2 \\
&= \|w_n - z\|^2 - 2\tau_n \langle w_n + q_n - z_n, w_n - z \rangle + \tau_n^2 \|w_n + q_n - z_n\|^2 \\
&\leq \|w_n - z\|^2 - \lambda_n \frac{(\|w_n - z_n\|^2 + \|G w_n\|^2)^2}{\|w_n + q_n - z_n\|^2} \\
&\quad + \frac{\lambda_n^2}{4} \frac{(\|w_n - z_n\|^2 + \|G w_n\|^2)^2}{\|w_n + q_n - z_n\|^2} \\
&= \|w_n - z\|^2 - \lambda_n (1 - \frac{\lambda_n}{4}) \frac{(\|w_n - z_n\|^2 + \|G w_n\|^2)^2}{\|w_n + q_n - z_n\|^2}.
\end{aligned} \tag{10}$$

This implies that $\lim_{n\to\infty} \|w_n - z\|$ exists. Thus the sequence $\{w_n\}$ is bounded and so are the sequences $\{G w_n\}$ and $\{P_{S_i} w_n\}$, $i \in \Lambda$.
Next we show that $\omega_w(w_n) \subset \Omega$.
Since $\lim_{n\to\infty} \|w_n - z\|$ exists and

$$\lambda_n (1 - \frac{\lambda_n}{4}) \frac{(\|w_n - z_n\|^2 + \|G w_n\|^2)^2}{\|w_n + q_n - z_n\|^2} \leq \|w_n - z\|^2 - \|w_{n+1} - z\|^2,$$

together with the boundedness of the sequence $\{w_n + q_n - z_n\}$ and the definition of λ_n, it follows that

$$\lim_{n\to} \frac{(\|w_n - z_n\|^2 + \|G w_n\|^2)^2}{\|w_n + q_n - z_n\|^2} = 0,$$

which implies that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0 \quad and \quad \lim_{n\to\infty} \|Gw_n\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|w_n - P_{S_i} w_n\| = 0, i \in \Lambda \quad and \quad \lim_{n\to\infty} \|Gw_n\| = 0.$$

Since $\{w_n\}$ is bounded, let w^* be a weak cluster point of $\{w_n\}$ with subsequence $\{w_{n_i}\}$ weakly convergent to it.

$$\lim_{n\to\infty} \|w_{n_i} - P_{S_i} w_{n_i}\| = 0, i \in \Lambda \quad and \quad \lim_{n\to\infty} \|Gw_{n_i}\| = 0.$$

By Lemma 3, we get $w^* \in \Omega$, and by the arbitrariness of $w^* \in \omega_w(w_n)$, we deduce that $\omega_w(w_n) \subset \Omega$. Moreover, the conditions in Lemma 4 have also been satisfied, and the sequence $\{w_n\}$ generated by the Algorithm 1 converges weakly to some solution of the problem (3). The proof is completed. □

There is only weak convergence in Theorem 1. Next, we show a strong convergence theorem for solving the problem (3).

Next, we discuss the convergence of the iterative sequence $\{w_n\}$ generated by Algorithm 2 if it does not terminate in finite steps.

Algorithm 2: Gradient method 2

Take $u \in H$ and initial point $w_0 \in H$. Compute

$$\begin{aligned} z_n &= P_{S_{i(n)}} w_n, \\ q_n &= G^* G w_n, \end{aligned} \tag{11}$$

where

$$i(n) = \{i | \max_{i\in\Lambda} \|w_n - P_{S_i} w_n\|, \Lambda = \{1, 2, \cdots, t\}\}.$$

If

$$\|w_n + q_n - z_n\| = 0,$$

then stop. w_n is the solution (by Remark 1). Otherwise, calculate

$$w_{n+1} = \alpha_n u + (1 - \alpha_n)(w_n - \tau_n(w_n + q_n - z_n)), \tag{12}$$

where $\alpha_n \in (0, 1)$, $n \geq 0$ and

$$\tau_n = \lambda_n \frac{\|w_n - z_n\|^2 + \|Gw_n\|^2}{2\|w_n + q_n - z_n\|^2},$$

in which $\lambda_n \in (0, 4)$.

Theorem 2. *Suppose that* $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$, *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 4$. *Taking* $u \in H$ *and initial point* $w_0 \in H$ *arbitrarily, then the sequence* $\{w_n\}$ *generated by the Algorithm 2 converges strongly to* $z = P_\Omega u$.

Proof. Let $u_n = w_n - \tau_n(w_n + q_n - z_n)$, for $n \geq 0$. From the process (10) in Theorem 1, we get

$$\|u_n - z\|^2 \leq \|w_n - z\|^2 - \lambda_n (1 - \frac{\lambda_n}{4}) \frac{(\|w_n - z_n\|^2 + \|Gw_n\|^2)^2}{\|w_n + q_n - z_n\|^2} \tag{13}$$

by the definition of λ_n, that is, $\|u_n - z\| \leq \|w_n - z\|$. Thus

$$\begin{aligned}\|w_{n+1} - z\| &= \|\alpha_n u + (1-\alpha_n)u_n - z\| \\ &\leq \alpha_n\|u - z\| + (1-\alpha_n)\|u_n - z\| \\ &\leq \alpha_n\|u - z\| + (1-\alpha_n)\|w_n - z\| \\ &\leq \max\{\|w_n - z\|, \|u - z\|\}.\end{aligned}$$

By induction, we derive

$$\|w_{n+1} - z\| \leq \max\{\|w_0 - z\|, \|u - z\|\},$$

which means that the sequence $\{w_n\}$ is bounded and so are the sequences $\{Gw_n\}$ and $\{P_{S_i}w_n\}$, $i \in \Lambda$. By a simple derivation,

$$\begin{aligned}\|w_{n+1} - z\|^2 &= \|\alpha_n(u - z) + (1-\alpha_n)(u_n - z)\|^2 \\ &\leq (1-\alpha_n)\|u_n - z\|^2 + 2\alpha_n\langle u - z, w_{n+1} - z\rangle.\end{aligned}$$

Then by (13),

$$\begin{aligned}\|w_{n+1} - z\|^2 &\leq (1-\alpha_n)\|w_n - z\|^2 + 2\alpha_n\langle u - z, w_{n+1} - z\rangle \\ &\quad -(1-\alpha_n)\lambda_n(1 - \frac{\lambda_n}{4})\frac{(\|w_n - z_n\|^2 + \|Gw_n\|^2)^2}{\|w_n + q_n - z_n\|^2} \\ &= (1-\alpha_n)\|w_n - z\|^2 + \alpha_n[2\langle u - z, w_{n+1} - z\rangle \\ &\quad -\frac{(1-\alpha_n)}{\alpha_n}\lambda_n(1 - \frac{\lambda_n}{4})\frac{(\|w_n - z_n\|^2 + \|Gw_n\|^2)^2}{\|w_n + q_n - z_n\|^2}].\end{aligned} \tag{14}$$

Let

$$\theta_n = \|w_n - z\|^2,$$

$$\delta_n = 2\langle u - z, w_{n+1} - z\rangle - \frac{(1-\alpha_n)}{\alpha_n}\lambda_n(1 - \frac{\lambda_n}{4})\frac{(\|w_n - z_n\|^2 + \|Gw_n\|^2)^2}{\|w_n + q_n - z_n\|^2}.$$

Then the inequality (14) equals

$$\theta_{n+1} \leq (1-\alpha_n)\theta_n + \alpha_n\delta_n, \tag{15}$$

and also

$$0 \leq \theta_{n+1} \leq (1-\alpha_n)\theta_n + \alpha_n\delta_n, n \geq 0.$$

It follows that

$$\delta_n \leq 2\langle u - z, w_{n+1} - z\rangle \leq 2\|u - z\|\|w_{n+1} - z\|.$$

So

$$\limsup_{n\to\infty} \delta_n < \infty.$$

Next, we show that $\limsup_{n\to\infty} \delta_n \geq -1$. Otherwise, if $\limsup_{n\to\infty} \delta_n < -1$, then by the definition of the supremum, there exists m such that $\delta_n \leq -1$ for all $n \geq m$. It follows that for all $n \geq m$,

$$\begin{aligned}\theta_{n+1} &\leq (1-\alpha_n)\theta_n + \alpha_n\delta_n \\ &= \theta_n + \alpha_n(\delta_n - \theta_n) \\ &\leq \theta_n - \alpha_n.\end{aligned}$$

Thus

$$\theta_{n+1} \leq \theta_m - \Sigma_{i=m}^n \alpha_i.$$

Hence, taking lim sup as $n \to \infty$ in the above inequality, we obtain

$$0 \leq \limsup_{n\to\infty} \theta_{n+1} \leq \theta_m - \limsup \Sigma_{i=m}^{n} \alpha_i = -\infty,$$

which is a contradiction. Therefore, $\limsup_{n\to\infty} \delta_n \geq -1$, and it is finite. By the boundedness of $\{\delta_n\}$, we can take a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\begin{aligned} \limsup_{n\to\infty} \delta_n &= \lim_{k\to\infty} \delta_{n_k} \\ &= \lim_{k\to\infty} [2\langle u - z, w_{n_k+1} - z\rangle \\ &\quad - \frac{(1-\alpha_{n_k})}{\alpha_{n_k}} \lambda_{n_k} (1 - \frac{\lambda_{n_k}}{4}) \frac{(\|w_{n_k} - z_{n_k}\|^2 + \|Gw_{n_k}\|^2)^2}{\|w_{n_k} + q_{n_k} - z_{n_k}\|^2}]. \end{aligned}$$

Since the sequence $\{w_{n_k+1}\}$ is bounded, there exists a subsequence of $\{w_{n_k+1}\}$. Without loss of generality, we may assume it's $\{w_{n_k+1}\}$ itself, such that $\lim_{k\to\infty}\langle u - z, w_{n_k+1} - z\rangle$ exists. Consequently, the following limit exists:

$$\lim_{k\to\infty} -\frac{(1-\alpha_{n_k})}{\alpha_{n_k}} \lambda_{n_k} (1 - \frac{\lambda_{n_k}}{4}) \frac{(\|w_{n_k} - z_{n_k}\|^2 + \|Gw_{n_k}\|^2)^2}{\|w_{n_k} + q_{n_k} - z_{n_k}\|^2}.$$

Together with the definitions of α_n and λ_n, it shows that

$$\lim_{k\to\infty} \frac{(\|w_{n_k} - z_{n_k}\|)^2 + \|Gw_{n_k}\|^2}{\|w_{n_k} + q_{n_k} - z_{n_k}\|^2} = 0,$$

which yields

$$\lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| = 0 \quad \textit{and} \quad \lim_{k\to\infty} \|Gw_{n_k}\| = 0.$$

Following the proof procedure of Theorem 1, we conclude that $\omega_w(w_{n_k}) \subset \Omega$. Since

$$\begin{aligned} \|w_{n_k+1} - w_{n_k}\| &= \|\alpha_{n_k} u + (1-\alpha_{n_k}) u_{n_k} - w_{n_k}\| \\ &\leq \alpha_{n_k}\|u - w_{n_k}\| + (1-\alpha_{n_k})\|u_{n_k} - w_{n_k}\| \\ &= \alpha_{n_k}\|u - w_{n_k}\| + (1-\alpha_{n_k})\tau_{n_k}\|w_{n_k} + q_{n_k} - z_{n_k}\| \\ &= \alpha_{n_k}\|u - w_{n_k}\| + (1-\alpha_{n_k})\lambda_{n_k} \frac{\|w_{n_k} - z_{n_k}\|^2 + \|Gw_{n_k}\|^2}{\|w_{n_k} + q_{n_k} - z_{n_k}\|} \\ &\to 0, \end{aligned}$$

assume that $w_{n_k+1} \rightharpoonup w^* \in \Omega$. Then

$$\begin{aligned} \limsup_{n\to\infty} \delta_n &= \lim_{k\to\infty} \delta_{n_k} \\ &= \lim_{k\to\infty} [2\langle u - z, w_{n_k+1} - z\rangle \\ &\quad - \frac{(1-\alpha_{n_k})}{\alpha_{n_k}} \lambda_{n_k} (1 - \frac{\lambda_{n_k}}{4}) \frac{(\|w_{n_k} - z_{n_k}\|^2 + \|Gw_{n_k}\|^2)^2}{\|w_{n_k} + q_{n_k} - z_{n_k}\|^2}] \\ &\leq \lim_{k\to\infty} 2\langle u - z, w_{n_k+1} - z\rangle \\ &= 2\langle u - z, w^* - z\rangle \\ &\leq 0, \end{aligned}$$

due to the fact that $z = P_\Omega u$ and Lemma 1. Finally, applying Lemma 2 to (15), we conclude that $w_n \to z$. The proof is completed. □

4. Numerical Experiments

In this section, we provide several numerical results of the MSSEP (2) to confirm the effectiveness of the suggested Algorithm 1. The whole program was written in Wolfram Mathematica (version 9.0). All of the numerical results were carried out on a personal Lenovo computer with Intel(R)Core(TM) i5-6600 CPU 3.30 GHz and RAM 8.00 GB.

The MSSEP with $C_1 = \{x \in R^2 | \|x - (-1,1)\| \leq 5\}$, $C_2 = \{x \in R^2 | \|x - (1,1)\| \leq 5\}$, $C_3 = \{x \in R^2 | \|x - (0,-3)\| \leq 5\}$, $Q_1 = \{y \in R^3 | \|y - (1,1,1)\| \leq 5\}$, $Q_2 = \{y \in R^3 | \|y - (0,0,0)\| \leq 5\}$, $Q_3 = \{y \in R^3 | \|y - (1,0,0)\| \leq 5\}$, $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$, $\Lambda = \{1,2,3\}$, $\lambda_n = 0.6$. We choose two initial values $x_0 = (2,2)$, $y_0 = (2,2,2)$ and $x_0 = (20,20)$, $y_0 = (10,10,10)$ and take the iterative steps n as the transverse axis and $\|Ax - By\|$ as the vertical axis in the figures below (Figures 1 and 2). We considered using the Algorithm 1 to solve this MSSEP.

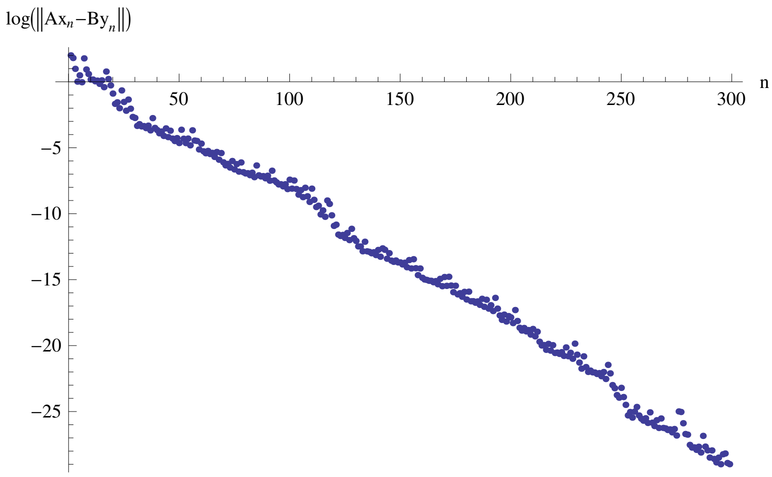

Figure 1. $x_0 = (2,2), y_0 = (2,2,2)$.

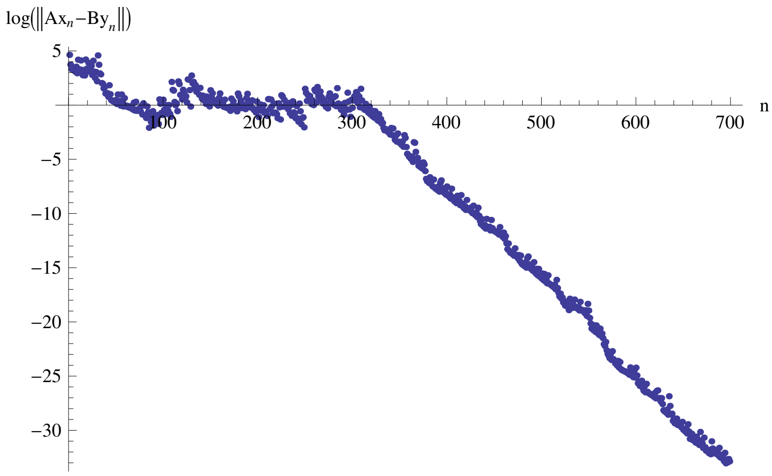

Figure 2. $x_0 = (20,20), y_0 = (10,10,10)$.

The figures above confirm the effectiveness of the proposed Algorithm 1 and also show that there is an approximately linear downward trend after finite steps, which means the convergence rate of the proposed Algorithm 1 may be fast enough.

Author Contributions: The main idea of this paper was proposed by D.T.; L.J. and L.S. reviewed all the steps of the initial manuscript. All authors approved the final manuscript.

Funding: This research was supported by NSFC Grants No. 11301379; No. 11226125.

Conflicts of Interest: The authors declare no conflict of interest.

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