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Generalized Mann Viscosity Implicit Rules for Solving Systems of Variational Inequalities with Constraints of Variational Inclusions and Fixed Point Problems

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Abstract: In this work, let X be Banach space with a uniformly convex and q-uniformly smooth structure, where $1 < q \leq 2$. We introduce and consider a generalized Mann-like viscosity implicit rule for treating a general optimization system of variational inequalities, a variational inclusion and a common fixed point problem of a countable family of nonexpansive mappings in X. The generalized Mann-like viscosity implicit rule investigated in this work is based on the Korpelevich's extragradient technique, the implicit viscosity iterative method and the Mann's iteration method. We show that the iterative sequences governed by our generalized Mann-like viscosity implicit rule converges strongly to a solution of the general optimization system.

Keywords: generalized Mann-like viscosity rule; system of variational inequalities; variational inclusions; nonexpansive mappings; strong convergence; uniform convexity; uniform smoothness

MSC: 47H05; 47H09; 49J40

1. Introduction

Throughout this work, we always suppose that C is a non-empty, convex and closed subset of a real Banach space X. X^* will be used to present the dual space of space X. In this present work, we let the norms of X and X^* be presented by the denotation $\|\cdot\|$. Let T be a nonlinear self mapping with fixed points defined on subset C.

One use $\langle \cdot, \cdot \rangle$ to denote the duality pairing. The possible set-valued normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) := \{\phi \in X^* : \langle \phi, x \rangle = \|\phi\|^2 = \|x\|^2\}, \quad \forall x \in X.$$

Banach space X is said to be a smooth space (has a Gâteaux differentiable norm) if $\lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $\|x\| = \|y\| = 1$. J is norm-to-weak* continuous single-valued map in such a space. X is also said to be a uniformly smooth space (has a uniformly Fréchet differentiable norm) if the above limit is attained uniformly for $\|x\| = \|y\| = 1$ and J is norm-to-norm uniformly continuous on bounded sets in such a space. X is said to be a strictly convex space if $\|(1-\lambda)x + \lambda y\| < 1$, $\forall \lambda \in (0,1)$ for all $\|x\| = \|y\| = 1$. Space X is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, we have a constant $\delta > 0$ such that $\|\frac{x+y}{2}\| > 1-\delta \Rightarrow \|x-y\| < \varepsilon$ for all $\|x\| = \|y\| = 1$. A uniformly convex Banach space yields a strictly convex Banach space. Under the reflexive framework, X is strictly convex if and only if X^* is smooth.

Next, we suppose X is smooth, i.e., J is single-valued. Let $A_1, A_2 : C \to X$ be two nonlinear single-valued mappings. One is concerned with the problem of approximating $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* + \mu_1 A_1 y^* - y^*, J(x^* - x)\rangle \leq 0, & \forall x \in C, \\ \langle y^* + \mu_2 A_2 x^* - x^*, J(y^* - x)\rangle \leq 0, & \forall x \in C, \end{cases} \tag{1}$$

with two real positive constants μ_1 and μ_2. This optimization system is called a general system of variational inequalities (GSVI). In particular, in the case that $X = H$ is Hilbert, then GSVI (1) is reduced to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* + \mu_1 A_1 y^* - y^*, x^* - x\rangle \geq 0, & \forall x \in C, \\ \langle y^* + \mu_2 A_2 x^* - x^*, y^* - x\rangle \geq 0, & \forall x \in C, \end{cases}$$

with two real positive constants μ_1 and μ_2. This was introduced and studied in [1]. Additionally, if $A = A_1 = A_2$ and $x^* = y^*$, then GSVI (1) becomes the variational problem of finding $x^* \in C$ such that $\langle Ax^*, J(x - x^*)\rangle \geq 0$, $\forall x \in C$. In 2006, Aoyama, Iiduka and Takahashi [2] proposed an iterative scheme of finding its approximate solutions and claimed the weak convergence of the iterative sequences governed by the proposed algorithm. Recently, many researchers investigated the variational inequality problem through gradient-based or splitting-based methods; see, e.g., [3–13]. Some stability results can be found at [14,15].

In 2013, Ceng, Latif and Yao [16] analyzed and introduced an implicit computing method by using a double-step relaxed gradient idea in the setting of 2-uniformly smooth and uniformly convex space X with 2-uniform smoothness coefficient κ_2. Let $\Pi_C : X \to C$ be a retraction, which is both sunny and nonexpansive. Let $f : C \to C$ be a contraction with constant $\delta \in (0, 1)$. Let the mapping $A_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self single-valued mappings on C such that $\Omega = \cap_{n=0}^{\infty}\text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \neq \emptyset$, where $\text{GSVI}(C, A_1, A_2)$ stands for the set of fixed points the mapping $G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$. For a arbitrary initial $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = \alpha_n f(y_n) + (1 - \alpha_n)\Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n, \quad \forall n \geq 0, \end{cases}$$

with $0 < \mu_i < \frac{2\alpha_i}{\kappa_2}$ for $i = 1, 2$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $(0, 1)$ satisfying the restrictions: $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \beta_n > o$. They got convergence analysis of $\{x_n\}$ to $x^* \in \Omega$, which treats the variational inequality: $\langle (I - f)x^*, J(x^* - p)\rangle \leq 0$, $\forall p \in \Omega$. Recently, projection-like methods, including sunny nonexpansive retractions, have largely studied in Hilbert and Banach spaces; see, e.g., [17–24] and the references therein.

2. Preliminaries

Next, we let X be a space with uniformly convex and q-uniformly smooth structures. Then the following inequality holds:

$$\|x + y\|^q \leq \kappa_q \|y\|^q + \|x\|^q + q\langle y, \mathfrak{J}_q(x)\rangle, \quad \forall x, y \in E,$$

where κ_q is the smoothness coefficient. Let $\Pi_C, A_1, A_2, G, \{S_n\}_{n=0}^{\infty}$ be the same mappings as above. Assume that $\Omega = \cap_{n=0}^{\infty}\text{Fix}(S_n) \cap \text{GSVI}(C, A_1, A_2) \neq \emptyset$. Suppose that $F : C \to X$ is a η-strongly accretive operator with constants $k, \eta > 0$ and k-Lipschitzian, and $f : C \to X$ is L-Lipschitzian mapping. Assume $0 < \rho < (\frac{q\eta}{\kappa_q k^q})^{\frac{1}{q-1}}$, $0 < \mu_i < (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$, $i = 1, 2$, and $0 \leq \gamma L < \tau$, where $\tau = \rho(\eta - \frac{\kappa_q \rho^{q-1} k^q}{q})$.

Recently, Song and Ceng [25] proposed and considered a very general iterative scheme by the modified relaxed extragradient method, i.e., for arbitrary initial $x_0 \in C$, we generate $\{x_n\}$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n\Pi_C(I-\mu_1 A_1)\Pi_C(I-\mu_2 A_2)x_n, \\ x_{n+1} = \Pi_C[((1-\gamma_n)I - \alpha_n\rho F)S_n y_n + \gamma\alpha_n f(x_n) + \gamma_n x_n], \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the conditions: (i) $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ and $\alpha_n \to 0$; (ii) $\limsup_{n\to\infty}\gamma_n < 1$, $\liminf_{n\to\infty}\gamma_n > 0$, $\sum_{n=0}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty$; and (iii) $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\liminf_{n\to\infty}\beta_n > 0$. The authors claimed convergence of $\{x_n\}$ to $x^* \in \Omega$, which deals with the variational inequality: $\langle(\rho F - \gamma f)x^*, J(x^*-p)\rangle \leq 0$, $\forall p \in \Omega$.

On the other hand, Let $A : X \to X$ be an α-inverse-strongly accretive operator, $B : X \to 2^X$ be an m-accretive operator, $f : X \to X$ be a contraction with constant $\delta \in (0,1)$. Assume that the inclusion of finding $x^* \in X$ such that $0 \in (A+B)x^*$, has a solution, i.e., $\Omega = (A+B)^{-1}0 \neq \emptyset$. In 2017, Chang et al. [26] introduced and studied a generalized viscosity implicit rule, i.e., for arbitrary initial $x_0 \in X$, we generate $\{x_n\}$ by

$$x_{n+1} = (1-\alpha_n)J_\lambda^B(I-\lambda A)(t_n x_n + (1-t_n)x_{n+1}) + \alpha_n f(x_n), \quad \forall n \geq 0,$$

where $J_\lambda^B = (I+\lambda B)^{-1}$, $\{t_n\}, \{\alpha_n\} \subset [0,1]$ and $\lambda \in (0,\infty)$ satisfying the conditions: (i) $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$; (ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$; (iii) $0 < \varepsilon \leq t_n \leq t_{n+1} < 1$, $\forall n \geq 0$; and (iv) $0 < \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$. These authors studied and proved convergence of $\{x_n\}$ to $x^* \in \Omega$, solving the inequality: $\langle(I-f)x^*, J(x^*-p)\rangle \leq 0$, $\forall p \in \Omega$. For recent results, we refer the reader to [27–34]. The purpose of this work is to approximate a common solution of GSVI (1), a variational inclusion and a common fixed point problem of a countable family of nonexpansive mappings in spaces with uniformly convex and q-uniformly smooth structures. This paper introduces and considers a generalized Mann viscosity implicit rule, based on the Korpelevich's extragradient method, the implicit approximation method and the Mann's iteration method. We investigate norm convergence of the sequences generated by the generalized Mann viscosity implicit rule to a common solution of the GSVI, VI and CFPP, which solves a hierarchical variational inequality. Our results improve and extend the results reported recently, e.g., Ceng et al. [16], Song and Ceng [25] and Chang et al. [26].

Next, for simplicity, we employ $x_n \rightharpoonup x$ (resp., $x_n \to x$) to present the weak (resp., strong) convergence of the sequence $\{x_n\}$ to x. It is known that $J(tx) = tJ(x)$ and $J(-x) = -J(x)$ for all $t > 0$ and $x \in X$. Then the convex modulus of X is defined by $\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : x, y \in U, \|x-y\| \geq \epsilon\}$, $\forall \epsilon \in [0,2]$. X is said to be uniformly convex if $\delta_X(0) = 0$, and $\delta_X(\epsilon) > 0$ for each $\epsilon \in (0,2]$. Let q be a fixed real number with $q > 1$. Then a Banach space X is said to be q-uniformly convex if $\delta_X(t) \geq ct^q$, $\forall t \in (0,2]$, where $c > 0$. Each Hilbert space H is 2-uniformly convex, while L^p and ℓ_p spaces are $\max\{2,p\}$-uniformly convex for each $p > 1$.

Proposition 1. [35] *Let X be space with smooth and uniformly convex structures, and $r > 0$. Then $g(0) = 0$ and $g(\|x-y\|) \leq \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2$ for all $x, y \in B_r = \{y \in X : \|y\| \leq r\}$, where $g : [0,2r] \to \mathbf{R}$ is a continuous, strictly increasing, and convex function.*

Let $\rho_X : [0,\infty) \to [0,\infty)$ be the smooth modulus of X defined by

$$\rho_X(t) = \sup\{\frac{\|x+y\| + \|x-y\| - 2}{2} : \|x\| = 1, \|y\| \leq t\}.$$

A Banach space X is said to be q-uniformly smooth if $\rho_X(t) \leq ct^q$, $\forall t > 0$, where $c > 0$. It is known that each Hilbert, L^p and ℓ_p spaces are uniformly smooth where $p > 1$. More precisely, each

Hilbert space H is 2-uniformly smooth, while L^p and ℓ_p spaces are $\min\{2,p\}$-uniformly smooth for each $p > 1$. Let $q > 1$. $J_q : X \to 2^{X^*}$, the duality mapping, is defined by

$$J_q(x) := \{\phi \in X^* : \langle x, \phi \rangle = \|x\|^q \text{ and } \|\phi\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

It is quite easy to see that $J_q(x) = J(x)\|x\|^{q-2}$, and if $X = H$, then $J_2 = J = I$ the identity mapping of H.

Proposition 2. [35] *Let $q \in (1,2]$ a given real number and let X be uniformly smooth with order q. Then $\|x+y\|^q - q\langle y, J_q(x)\rangle \le \|x\|^q + \kappa_q\|y\|^q$, $\forall x,y \in X$, where κ_q is the real smooth constant. In particular, if X is uniformly smooth with order 2, then $\|x+y\|^2 - 2\langle y, J(x)\rangle \le \|x\|^2 + \kappa_2\|y\|^2$, $\forall x,y \in X$.*

Using the structures of subdifferentials, we obtain the following tool.

Lemma 1. *Let $q > 1$ and X be a real normed space with the generalized duality mapping J_q. Then, for any given $x,y \in X$, $\|x+y\|^q - q\langle y, j_q(x+y)\rangle \le \|x\|^q$, $\forall j_q(x+y) \in J_q(x+y)$.*

Let D be a set in set C and let Π map C into D. We say that Π is sunny if $\Pi(x) = \Pi[t(x - \Pi(x)) + \Pi(x)]$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. We say Π is a retraction if $\Pi = \Pi^2$. We say that a subset D of C is a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Proposition 3. [36] *Let X be smooth, D be a non-empty set in C and Π be a retraction onto D. (i) Π is nonexpansive sunny; (ii) $\langle x - y, J(\Pi(x) - \Pi(y))\rangle \ge \|\Pi(x) - \Pi(y)\|^2$, $\forall x,y \in C$; (iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \le 0$, $\forall x \in C, y \in D$. Then the above relations are equivalent to each other.*

Let $A : C \to 2^X$ be a set-valued operator with $Ax \neq \emptyset$, $\forall x \in C$. Let $q > 1$. An operator A is accretive if for each $x,y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that $\langle j_q(x-y), u - v\rangle \ge 0$, $\forall u \in Ax, v \in Ay$. An accretive operator A is inverse-strongly accretive of order q, i.e., α-inverse-strongly accretive, if for each $x,y \in C$, there exist $\alpha > 0$ such that $\langle u - v, j_q(x-y)\rangle \ge \alpha\|Ax - Ay\|^q$, $\forall u \in Ax, v \in Ay$, where $j_q(x-y) \in J_q(x-y)$. In a Hilbert space H, $A : C \to H$ is called α-inverse-strongly monotone.

Operator A is said to be m-accretive if and only if $(I + \lambda A)C = X$ for all $\lambda > 0$ and A is accretive. One defines the mapping $J_\lambda^A : (I + \lambda A)C \to C$ by $J_\lambda^A = (I + \lambda A)^{-1}$ with real constant $\lambda > 0$. Such J_λ^A is called the resolvent mapping of A for each $\lambda > 0$.

Lemma 2. [37] *The following statements hold:*

(i) *the resolvent identity: $J_\lambda^A x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^A x)$, $\forall \lambda, \mu > 0$, $x \in X$;*
(ii) *if J_λ^A is a resolvent of A for $\lambda > 0$, then J_λ^A is a single-valued nonexpansive mapping with $\mathrm{Fix}(J_\lambda^A) = A^{-1}0$, where $A^{-1}0 = \{x \in C : 0 \in Ax\}$;*
(iii) *in a Hilbert space H, an operator A is maximal monotone iff it is m-accretive.*

Let $A : C \to X$ be an α-inverse-strongly accretive mapping and $B : C \to 2^X$ be an m-accretive operator. In the sequel, one will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)$, $\forall \lambda > 0$. The following statements (see [38]) hold:

(i) $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$, $\forall \lambda > 0$;
(ii) $\|x - T_\lambda x\| \le 2\|x - T_s x\|$ for $0 < \lambda \le s$ and $x \in X$.

Proposition 4. [38] *Let X be a Banach space with the uniformly convex and q-uniformly smooth structures with $1 < q \leq 2$. Assume that $A : C \to X$ is a α-inverse-strongly accretive single-valued mapping and $B : C \to 2^X$ is an m-accretive operator. Then*

$$\|T_\lambda x - T_\lambda y\|^q \leq \|x-y\|^q - \lambda(\alpha q - \lambda^{q-1}\kappa_q)\|Ax - Ay\|^q - \phi(\|(I-J_\lambda^B)(I-\lambda A)x - (I-J_\lambda^B)(I-\lambda A)y\|),$$

for all $x, y \in \widetilde{B}_r := \{x \in C : \|x\| \leq r\}$, where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\phi(0) = 0$ is a convex, strictly increasing and continuous function, λ and r two positive real constants, κ_q is the real smooth constant of X, and T_λ and J_λ^B are resolvent operators defined as above. In particular, if $0 < \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$, then T_λ is nonexpansive.

Lemma 3. [39] *Let X be uniformly smooth, T be single-valued nonexpansivitity on C with $\mathrm{Fix}(T) \neq \emptyset$, and $f : C \to C$ be a any contraction. For each $t \in (0,1)$, one employs $z_t \in C$ to present the unique fixed point of the new contraction $C \ni z \mapsto tf(z) + (1-t)Tz$ on C, i.e., $z_t = (1-t)Tz_t + tf(z_t)$. Then $\{z_t\}$ converges to $x^* \in \mathrm{Fix}(T)$ in norm, which deals with the variational inequality: $\langle (I-f)x^*, J(x^* - p)\rangle \leq 0$, $\forall p \in \mathrm{Fix}(T)$.*

Lemma 4. [25] *Let X be a uniformly smooth with order q. Suppose that Π_C is a sunny nonexpansive retraction from X onto C. Let the mapping $A_i : C \to X$ be α_i-inverse-strongly accretive of order q for $i = 1, 2$. Let the mapping $G : C \to C$ be defined as $Gx := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$, $\forall x \in C$. If $0 < \mu_i \leq (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$, then $G : C \to C$ is nonexpansive. For given $(x^*, y^*) \in C \times C$, (x^*, y^*) is a solution of GSVI (1) if and only if $x^* = \Pi_C(y^* - \mu_1 A_1 y^*)$ where $y^* = \Pi_C(x^* - \mu_2 A_2 x^*)$, i.e., $x^* = Gx^*$.*

Lemma 5. [40] *Let $\{S_n\}_{n=0}^\infty$ be a mapping sequence on C. Suppose that $\sum_{n=1}^\infty \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty$. Then $\{S_n x\}$ converges to some point of C in norm for each $x \in C$. Besides, we present S, a self-mapping, on C by $Sx = \lim_{n\to\infty} S_n x$, $\forall x \in C$. Then $\lim_{n\to\infty} \sup\{\|S_n x - Sx\| : x \in C\} = 0$.*

Lemma 6. [41] *Let X be Banach space. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$. Let $x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n$, $\forall n \geq 0$ and $\limsup_{n\to\infty}(\|y_n - y_{n+1}\| - \|x_{n+1} - x_n\|) \leq 0$, where $\{x_n\}$ and $\{y_n\}$ be bounded sequences in X. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 7. [42] *Let X be strictly convex, and $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings on C. Suppose that $\cap_{n=0}^\infty \mathrm{Fix}(T_n) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^\infty \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^\infty \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \cap_{n=0}^\infty \mathrm{Fix}(T_n)$ holds.*

Lemma 8. [43] *Let $\{a_n\}$ be a non-negative number sequence of with $a_{n+1} \leq a_n(1-\lambda_n) + \lambda_n\gamma_n$, $\forall n \geq 1$, where $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences such that (a) $\sum_{n=1}^\infty |\lambda_n\gamma_n| < \infty$ (or $\limsup_{n\to\infty} \gamma_n \leq 0$) and (b) $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=1}^\infty \lambda_n = \infty$. Then a_n goes to zero as n goes to the infinity.*

Lemma 9. [7,35] *Let X be uniformly convex, and the ball $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then*

$$\|\alpha x + \beta y + \gamma y\|^2 + \alpha\beta g(\|x-y\|) \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2$$

for all $x, y, z \in B_r$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, where $g : [0,\infty) \to [0,\infty)$ is a convex, continuous and strictly increasing function.

3. Iterative Algorithms and Convergence Criteria

Space X presents a real Banach space and its topological dual is X^*, and C is a non-empty convex and closed set in space X. We are now ready to state and prove the main results in this paper.

Theorem 1. *Let X be uniformly convex and uniformly smooth with the constant $1 < q \leq 2$. Let Π_C be a nonexpansive sunny retraction from X onto C. Assume that the mappings $A, A_i : C \to X$ are inverse-strongly accretive of order q and α_i-inverse-strongly accretive of order q, respectively for $i = 1, 2$. Let $B : C \to 2^X$ be an* *m-accretive operator, and let* $\{S_n\}_{n=0}^{\infty}$ *be a countable family of nonexpansive single-valued self-mappings on* C *such that* $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A+B)^{-1}0 \neq \emptyset$ *where* $\mathrm{GSVI}(C, A_1, A_2)$ *is the fixed point set of* $G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$ *with* $0 < \mu_i < (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ *for* $i = 1, 2$. *Let* $f : C \to C$ *be a contraction with constant* $\delta \in (0, 1)$. *For arbitrary initial* $x_0 \in C$, $\{x_n\}$ *is a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(y_n) + \gamma_n J_{\lambda_n}^B (I - \lambda_n A)(t_n x_n + (1 - t_n) y_n) + \beta_n x_n, \\ v_n = \Pi_C(I - \mu_1 A_1)\Pi_C(y_n - \mu_2 A_2 y_n), \\ x_{n+1} = (1 - \delta_n) S_n v_n + \delta_n x_n, \quad n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ *satisfy*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = \lim_{n\to\infty} |\gamma_n - \gamma_{n-1}| = 0$;
(iii) $\lim_{n\to\infty} |t_n - t_{n-1}| = 0$ *and* $\liminf_{n\to\infty} \gamma_n(1 - t_n) > 0$;
(iv) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\limsup_{n\to\infty} \delta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;
(v) $0 < \bar{\lambda} \leq \lambda_n$, $\forall n \geq 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|(S_n - S_{n-1})x\| < \infty$ *for any bounded set* D, *which is subset of* C *and let* S *be a self-mapping* $Sx = \lim_{n\to\infty} S_n x$, $\forall x \in C$ *and suppose that* $\mathrm{Fix}(S) = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. *Then* $x_n \to x^* \in \Omega$, *which solves* $\langle (I - f)x^*, J(x^* - p)\rangle \leq 0$, $\forall p \in \Omega$ *uniquely.*

Proof. Set $u_n = \Pi_C(y_n - \mu_2 A_2 y_n)$. It is not hard to find that our scheme can be re-written by

$$\begin{cases} y_n = \alpha_n f(y_n) + \beta_n x_n + \gamma_n T_n(t_n x_n + (1 - t_n) y_n), \\ x_{n+1} = (1 - \delta_n) S_n G y_n + \delta_n x_n, \quad n \geq 0, \end{cases} \tag{2}$$

where $T_n := J_{\lambda_n}^B (I - \lambda_n A)$, $\forall n \geq 0$. By condition (v) and Proposition 4, one observes that $T_n : C \to C$ is a nonexpansive mapping for each $n \geq 0$. Since $\alpha_n + \beta_n + \gamma_n = 1$, we know that

$$\alpha_n\delta + \gamma_n(1 - t_n) + \beta_n + \gamma_n t_n = \alpha_n\delta + \gamma_n + \beta_n = 1 - \alpha_n(1 - \delta), \quad \forall n \geq 0.$$

One first claims that the sequence $\{x_n\}$ generated by (2) is well defined. Indeed, for each fixed $x_n \in C$, one defines a mapping $F_n : C \to C$ by $F_n(x) = \alpha_n f(x) + \beta_n x_n + \gamma_n T_n(t_n x_n + (1 - t_n)x)$, $\forall x \in C$. Then, one gets, for any $x, y \in C$,

$$\begin{aligned} &\|F_n(x) - F_n(y)\| \leq \alpha_n \|f(x) - f(y)\| + \gamma_n \|T_n(t_n x_n + (1 - t_n)x) - T_n(t_n x_n + (1 - t_n)y)\| \\ &\leq \alpha_n\delta\|x - y\| + \gamma_n(1 - t_n)\|x - y\| = (\alpha_n\delta + \gamma_n(1 - t_n))\|x - y\| \leq (1 - \alpha_n(1 - \delta))\|x - y\|. \end{aligned}$$

This implies that F_n is a strictly contraction operator. Hence the Banach fixed-point theorem ensures that there is a unique fixed point $y_n \in C$ satisfying

$$y_n = \alpha_n f(y_n) + \beta_n x_n + \gamma_n T_n(t_n x_n + (1 - t_n) y_n).$$

Next, one claims that $\{x_n\}$ is bounded. Indeed, arbitrarily take a fixed $p \in \Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A + B)^{-1}0$. One knows that $S_n p = p$, $Gp = p$ and $T_n p = p$. Moreover, by using the nonexpansivity of T_n, we have

$$\begin{aligned} \|y_n - p\| &\leq \alpha_n(\|f(y_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|T_n(t_n x_n + (1 - t_n)y_n) - p\| \\ &\leq \alpha_n(\delta\|y_n - p\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n[t_n\|x_n - p\| + (1 - t_n)\|y_n - p\|] \\ &= (\alpha_n\delta + \gamma_n(1 - t_n))\|y_n - p\| + (\beta_n + \gamma_n t_n)\|x_n - p\| + \alpha_n\|p - f(p)\|, \end{aligned}$$

which hence implies that

$$\begin{aligned}\|y_n - p\| &\le \tfrac{\gamma_n t_n+\beta_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|x_n - p\| + \tfrac{\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p) - p\| \\ &= \tfrac{1-\alpha_n(1-\delta)-(\alpha_n\delta+\gamma_n(1-t_n))}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|x_n - p\| + \tfrac{\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p) - p\| \\ &= (1 - \tfrac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n - p\| + \tfrac{\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p) - p\|.\end{aligned} \tag{3}$$

Thus, from (2) and (3), we have

$$\begin{aligned}&\|x_{n+1} - p\| \le \delta_n\|x_n - p\| + (1-\delta_n)\|S_nGy_n - p\| \le (1-\delta_n)\|y_n - p\| + \delta_n\|x_n - p\| \\ &\le \delta_n\|x_n - p\| + (1-\delta_n)\{(1 - \tfrac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n - p\| + \tfrac{\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p) - p\|\} \\ &= [1 - \tfrac{(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\alpha_n]\|x_n - p\| + \tfrac{(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\alpha_n\tfrac{\|f(p)-p\|}{1-\delta}.\end{aligned}$$

By induction, we get that $\{x_n\}$ is bounded. Please note that G is non-expansive thanks to Lemma 4. Using (3) and the nonexpansivity of $I - \mu_1 A_1,\ I - \mu_2 A_2, S_n, T_n$ and G, it is guaranteed that $\{u_n\}, \{v_n\}, \{y_n\}, \{Gy_n\}, \{S_n v_n\}$ and $\{T_n z_n\}$ are bounded too, where $u_n := \Pi_C(I - \mu_2 A_2)y_n,\ v_n := \Pi_C(I - \mu_1 A_1)u_n$ and $z_n := t_n x_n + (1 - t_n)y_n$ for all $n \ge 0$. Thanks to (2), we have

$$\begin{cases} z_n = t_n(x_n - y_n) + y_n, \\ z_{n-1} = t_{n-1}(x_{n-1} - y_{n-1}) + y_{n-1}, \quad \forall n \ge 1, \end{cases}$$

and

$$\begin{cases} y_n = \alpha_n f(y_n) + \beta_n x_n + \gamma_n T_n z_n, \\ y_{n-1} = \alpha_{n-1} f(y_{n-1}) + \beta_{n-1} x_{n-1} + \gamma_{n-1} T_{n-1} z_{n-1}, \quad \forall n \ge 1, \end{cases}$$

Simple calculations show that

$$z_n - z_{n-1} = (t_n - t_{n-1})(x_{n-1} - y_{n-1}) + (1 - t_n)(y_n - y_{n-1}) + t_n(x_n - x_{n-1}),$$

and

$$\begin{aligned}y_n - y_{n-1} &= (\alpha_n - \alpha_{n-1})f(y_{n-1}) + \alpha_n(f(y_n) - f(y_{n-1})) + \beta_n(x_n - x_{n-1}) \\ &\quad + (\beta_n - \beta_{n-1})x_{n-1} + \gamma_n(T_n z_n - T_{n-1}z_{n-1}) + (\gamma_n - \gamma_{n-1})T_{n-1}z_{n-1}.\end{aligned} \tag{4}$$

It follows from the resolvent identity that

$$\begin{aligned}&\|T_n z_n - T_{n-1}z_{n-1}\| \le \|T_n z_n - T_n z_{n-1}\| + \|T_n z_{n-1} - T_{n-1}z_{n-1}\| \\ &\le \|z_n - z_{n-1}\| + \|J^B_{\lambda_n}(I - \lambda_n A)z_{n-1} - J^B_{\lambda_{n-1}}(I - \lambda_{n-1}A)z_{n-1}\| \\ &\le \|z_n - z_{n-1}\| + \|J^B_{\lambda_n}(I - \lambda_n A)z_{n-1} - J^B_{\lambda_{n-1}}(I - \lambda_n A)z_{n-1}\| \\ &\quad + \|J^B_{\lambda_{n-1}}(I - \lambda_n A)z_{n-1} - J^B_{\lambda_{n-1}}(I - \lambda_{n-1}A)z_{n-1}\| \\ &= \|z_n - z_{n-1}\| + \|J^B_{\lambda_{n-1}}(\tfrac{\lambda_{n-1}}{\lambda_n}I + (1 - \tfrac{\lambda_{n-1}}{\lambda_n})J^B_{\lambda_n})(I - \lambda_n A)z_{n-1} - J^B_{\lambda_{n-1}}(I - \lambda_n A)z_{n-1}\| \\ &\quad + \|J^B_{\lambda_{n-1}}(I - \lambda_n A)z_{n-1} - J^B_{\lambda_{n-1}}(I - \lambda_{n-1}A)z_{n-1}\| \\ &\le t_n\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\|x_{n-1} - y_{n-1}\| + (1 - t_n)\|y_n - y_{n-1}\| \\ &\quad + |1 - \tfrac{\lambda_{n-1}}{\lambda_n}|\|J^B_{\lambda_n}(I - \lambda_n A)z_{n-1} - (I - \lambda_n A)z_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| \\ &\le t_n\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\|x_{n-1} - y_{n-1}\| + (1 - t_n)\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1,\end{aligned} \tag{5}$$

where

$$\sup_{n\ge1}\{\frac{\|J^B_{\lambda_n}(I - \lambda_n A)z_{n-1} - (I - \lambda_n A)z_{n-1}\|}{\bar{\lambda}} + \|Az_{n-1}\|\} \le M_1$$

for some $M_1 > 0$. This together with (4), implies that

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + \beta_n\|x_{n-1} - x_n\| + \alpha_n\|f(y_n) - f(y_{n-1})\| \\ &\quad + \gamma_n\|T_n z_n - T_{n-1}z_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|T_{n-1}z_{n-1}\| + |\beta_{n-1} - \beta_n|\|x_{n-1}\| \\ &\leq \alpha_n\delta\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(y_{n-1})\| + \beta_n\|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\{t_n\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\|x_{n-1} - y_{n-1}\| \\ &\quad + (1 - t_n)\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1\} + |\gamma_n - \gamma_{n-1}|\|T_{n-1}z_{n-1}\| \\ &\leq (\alpha_n\delta + \gamma_n(1 - t_n))\|y_n - y_{n-1}\| + (\beta_n + \gamma_n t_n)\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| \\ &\quad + |\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n| + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_2,\end{aligned}$$

where

$$\sup_{n\geq 0}\{\|f(y_n)\| + \|x_n\| + \|y_n\| + M_1 + \|T_n z_n\|\} \leq M_2$$

for some $M_2 > 0$. So it follows that

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \frac{\beta_n + \gamma_n t_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|x_{n-1} - x_n\| + \frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}(|\alpha_n - \alpha_{n-1}| \\ &\quad + |\beta_n - \beta_{n-1}| + |\gamma_{n-1} - \gamma_n| + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_2 \\ &= (1 - \frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_{n-1} - x_n\| + \frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}(|\alpha_n - \alpha_{n-1}| \\ &\quad + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_2 \\ &\leq \|x_n - x_{n-1}\| + \frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\ &\quad + |\gamma_{n-1} - \gamma_n| + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_2.\end{aligned}$$

Hence we get

$$\begin{aligned}&\|S_nGy_n - S_{n-1}Gy_{n-1}\| \leq \|S_nGy_n - S_nGy_{n-1}\| + \|S_{n-1}Gy_{n-1} - S_nGy_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + \|S_nGy_{n-1} - S_{n-1}Gy_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + \frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}(|\beta_{n-1} - \beta_n| + |\alpha_n - \alpha_{n-1}| \\ &\quad + |\gamma_n - \gamma_{n-1}| + |t_{n-1} - t_n| + |\lambda_n - \lambda_{n-1}|)M_2 + \|S_nGy_{n-1} - S_{n-1}Gy_{n-1}\|.\end{aligned}$$

Consequently,

$$\begin{aligned}\|S_nGy_n - S_{n-1}Gy_{n-1}\| - \|x_n - x_{n-1}\| &\leq \frac{1}{1-(\alpha_n\delta+\gamma_n(1-t_n))}(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\ &\quad + |\gamma_n - \gamma_{n-1}| + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_2 + \|S_nGy_{n-1} - S_{n-1}Gy_{n-1}\|.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\sup_{x\in D}\|(S_n - S_{n-1})x\| < \infty$ for bounded subset $D = \{Gy_n : n \geq 0\}$ of C (due to the assumption), we know that $\lim_{n\to\infty}\|(S_nG - S_{n-1}G)y_{n-1}\| = 0$. Please note that $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\lambda_n = \lambda$ and $\liminf_{n\to\infty}\gamma_n(1 - t_n) > 0$. Thus, from $|\beta_n - \beta_{n-1}| \to 0$, $|\gamma_n - \gamma_{n-1}| \to 0$ and $|t_n - t_{n-1}| \to 0$ as $n \to \infty$ (due to conditions (ii), (iii)), we get

$$\limsup_{n\to\infty}(\|S_nGy_n - S_{n-1}Gy_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0.$$

So it follows from condition (iv) and Lemma 6 that $\lim_{n\to\infty}\|S_nGy_n - x_n\| = 0$. Hence we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \delta_n)\|S_nGy_n - x_n\| = 0. \tag{6}$$

Let $\bar{p} := \Pi_C(I - \mu_2 A_2)p$. Please note that $u_n = \Pi_C(I - \mu_2 A_2)y_n$ and $v_n = \Pi_C(I - \mu_1 A_1)u_n$. Then $v_n = Gy_n$. From Proposition 4 (see also Lemma 2.13 in [25]), we have

$$\begin{aligned}\|u_n - \bar{p}\|^q &= \|\Pi_C(I - \mu_2 A_2)y_n - \Pi_C(I - \mu_2 A_2)p\|^q \\ &\leq \|(I - \mu_2 A_2)y_n - (I - \mu_2 A_2)p\|^q \\ &\leq \|y_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q,\end{aligned} \tag{7}$$

and

$$\begin{aligned} \|v_n - p\|^q &= \|\Pi_C(I-\mu_1 A_1)u_n - \Pi_C(I-\mu_1 A_1)\bar{p}\|^q \\ &\le \|(I-\mu_1 A_1)u_n - (I-\mu_1 A_1)\bar{p}\|^q \\ &\le \|u_n - \bar{p}\|^q - \mu_1(q\alpha_1 - \kappa_q \mu_1^{q-1})\|A_1 u_n - A_1 \bar{p}\|^q. \end{aligned} \tag{8}$$

Substituting (7) to (8), we obtain

$$\begin{aligned} \|v_n - p\|^q &\le \|y_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q \mu_2^{q-1})\|A_2 y_n - A_2 p\|^q \\ &\quad - \mu_1(q\alpha_1 - \kappa_q \mu_1^{q-1})\|A_1 u_n - A_1 \bar{p}\|^q. \end{aligned} \tag{9}$$

According to Proposition 4, we obtain from (2) that $\|z_n - p\|^q \le t_n\|x_n - p\|^q + (1-t_n)\|y_n - p\|^q$, and hence

$$\begin{aligned} &\|y_n - p\|^q = \|\beta_n(p - x_n) + \alpha_n(f(p) - f(y_n)) + \gamma_n(p - T_n z_n) + \alpha_n(p - f(p))\|^q \\ &\le \|\alpha_n(f(y_n) - f(p)) + \beta_n(x_n - p) + \gamma_n(T_n z_n - p)\|^q + q\alpha_n\langle f(p) - p, J_q(y_n - p)\rangle \\ &\le \alpha_n\|f(y_n) - f(p)\|^q + \beta_n\|x_n - p\|^q + \gamma_n\|T_n z_n - p\|^q + q\alpha_n\langle f(p) - p, J_q(y_n - p)\rangle \\ &\le \alpha_n\delta\|y_n - p\|^q + \beta_n\|x_n - p\|^q + \gamma_n[t_n\|x_n - p\|^q + (1-t_n)\|y_n - p\|^q] \\ &\quad + q\alpha_n\|f(p) - p\|\|y_n - p\|^{q-1}, \end{aligned}$$

which immediately yields

$$\|y_n - p\|^q \le (1 - \frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta + \gamma_n(1-t_n))})\|x_n - p\|^q + \frac{q\alpha_n}{1-(\alpha_n\delta + \gamma_n(1-t_n))}\|f(p) - p\|\|y_n - p\|^{q-1}.$$

This, together with the convexity of $\|\cdot\|^q$ and (9), leads to

$$\begin{aligned} &\|x_{n+1} - p\|^q = \|\delta_n(x_n - p) + (1-\delta_n)(S_n G y_n - p)\|^q \\ &\le \delta_n\|x_n - p\|^q + (1-\delta_n)\|S_n v_n - p\|^q \\ &\le \delta_n\|x_n - p\|^q + (1-\delta_n)\{\|y_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q \\ &\quad - \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1 u_n - A_1\bar{p}\|^q\} \\ &\le \delta_n\|x_n - p\|^q + (1-\delta_n)\{(1 - \tfrac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n - p\|^q + \tfrac{q\|f(p)-p\|\|y_n-p\|^{q-1}}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\alpha_n \\ &\quad - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q - \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1 u_n - A_1\bar{p}\|^q\} \\ &= (1 - \tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n - p\|^q + \tfrac{q(1-\delta_n)\|f(p)-p\|\|y_n-p\|^{q-1}}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\alpha_n \\ &\quad - (1-\delta_n)[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1 u_n - A_1\bar{p}\|^q] \\ &\le \|x_n - p\|^q - (1-\delta_n)[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q \\ &\quad + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1 u_n - A_1\bar{p}\|^q] + \alpha_n M_3, \end{aligned} \tag{10}$$

where

$$\sup_{n\ge 0}\{\frac{q(1-\delta_n)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p) - p\|\|y_n - p\|^{q-1}\} \le M_3$$

for some $M_3 > 0$. So it follows from (10) and Proposition 2 that

$$\begin{aligned} &(1-\delta_n)[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1 u_n - A_1\bar{p}\|^q] \\ &\le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n M_3 \\ &\le q\|x_n - x_{n+1}\|\|x_{n+1} - p\|^{q-1} + \kappa_q\|x_n - x_{n+1}\|^q + \alpha_n M_3. \end{aligned}$$

Since $0 < \mu_i < (\frac{q\alpha_i}{\kappa_q})^{\frac{1}{q-1}}$ for $i = 1, 2$, from conditions (i), (iv) and (6) we get

$$\lim_{n\to\infty}\|A_2 y_n - A_2 p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|A_1 u_n - A_1\bar{p}\| = 0. \tag{11}$$

Using Proposition 1, we have

$$\begin{aligned}&\|u_n-\bar{p}\|^2=\|\Pi_C(I-\mu_2A_2)y_n-\Pi_C(I-\mu_2A_2)p\|^2\\&\le\langle(I-\mu_2A_2)y_n-(I-\mu_2A_2)p,J(u_n-\bar{p})\rangle\\&=\langle y_n-p,J(u_n-\bar{p})\rangle+\mu_2\langle A_2p-A_2y_n,J(u_n-\bar{p})\rangle\\&\le\tfrac{1}{2}[\|y_n-p\|^2+\|u_n-\bar{p}\|^2-g_1(\|y_n-u_n-(p-\bar{p})\|)]+\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|,\end{aligned}$$

where g_1 is given by Proposition 1. This yields

$$\|u_n-\bar{p}\|^2\le\|y_n-p\|^2-g_1(\|y_n-u_n-(p-\bar{p})\|)+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|. \tag{12}$$

In the same way, we derive

$$\begin{aligned}\|v_n-p\|^2&=\|\Pi_C(I-\mu_1A_1)u_n-\Pi_C(I-\mu_1A_1)\bar{p}\|^2\\&\le\langle(I-\mu_1A_1)u_n-(I-\mu_1A_1)\bar{p},J(v_n-p)\rangle\\&=\langle u_n-\bar{p},J(v_n-p)\rangle+\mu_1\langle A_1\bar{p}-A_1u_n,J(v_n-p)\rangle\\&\le\tfrac{1}{2}[\|u_n-\bar{p}\|^2+\|v_n-p\|^2-g_2(\|u_n-v_n+(p-\bar{p})\|)]+\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|,\end{aligned}$$

where g_2 is given by Proposition 1. This yields

$$\|v_n-p\|^2\le\|u_n-\bar{p}\|^2-g_2(\|u_n-v_n+(p-\bar{p})\|)+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|. \tag{13}$$

Substituting (12) for (13), we get

$$\begin{aligned}\|v_n-p\|^2&\le\|y_n-p\|^2-g_1(\|y_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|)\\&\quad+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|.\end{aligned} \tag{14}$$

Please note that $\|\cdot\|^2$ is convex. Using Proposition 1, Lemmas 1 and 9, one concludes

$$\begin{aligned}\|y_n-p\|^2&\le\|\beta_n(p-x_n)+\alpha_n(f(p)-f(y_n))+\gamma_n(p-T_nz_n)\|^2+2\alpha_n\langle f(p)-p,J(y_n-p)\rangle\\&\le\alpha_n\|f(p)-f(y_n)\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|T_nz_n-p\|^2-\beta_n\gamma_ng_3(\|x_n-T_nz_n\|)\\&\quad+2\alpha_n\langle f(p)-p,J(y_n-p)\rangle\\&\le\alpha_n\delta\|p-y_n\|^2+\beta_n\|x_n-p\|^2+\gamma_n(t_n\|x_n-p\|^2+(1-t_n)\|y_n-p\|^2)\\&\quad+2\alpha_n\|f(p)-p\|\|p-y_n\|-\beta_n\gamma_ng_3(\|x_n-T_nz_n\|),\end{aligned}$$

which immediately sends

$$\begin{aligned}\|y_n-p\|^2&\le\Big(1-\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\Big)\|x_n-p\|^2+\frac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p)-p\|\|y_n-p\|\\&\quad-\frac{\beta_n\gamma_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}g_3(\|x_n-T_nz_n\|).\end{aligned}$$

This together with (2) and (14) leads to

$$\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\delta_n(x_n-p)+(1-\delta_n)(S_nGy_n-p)\|^2 \\
&\le \delta_n\|p-x_n\|^2+(1-\delta_n)\|S_nv_n-p\|^2 \\
&\le \delta_n\|p-x_n\|^2+(1-\delta_n)\{\|y_n-p\|^2-g_1(\|y_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|) \\
&\quad +2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\} \\
&\le \delta_n\|x_n-p\|^2+(1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n-p\|^2 \\
&\quad +\tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p)-p\|\|y_n-p\|-\tfrac{\beta_n\gamma_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}g_3(\|x_n-T_nz_n\|) \\
&\quad -g_1(\|y_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|) \\
&\quad +2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\} \\
&\le (1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n-p\|^2+\tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p)-p\|\|y_n-p\| \\
&\quad -(1-\delta_n)[\tfrac{\beta_n\gamma_ng_3(\|x_n-T_nz_n\|)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}+g_1(\|y_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|)] \\
&\quad +2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| \\
&\le \|x_n-p\|^2-(1-\delta_n)[\tfrac{\beta_n\gamma_ng_3(\|x_n-T_nz_n\|)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}+g_1(\|y_n-u_n-(p-\bar{p})\|) \\
&\quad +g_2(\|u_n-v_n+(p-\bar{p})\|)]+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| \\
&\quad +\tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p)-p\|\|y_n-p\|,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
&(1-\delta_n)[\tfrac{\beta_n\gamma_ng_3(\|x_n-T_nz_n\|)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}+g_1(\|y_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|)] \\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| \\
&\quad +\tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p)-p\|\|y_n-p\| \\
&\le (\|p-x_n\|+\|p-x_{n+1}\|)\|x_n-x_{n+1}\|+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\| \\
&\quad +2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\|f(p)-p\|\|y_n-p\|.
\end{aligned}$$

Using (6) and (11), from $\liminf_{n\to\infty}\beta_n\gamma_n>0$, and $\liminf_{n\to\infty}(1-\delta_n)>0$, we have

$$\lim_{n\to\infty}g_1(\|y_n-u_n-(p-\bar{p})\|)=\lim_{n\to\infty}g_2(\|u_n-v_n+(p-\bar{p})\|)=\lim_{n\to\infty}g_3(\|x_n-T_nz_n\|)=0.$$

Using the properties of g_1,g_2 and g_3, we deduce that

$$\lim_{n\to\infty}\|y_n-u_n-(p-\bar{p})\|=\lim_{n\to\infty}\|u_n-v_n+(p-\bar{p})\|=\lim_{n\to\infty}\|x_n-T_nz_n\|=0. \tag{15}$$

From (15) we get

$$\|y_n-Gy_n\|=\|y_n-v_n\|\le\|y_n-u_n-(p-\bar{p})\|+\|u_n-v_n+(p-\bar{p})\|\to 0\quad(n\to\infty). \tag{16}$$

Meantime, again from (2) we have $y_n-x_n=\alpha_n(f(y_n)-x_n)+\gamma_n(T_nz_n-x_n)$. Hence from (15) we get $\|y_n-x_n\|\le\alpha_n\|f(y_n)-x_n\|+\|T_nz_n-x_n\|\to 0$ $(n\to\infty)$. This together with (16), implies that

$$\begin{aligned}
\|x_n-Gx_n\| &\le \|x_n-y_n\|+\|y_n-Gy_n\|+\|Gy_n-Gx_n\| \\
&\le \|y_n-Gy_n\|+2\|x_n-y_n\|\to 0\quad(n\to\infty).
\end{aligned} \tag{17}$$

Next, one claims that $\|x_n-Sx_n\|\to 0$, $\|x_n-T_\lambda x_n\|\to 0$ and $\|x_n-Wx_n\|\to 0$ as $n\to\infty$, where $Sx=\lim_{n\to\infty}S_nx$, $\forall x\in C$, $T_\lambda=J_\lambda^B(I-\lambda A)$ and $Wx=\theta_1Sx+\theta_2Gx+\theta_3T_\lambda x$, $\forall x\in C$ for constants $\theta_1,\theta_2,\theta_3\in(0,1)$ satisfying $\theta_1+\theta_2+\theta_3=1$. Indeed, since $x_{n+1}=\delta_nx_n+(1-\delta_n)S_nGy_n$ leads to $\|S_nGy_n-x_n\|=\frac{1}{1-\delta_n}\|x_{n+1}-x_n\|$, we deduce from (17), $\liminf_{n\to\infty}(1-\delta_n)>0$ and $x_n-y_n\to 0$ that

$$\begin{aligned}
\|S_nx_n-x_n\| &\le \|S_nx_n-S_nGx_n\|+\|S_nGx_n-S_nGy_n\|+\|S_nGy_n-x_n\| \\
&\le \|x_n-Gx_n\|+\|x_n-y_n\|+\tfrac{1}{1-\delta_n}\|x_{n+1}-x_n\|\to 0\quad(n\to\infty),
\end{aligned}$$

which implies that

$$\|Sx_n - x_n\| \le \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \to 0 \quad (n \to \infty). \tag{18}$$

Furthermore, using the similar arguments to those of (5), we obtain

$$\begin{aligned}\|T_n z_n - T_\lambda z_n\| &\le |1 - \tfrac{\lambda}{\lambda_n}|\|J^B_{\lambda_n}(I - \lambda_n A)z_n - (I - \lambda_n A)z_n\| + |\lambda_n - \lambda|\|Az_n\| \\ &= |1 - \tfrac{\lambda}{\lambda_n}|\|T_n z_n - (I - \lambda_n A)z_n\| + |\lambda_n - \lambda|\|Az_n\|.\end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$ and the sequences $\{z_n\}, \{T_n z_n\}, \{Az_n\}$ are bounded, we get

$$\lim_{n\to\infty} \|T_n z_n - T_\lambda z_n\| = 0. \tag{19}$$

Taking into account condition (v), i.e., $0 < \bar{\lambda} \le \lambda_n$, $\forall n \ge 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, we know that $0 < \bar{\lambda} \le \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$. So it follows from Proposition 4 that $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$ and $T_\lambda : C \to C$ is nonexpansive. Therefore, we deduce from (15), (19) and $x_n - y_n \to 0$ that

$$\begin{aligned}\|T_\lambda x_n - x_n\| &\le \|T_\lambda x_n - T_\lambda z_n\| + \|T_\lambda z_n - T_n z_n\| + \|T_n z_n - x_n\| \\ &\le \|x_n - z_n\| + \|T_\lambda z_n - T_n z_n\| + \|T_n z_n - x_n\| \\ &\le \|x_n - y_n\| + \|T_\lambda z_n - T_n z_n\| + \|T_n z_n - x_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{20}$$

We now define the mapping $Wx = \theta_1 Sx + \theta_2 Gx + \theta_3 T_\lambda x$, $\forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0,1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. So by using Lemma 7, we know that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \Omega$. One observes that

$$\begin{aligned}\|x_n - Wx_n\| &= \|\theta_1(x_n - Sx_n) + \theta_2(x_n - Gx_n) + \theta_3(x_n - T_\lambda x_n)\| \\ &\le \theta_1\|x_n - Sx_n\| + \theta_2\|x_n - Gx_n\| + \theta_3\|x_n - T_\lambda x_n\|.\end{aligned} \tag{21}$$

From (17), (18), (20) and (21), we get

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0.$$

The next step is to claim

$$\limsup_{n\to\infty} \langle J(x_n - x^*), f(x^*) - x^* \rangle \le 0, \tag{22}$$

with $x^* =$s-$\lim_{n\to\infty} x_t$, where x_t is a fixed point of $x \mapsto tf(x) + (1-t)Wx$ for each $t \in (0,1)$. Please note that the existence of x^* ($x^* \in \mathrm{Fix}(W)$) is from Lemma 3. Indeed, the Banach contraction mapping principle guarantees that for each $t \in (0,1)$, x_t satisfies $x_t = tf(x_t) + (1-t)Wx_t$. Hence we have $\|x_t - x_n\| = \|(Wx_t - x_n)(1-t) + (f(x_t) - x_n)t\|$. Using the known subdifferential inequality (see [7]), we conclude that

$$\begin{aligned}\|x_n - x_t\|^2 &\le 2t\langle x_n - f(x_t), J(x_n - x_t)\rangle + (1-t)^2\|Wx_t - x_n\|^2 \\ &\le 2t\langle x_n - f(x_t), J(x_n - x_t)\rangle + (1-t)^2(\|Wx_n - x_n\| + \|Wx_t - Wx_n\|)^2 \\ &\le 2t\langle x_n - f(x_t), J(x_n - x_t)\rangle + (1-t)^2(\|x_n - x_t\| + \|x_n - Wx_n\|)^2 \\ &= (t^2 - 2t + 1)\|x_n - x_t\|^2 + 2t\langle x_t - f(x_t), J(x_n - x_t)\rangle + f_n(t) + 2t\|x_n - x_t\|^2,\end{aligned} \tag{23}$$

where

$$f_n(t) = (\|x_n - Wx_n\| + 2\|x_n - x_t\|)\|x_n - Wx_n\|(1-t)^2 \to 0 \quad (n \to \infty). \tag{24}$$

It follows from (23) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t} f_n(t). \tag{25}$$

Using both (25) and (24), we derive

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq \frac{t}{2} M_4, \tag{26}$$

where $\sup\{\|x_t - x_n\|^2 : t \in (0,1) \text{ and } n \geq 0\} \leq M_4$ for some $M_4 > 0$. Taking $t \to 0$ in (26), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq 0.$$

On the other hand, we have

$$\begin{aligned}
&\langle f(x^*) - x^*, J(x_n - x^*)\rangle = \langle f(x^*) - x^*, J(x_n - x^*)\rangle - \langle f(x^*) - x^*, J(x_n - x_t)\rangle \\
&\quad + \langle f(x^*) - x^*, J(x_n - x_t)\rangle - \langle f(x^*) - x_t, J(x_n - x_t)\rangle + \langle f(x^*) - x_t, J(x_n - x_t)\rangle \\
&\quad - \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \langle f(x_t) - x_t, J(x_n - x_t)\rangle \\
&= \langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle + \langle x_t - x^*, J(x_n - x_t)\rangle \\
&\quad + \langle x_t - f(x_t), J(x_t - x_n)\rangle + \langle f(x^*) - f(x_t), J(x_n - x_t)\rangle.
\end{aligned}$$

So it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle &\leq \limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle \\
&\quad + (1+\delta)\|x_t - x^*\|\limsup_{n\to\infty}\|x_n - x_t\| + \limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned}$$

Taking into account that $x_t \to x^*$ yield

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \\
&\leq \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle.
\end{aligned}$$

Using the property on nonlinear mapping J yields (22). Please note that $x_n - y_n \to 0$ implies $J(y_n - x^*) - J(x_n - x^*) \to 0$. Thus, we conclude from (22) that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(y_n - x^*)\rangle = \limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \leq 0. \tag{27}$$

One observes that

$$\begin{aligned}
&\|y_n - x^*\|^2 = \|\alpha_n(f(y_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(T_n z_n - x^*) + \alpha_n(f(x^*) - x^*)\|^2 \\
&\leq \alpha_n\|f(y_n) - f(x^*)\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|z_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle \\
&\leq \alpha_n\delta\|y_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n(t_n\|x_n - x^*\|^2 + (1-t_n)\|y_n - x^*\|^2) \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, J(y_n - x^*)\rangle,
\end{aligned}$$

which hence yields

$$\|y_n - x^*\|^2 \quad \leq \left(1 - \tfrac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\right)\|x_n - x^*\|^2 + \tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\langle f(x^*) - x^*, J(y_n - x^*)\rangle. \tag{28}$$

By the convexity of $\|\cdot\|^2$, the nonexpansivity of S_nG and (28), we get

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 = \|(x_n-x^*)\delta_n + (S_nGy_n - x^*)(1-\delta_n)\|^2\\ &\le \delta_n\|x_n-x^*\|^2 + (1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n-x^*\|^2\\ &\quad + \tfrac{2\alpha_n}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\langle f(x^*)-x^*, J(y_n-x^*)\rangle\}\\ &= (1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))})\|x_n-x^*\|^2 + \tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\cdot\tfrac{2\langle f(x^*)-x^*, J(y_n-x^*)\rangle}{1-\delta}.\end{aligned} \tag{29}$$

Since $\liminf_{n\to\infty}\frac{(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))} > 0$, $\{\frac{\alpha_n(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\}\subset(0,1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, we know that $\{\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))}\}\subset(0,1)$ and $\sum_{n=0}^{\infty}\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-(\alpha_n\delta+\gamma_n(1-t_n))} = \infty$. Using (27) and Lemma 8, we conclude from (29) that $\|x_n-x^*\|\to 0$ as $n\to\infty$. This proof is now complete. □

Remark 1. *From the related associated results in Ceng et al. [16], Song and Ceng [25], our obtained results extend and improve and them in the following ways:*

(i) The approximating problem of $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)$ *in [[16], Theorem 3.1] is moved to devise our approximating problem* $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)\cap(A+B)^{-1}0$ *where* $(A+B)^{-1}0$ *is the solution set of the VI:* $0\in(A+B)x$. *The implicit (two-step) relaxed extragradient method in [[16], Theorem 3.1] is extended to develop our generalized Mann viscosity implicit rule in Theorem 1. That is, two iterative steps* $y_n = (1-\alpha_n)Gx_n + \alpha_n f(y_n)$ *and* $x_{n+1} = (1-\beta_n)S_ny_n + \beta_n x_n$ *in [[16], Theorem 3.1] is refined to develop our two iterative steps* $y_n = \alpha_n f(y_n) + \beta_n x_n + \gamma_n T_n(t_nx_n + (1-t_n)y_n)$ *and* $x_{n+1} = \delta_n x_n + (1-\delta_n)S_nGy_n$*, where* $T_n = J^B_{\lambda_n}(I-\lambda_nA)$*. In addition, uniformly convex and* 2*-uniformly smooth restructures in [[16], Theorem 3.1] is generalized to the structures of uniformly convex and q-uniformly smooth for* $1<q\le 2$*.*

(ii) The problem of finding an element of $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)$ *in [[25], Theorem 3.1] is generalized to devise our approximating problem on the element in* $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{GSVI}(C,A_1,A_2)\cap(A+B)^{-1}0$*, where* $(A+B)^{-1}0$ *is the solution set of the VI:* $0\in(A+B)x$*. The modified relaxed extragradient method in [[25], Theorem 3.1] is extended to develop our generalized Mann viscosity implicit rule in Theorem 1. That is, two iterative steps* $y_n = (1-\beta_n)x_n + \beta_n Gx_n$ *and* $x_{n+1} = \Pi_C[\alpha_n\gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\rho F)S_ny_n]$ *in [[25], Theorem 3.1] is extended to develop our two iterative steps* $y_n = \alpha_n f(y_n) + \beta_n x_n + \gamma_n T_n(t_nx_n + (1-t_n)y_n)$ *and* $x_{n+1} = \delta_n x_n + (1-\delta_n)S_nGy_n$*, where* $T_n = J^B_{\lambda_n}(I-\lambda_nA)$*.*

Next, Theorem 1 is applied to solve the GSVI, VIP and FPP in an illustrating example. Let $C = [-2,2]$ and $H = \mathbf{R}$ with the inner product $\langle a,b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial point x_0 is randomly chosen in C. We define $f(x) = \frac{1}{2}x$, $Sx = \sin x$ and $A_1x = A_2x = Ax = \frac{2}{3}x + \frac{1}{4}\sin x$ for all $x\in C$. Then f is $\frac{1}{2}$-contraction, S is a nonexpansive self-mapping on C with $\mathrm{Fix}(S) = \{0\}$ and A is $\frac{11}{12}$-Lipschitzian and $\frac{5}{12}$-strongly monotone mapping. Indeed, we observe that

$$\|Ax-Ay\| \le \frac{2}{3}\|x-y\| + \frac{1}{4}\|\sin x - \sin y\| \le (\frac{2}{3}+\frac{1}{4})\|x-y\| = \frac{11}{12}\|x-y\|,$$

and

$$\langle Ax-Ay, x-y\rangle = \frac{2}{3}\langle x-y, x-y\rangle + \frac{1}{4}\langle \sin x - \sin y, x-y\rangle \ge (\frac{2}{3}-\frac{1}{4})\|x-y\|^2 = \frac{5}{12}\|x-y\|^2.$$

This ensures that $\langle Ax-Ay, x-y\rangle \ge \frac{60}{121}\|Ax-Ay\|^2$. So it follows that $A_1 = A_2 = A$ is $\frac{60}{121}$-inverse-strongly monotone, and hence $\alpha_1 = \alpha_2 = \alpha = \frac{60}{121}$. Therefore, it is easy to see that $\Omega = \mathrm{Fix}(S)\cap\mathrm{GSVI}(C,A_1,A_2)\cap\mathrm{VI}(C,A) = \{0\} \ne \emptyset$. Let $\mu_1 = \mu_2 = \alpha = \frac{60}{121}$. Putting $\alpha_n = \frac{1}{2(n+2)}$, $\beta_n = \frac{1}{2} - \frac{1}{2(n+2)}$, $\gamma_n = \frac{1}{2}$, $\delta_n = \frac{1}{2}$, $t_n = \frac{1}{2}$ and $\lambda_n = \bar{\lambda} = \alpha = \frac{60}{121}$, we know that the conditions (i)–(v) on the parameter sequences $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ all are satisfied. In this case, the iterative scheme in Theorem 1 can be rewritten as follows:

$$\begin{cases} y_n = \frac{1}{2(n+2)} \cdot \frac{1}{2} y_n + (\frac{1}{2} - \frac{1}{2(n+2)}) x_n + \frac{1}{2} P_C(I - \frac{60}{121} A)(\frac{x_n + y_n}{2}), \\ x_{n+1} = \frac{x_n + SGy_n}{2}, \end{cases}$$

where $G := P_C(I - \frac{60}{121} A) P_C(I - \frac{60}{121} A)$. Then, by Theorem 1, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{VI}(C, A)$.

4. Applications

In this section, we will apply the main result of this paper for solving some important optimization problems in the setting of Hilbert spaces.

4.1. Variational Inequality Problems

Let $A : C \to H$ be a single-valued nonself mapping. Recall the monotone variational inequality of getting the desired vector $x^* \in C$ with $\langle Ax^*, x - x^* \rangle \geq 0$, $\forall x \in C$, whose solution set of is $\mathrm{VI}(C, A)$. Let I_C be an indicator operator of C given by

$$I_C y = \begin{cases} 0 & \text{if } y \in C, \\ \infty & \text{if } y \notin C. \end{cases}$$

We denote the normal cone of C at u by $N_C(u)$, i.e., $N_C(u)$ is a set consists of such points which solve $\langle w, v - u \rangle \leq 0$, $\forall v \in C$. It is known that I_C is a convex, lower semi-continuous and proper function and the subdifferential ∂I_C is maximally monotone. For $\lambda > 0$, the resolvent mapping of ∂I_C is denoted by $J_\lambda^{\partial I_C}$, i.e., $J_\lambda^{\partial I_C} = (I + \lambda \partial I_C)^{-1}$. Please note that

$$\begin{aligned} \partial I_C(u) &= \{w \in H : I_C(v) + \langle w, v - u \rangle \leq I_C(u), \ \forall v \in C\} \\ &= \{w \in H : \langle w, v - u \rangle \leq 0 \ \forall v \in C\} = N_C(u), \ \forall u \in C. \end{aligned}$$

So we know that $u = J_\lambda^{\partial I_C}(x) \Leftrightarrow x - u \in \lambda N_C(u) \Leftrightarrow \langle x - u, v - u \rangle \leq 0, \ \forall v \in C \Leftrightarrow u = P_C(x)$. Hence we get $(A + \partial I_C)^{-1} 0 = \mathrm{VI}(C, A)$.

Next, putting $B = \partial I_C$ in Theorem 1, we can obtain the following result.

Theorem 2. *Let non-empty set C be a convex close in a Hilbert space X stated as Theorem 1. For $i = 1, 2$, mappings $A, A_i : C \to H$ are α-inverse-strongly monotone and α_i-inverse-strongly monotone, respectively. Let S be a nonexpansive singled-valued self-mapping on C. Suppose $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{VI}(C, A) \neq \emptyset$, where $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed-point set of $G := P_C(I - \mu_1 A_1) P_C(I - \mu_2 A_2)$ with $0 < \mu_i < 2\alpha_i$ for $i = 1, 2$. Let $f : C \to C$ be a strictly contraction with constant $\delta \in (0, 1)$. For arbitrary initial $x_0 \in C$, define $\{x_n\}$ by*

$$\begin{cases} y_n = \alpha_n f(y_n) + \gamma_n P_C(I - \lambda_n A)(t_n x_n + (1 - t_n) y_n) + \beta_n x_n, \\ x_{n+1} = (1 - \delta_n) SGy_n + \delta_n x_n, \quad n \geq 0, \end{cases}$$

where $0 < \bar{\lambda} \leq \lambda_n$, $\forall n \geq 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < 2\alpha$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ satisfy conditions (i)–(iii) as in Theorem 1 in Section 2. Then $x_n \to x^* \in \Omega$, which is the unique solution to the variational inequality: $\langle (I - f)x^*, x^* - p \rangle \leq 0$, $\forall p \in \Omega$.

4.2. Convex Minimization Problems

Let $g : H \to \mathbf{R}$ and $h : H \to \mathbf{R}$ be two functions, where g is convex smooth and h is proper convex and lower semicontinuous. The associated minima problem is to find $x^* \in H$ such that

$$g(x^*) + h(x^*) = \min_{x \in H} \{g(x) + h(x)\}. \tag{30}$$

By Fermat's rule, we know that the problem (30) is equivalent to the fact that finds $x^* \in H$ such that $0 \in \nabla g(x^*) + \partial h(x^*)$ with ∇g being the gradient of function g and ∂h being the subdifferential function of function h. It is also known that if ∇g is $\frac{1}{\alpha}$-Lipschitz continuous, then it is also α-inverse-strongly monotone. Next, putting $A = \nabla g$ and $B = \partial h$ in Theorem 1, we can obtain the following result.

Theorem 3. *Let $g : H \to \mathbf{R}$ be a convex and differentiable function whose gradient ∇g is $\frac{1}{\alpha}$-Lipschitz continuous and $h : H \to \mathbf{R}$ be a convex and lower semi-continuous function. $A_i : C \to H$ are supposed to be α_i-inverse-strongly monotone for $i = 1, 2$. Let S be a nonexpansive single-valued self-mapping on C such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (\nabla g + \partial h)^{-1}0 \neq \emptyset$ where $(\nabla g + \partial h)^{-1}0$ is the set of minima attained by $g + h$, and $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed point set of $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$ with $0 < \mu_i < 2\alpha_i$ for $i = 1, 2$. Let $f : C \to C$ be a strictly contraction with constant $\delta \in (0, 1)$. For arbitrary initial $x_0 \in C$, define $\{x_n\}$ by*

$$\begin{cases} y_n = \alpha_n f(y_n) + \gamma_n J_{\lambda_n}^{\partial h}(I - \lambda_n \nabla g)(t_n x_n + (1 - t_n)y_n) + \beta_n x_n, \\ x_{n+1} = (1 - \delta_n)SGy_n + \delta_n x_n, \quad n \geq 0, \end{cases}$$

where $0 < \bar{\lambda} \leq \lambda_n$, $\forall n \geq 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < 2\alpha$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ satisfy conditions (i)–(iii) as in Theorem 1 in Section 2. Then $x_n \to x^* \in \Omega$, which uniquely solves $\langle (I - f)x^*, x^* - p\rangle \leq 0$, $\forall p \in \Omega$.

4.3. Split Feasibility Problems

Let C and Q be non-empty convex closed sets in Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a linearly bounded operator with its adjoint T^*. Consider the split feasibility problem (SFP) of obtaining a desired point $x^* \in C$ and $Tx^* \in Q$. The SFP can be borrowed to model the radiation therapy. It is clear that the set of solutions of the SFP is $C \cap T^{-1}Q$. To solve the SFP, we can rewrite it as the following convexly constrained minimization problem:

$$\min_{x \in C} g(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2.$$

Please note that the function g is differentiable convex whose Lipschitz gradient is given by $\nabla g = T^*(I - P_Q)T$. Furthermore, ∇g is $\frac{1}{\|T\|^2}$-inverse-strongly monotone, where $\|T\|^2$ is the spectral radius of T^*T. Thus, x^* solves the SFP if and only if $x^* \in H_1$ such that

$$\begin{aligned} 0 \in \nabla g(x^*) + \partial I_C(x^*) \quad &\Leftrightarrow \quad x^* - \lambda \nabla g(x^*) \in (I + \lambda \partial I_C)x^* \\ &\Leftrightarrow \quad x^* = J_\lambda^{\partial I_C}(x^* - \lambda \nabla g(x^*)) \\ &\Leftrightarrow \quad x^* = P_C(x^* - \lambda \nabla g(x^*)). \end{aligned}$$

Next, putting $A = \nabla g$ and $B = \partial I_C$ in Theorem 1, we can obtain the following result:

Theorem 4. *Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with its adjoint T^*. Let the mapping $A_i : C \to H_1$ be α_i-inverse-strongly monotone for $i = 1, 2$. Let S be a nonexpansive self-mapping on C such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (C \cap T^{-1}Q) \neq \emptyset$ where $\mathrm{GSVI}(C, A_1, A_2)$ is the fixed point set of $G := P_C(I - \mu_1 A_1)P_C(I - \mu_2 A_2)$ with $0 < \mu_i < 2\alpha_i$ for $i = 1, 2$. Let $f : C \to C$ be a δ-contraction with constant $\delta \in (0, 1)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(y_n) + \gamma_n P_C(I - \lambda_n T^*(I - P_Q)T)(t_n x_n + (1 - t_n)y_n) + \beta_n x_n, \\ x_{n+1} = \delta_n x_n + (1 - \delta_n)SGy_n, \quad n \geq 0, \end{cases}$$

where $0 < \bar{\lambda} \leq \lambda_n$, $\forall n \geq 0$ and $\lim_{n\to\infty} \lambda_n = \lambda < \frac{2}{\|T\|^2}$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ satisfy conditions (i)–(iii) as in Theorem 1 in Section 2. Then $x_n \to x^* \in \Omega$, which uniquely solves $\langle (I - f)x^*, x^* - p\rangle \leq 0$, $\forall p \in \Omega$.

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