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Informal Norm in Hyperspace and Its Topological Structure

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Abstract: The hyperspace consists of all subsets of a vector space. Owing to a lack of additive inverse elements, the hyperspace cannot form a vector space. In this paper, we shall consider a so-called informal norm to the hyperspace in which the axioms regarding the informal norm are almost the same as the axioms of the conventional norm. Under this consideration, we shall propose two different concepts of open balls. Based on the open balls, we shall also propose the different types of open sets. In this case, the topologies generated by these different concepts of open sets are investigated.

Keywords: hyperspace; informal open sets; informal norms; null set; open balls

1. Introduction

The topic of set-valued analysis (or multivalued analysis) has been studied for an extensive period. A detailed discussion can refer to Aubin and Frankowska [1], and Hu and Papageorgiou [2,3]. Applications in nonlinear analysis can refer to Agarwal and O'Regan [4], Burachik and Iusem [5], and Tarafdar and Chowdhury [6]. More specific applications in differential inclusion can also refer to Aubin and Cellina [7]. On the other hand, the fixed point theory for set-valued mappings can refer to Górniewicz [8], and set-valued optimization can refer to Chen et al. [9], Khan et al. [10] and Hamel et al. [11]. Also, the set optimization that is different from the set-valued optimization can refer to Wu [12] and the references therein.

Let $\mathcal{P}(X)$ be the collection of all subsets of a vector space X. The set-valued analysis usually studies the mathematical structure in $\mathcal{P}(X)$ in which each element in $\mathcal{P}(X)$ is treated as a subset of X. In this paper, we shall treat each element of $\mathcal{P}(X)$ as a "point". In other words, each subset of X is compressed as a point, and the family $\mathcal{P}(X)$ is treated as a universal set. In this case, the original vector space X plays no role in the settings. Therefore, we want to endow a vector structure to $\mathcal{P}(X)$. Although we can define the vector addition and scalar multiplication in $\mathcal{P}(X)$ in the usual way, owing to lacking an additive inverse element, the family $\mathcal{P}(X)$ cannot form a vector space. In this paper, we shall endow a so-called informal norm to $\mathcal{P}(X)$ even though $\mathcal{P}(X)$ is not a vector space. Then, the conventional techniques of functional analysis and topological vector space based on the vector space can be used by referring to the monographs [13–23]. The main purpose of this paper is to study the topological structures of informally normed space $\mathcal{P}(X)$. Based on these topological structures, the potential applications in nonlinear analysis, differential inclusion and set-valued optimization (or set optimization) are possible after suitable formulation.

Given a (conventional) vector space X, we denote by $\mathcal{P}(X)$ the collection of all subsets of X. For any $A, B \in \mathcal{P}(X)$, the set addition is defined by

$$A \oplus B = \{a + b : a \in A \text{ and } b \in B\}.$$

Given a scalar λ in $\mathbb{R}$, the scalar multiplication in $\mathcal{P}(X)$ is defined by

$$\lambda A = \{\lambda a : a \in A\}.$$

The substraction between A and B is denoted and defined by

$$A \ominus B \equiv A \oplus (-B) = \{a - b : a \in A \text{ and } b \in B\}.$$

We denote by θ_X the zero element of X. Let $\theta_{\mathcal{P}(X)} = \{\theta_X\}$ be a singleton set. We see that

$$A \oplus \theta_{\mathcal{P}(X)} = A \oplus \{\theta_X\} = A,$$

which says that $\{\theta_X\}$ is the zero element of $\mathcal{P}(X)$. It is clear to see that $A \ominus A \neq \{\theta_X\}$, which says that $A \ominus A$ cannot be the zero element of $\mathcal{P}(X)$. That is to say, the additive inverse element of A in $\mathcal{P}(X)$ does not exist. Therefore, the hyperspace $\mathcal{P}(X)$ cannot form a vector space under the above set of addition and scalar multiplication. Since $A \ominus A$ is not the zero element, we consider the null set of $\mathcal{P}(X)$ defined by

$$\Omega = \{A \ominus A : A \in \mathcal{P}(X)\}, \tag{1}$$

which may be treated as a kind of "zero element" of $\mathcal{P}(X)$. It is clear to see that the null set is closed under the addition.

In this paper, we shall consider the so-called informal norm in $\mathcal{P}(X)$. The axioms of informal norm will be almost the same as the axioms of conventional norm. The only difference is that the null set will be involved in the axioms of informal norm. In order to study the topological structure of $(\mathcal{P}(X), \| \cdot \|)$, we need to consider the open balls. Let us recall that if $(X, \| \cdot \|)$ is a (conventional) normed hyperspace, then we see that

$$\{y : \| x - y \| < \epsilon\} = \{x + z : \| z \| < \epsilon\}$$

by taking $y = x + z$. However, for the space $(\mathcal{P}(X), \| \cdot \|)$ and $A, B, C \in \mathcal{P}(X)$, the following equality

$$\{B : \| A \ominus B \| < \epsilon\} = \{A \oplus C : \| C \| < \epsilon\}$$

does not hold. The reason is that, by taking $B = A \oplus C$, we can just have

$$\| A \ominus B \| = \| A \ominus (A \oplus C) \| = \| \omega \ominus C \| \neq \| C \|,$$

where $\omega = A \ominus A \in \Omega$. In this case, two types of open balls will be considered in $(\mathcal{P}(X), \| \cdot \|)$. Therefore, many types of open sets will also be considered. Based on the different types of openness, we shall study the topological structure of the normed hyperspace $(\mathcal{P}(X), \| \cdot \|)$.

In Section 2, many interesting properties in $\mathcal{P}(X)$ are presented in order to study the the topology generated by the so-called informal norm. In Section 3, we introduce the concept of informal norms and provide many useful properties for further investigation. In Section 4, we provide the non-intuitive properties for the open balls. In Section 5, we propose many types of informal open sets based on the different types of open balls. Finally, in Section 6, we investigate the topologies generated by these different types of open sets.

2. Hyperspaces

Since the null set Ω defined in (1) can be treated as a kind of "zero element", we propose the almost identical concept for elements in $\mathcal{P}(X)$ as follows.

Definition 1. *For any $A, B \in \mathcal{P}(X)$, the elements A and B are said to be almost identical if there exist $\omega_1, \omega_2 \in \Omega$ satisfying $A \oplus \omega_1 = B \oplus \omega_2$. In this case, we write $A \stackrel{\Omega}{=} B$.*

For $A \ominus B = C$, we cannot have $A = B \oplus C$. However, we can obtain $A \stackrel{\Omega}{=} B \oplus C$. Let $B \ominus B \equiv \omega \in \Omega$. Since $A \ominus B = C$, by adding B on both sides, we have $A \oplus \omega = B \oplus C$, which says that $A \stackrel{\Omega}{=} B \oplus C$.

Proposition 1. *Given any $A, B \in \mathcal{P}(X)$, we have the following properties.*

(i) *Suppose that $A \ominus B \in \Omega$. Then $A \stackrel{\Omega}{=} B$.*
(ii) *Suppose that $A \stackrel{\Omega}{=} B$. Then there exists $\omega \in \Omega$ satisfying $A \ominus B \oplus \omega \in \Omega$.*

Proof. To prove part (i), we first note that there exists $\omega_1 \in \Omega$ such that

$$A \ominus B = A \oplus (-B) = \omega_1.$$

By adding B on both sides, we obtain $A \oplus (-B) \oplus B = \omega_1 \oplus B$. Therefore, we have $A \oplus \omega_2 = \omega_1 \oplus B$, where $\omega_2 = B \ominus B \in \Omega$.

To prove part (ii), since $A \stackrel{\Omega}{=} B$, there exist $\omega_1, \omega_2 \in \Omega$ such that $A \oplus \omega_2 = \omega_1 \oplus B$. By adding $-B$ on both sides, we obtain $A \ominus B \oplus \omega_2 = \omega_1 \oplus \omega_3 \in \Omega$, where $\omega_3 = B \ominus B \in \Omega$. This completes the proof. □

Proposition 2. *The following statements hold true.*

(i) *Given any subset $\mathcal{A}$ of $\mathcal{P}(X)$, we have $\mathcal{A} \subseteq \mathcal{A} \oplus \Omega$.*
(ii) *We have $\Omega \oplus \Omega = \Omega$. Given any subset $\mathcal{A}$ of $\mathcal{P}(X)$, let $\bar{\mathcal{A}} = \mathcal{A} \oplus \Omega$. Then $\bar{\mathcal{A}} \oplus \Omega = \bar{\mathcal{A}}$.*
(iii) *Given any $\omega = B \ominus B \in \Omega$ for some $B \subseteq X$, we have $\omega = \omega_1 \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. If $B \neq \{\theta_X\}$ then we can take $\omega_1 \neq \{\theta_X\}$ and $\omega_2 \neq \{\theta_X\}$.*

Proof. To prove part (i), since $\theta_{\mathcal{P}(X)} \equiv \{\theta_X\} \in \Omega$, given any $A \in \mathcal{A}$, we have

$$A = A \oplus \{\theta_X\} = A \oplus \theta_{\mathcal{P}(X)} \in \mathcal{A} \oplus \Omega.$$

To prove part (ii), given any $\omega_1, \omega_2 \in \Omega$, we have $\omega_1 = A \ominus A$ and $\omega_2 = B \ominus B$ for some $A, B \in \mathcal{P}(X)$. Therefore we obtain

$$\omega_1 \oplus \omega_2 = A \ominus A \oplus B \ominus B = (A \oplus B) \ominus (A \oplus B) \in \Omega,$$

which says that $\Omega \oplus \Omega \subseteq \Omega$. Now, for any $\omega \in \Omega$, since $\theta_{\mathcal{P}(X)} \equiv \{\theta_X\} \in \Omega$, we have

$$\omega = \omega \oplus \{\theta_X\} = \omega \oplus \theta_{\mathcal{P}(X)} \in \Omega \oplus \Omega,$$

which says that $\Omega \subseteq \Omega \oplus \Omega$. Therefore we obtain $\Omega \oplus \Omega = \Omega$. On the other hand, we have

$$\bar{\mathcal{A}} \oplus \Omega = \mathcal{A} \oplus \Omega \oplus \Omega = \mathcal{A} \oplus \Omega = \bar{\mathcal{A}}.$$

To prove part (iii), given any $B \subseteq X$, we have $B = B_1 \oplus B_2$ for some subsets B_1 and B_2 of X. For example, we can take $B_1 = \{b\}$ and $B_2 = B \ominus \{b\}$ for some $b \in B$. Therefore we have

$$\omega = B \ominus B = (B_1 \oplus B_2) \ominus (B_1 \oplus B_2) = (B_1 \ominus B_1) \oplus (B_2 \ominus B_2) \equiv \omega_1 \oplus \omega_2.$$

This completes the proof. □

The following interesting results will be used for discussing the topological structure of informal normed hyperspace.

Proposition 3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be subsets of $\mathcal{P}(X)$. Then the following inclusion is satisfied:*

$$(\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega \subseteq [(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)] \,.$$

If we further assume that $\mathcal{A}_1 \oplus \Omega \subseteq \mathcal{A}_1$ and $\mathcal{A}_2 \oplus \Omega \subseteq \mathcal{A}_2$, then the following equality is satisfied:

$$[(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)] = (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega.$$

Proof. For $B \in (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega$, we have $B = A \oplus \omega$ with $A \in \mathcal{A}_i$ for $i = 1,2$ and $\omega \in \Omega$, which also says that $B \in [(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)]$, i.e., $(\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega \subseteq [(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)]$. Under the assumption, using part (i) of Proposition 2, we have

$$[(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)] \subseteq \mathcal{A}_1 \cap \mathcal{A}_2 \subseteq (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega.$$

This completes the proof. □

3. Informal Norms

Many kinds of informal norms on $\mathcal{P}(X)$ are proposed below.

Definition 2. *Consider the nonnegative real-valued function $\| \cdot \|: \mathcal{P}(X) \to \mathbb{R}_+$ and the following conditions:*

(i) $\| \lambda A \| = |\lambda| \, \| A \|$ *for any* $A \in \mathcal{P}(X)$ *and* $\lambda \in \mathbb{R}$.
(i′) $\| \lambda A \| = |\lambda| \, \| A \|$ *for any* $A \in \mathcal{P}(X)$ *and* $\lambda \in \mathbb{R}$ *with* $\lambda \neq 0$.
(ii) $\| A \oplus B \| \leq \| A \| + \| B \|$ *for any* $A, B \in \mathcal{P}(X)$.
(iii) $\| A \| = 0$ *implies* $A \in \Omega$.

The informal norm $\| \cdot \|$ is said to satisfy the null condition when condition (iii) is replaced by $\| A \| = 0$ if and only if $A \in \Omega$.

Different kinds of informal normed hyperspaces are defined below.

- *The ordered pair $(\mathcal{P}(X), \| \cdot \|)$ is said to be an informal pseudo-seminormed hyperspace when conditions (i′) and (ii) are satisfied.*
- *The ordered pair $(\mathcal{P}(X), \| \cdot \|)$ is said to be an informal seminormed hyperspace when conditions (i) and (ii) are satisfied.*
- *The ordered pair $(\mathcal{P}(X), \| \cdot \|)$ is said to be an informal pseudo-normed hyperspace when conditions (i′), (ii) and (iii) are satisfied.*
- *The ordered pair $(\mathcal{P}(X), \| \cdot \|)$ is said to be an informal normed hyperspace when conditions (i), (ii) and (iii) are satisfied.*

We further consider the following conditions:

- *The informal norm $\| \cdot \|$ is said to satisfy the null super-inequality when $\| A \oplus \omega \| \geq \| A \|$ for any $A \in \mathcal{P}(X)$ and $\omega \in \Omega$.*
- *The informal norm $\| \cdot \|$ is said to satisfy the null sub-inequality when $\| A \oplus \omega \| \leq \| A \|$ for any $A \in \mathcal{P}(X)$ and $\omega \in \Omega$.*
- *The informal norm $\| \cdot \|$ is said to satisfy the null equality when $\| A \oplus \omega \| = \| A \|$ for any $A \in \mathcal{P}(X)$ and $\omega \in \Omega$.*

Example 1. *Let $(X, \| \cdot \|_X)$ be a (conventional) normed space. Given any element $A \in \mathcal{P}(X)$, we define*

$$\| A \| = \sup_{a \in A} \| a \|_X \,.$$

We are going to claim that $(\mathcal{P}(X), \| \cdot \|)$ *is an informal normed hyperspace.*

- *If* $A = \{\theta\}$, *then we have* $\| A \| = 0$. *If* $\| A \| = 0$, *then also we have* $\| a \|_X = 0$ *for all* $a \in A$, *i.e.,* $A = \{\theta\}$. *Therefore, we obtain that* $\| A \| = 0$ *if and only if* $A = \{\theta\} \in \Omega$.
- *We have*
$$\| \lambda A \| = \sup_{a \in \lambda A} \| a \|_X = \sup_{b \in A} \| \lambda b \|_X = |\lambda| \sup_{b \in A} \| b \|_X = |\lambda| \| A \| .$$
- *We want to prove the triangle inequality* $\| A \oplus B \| \leq \| A \| + \| B \|$. *Let*
$$\zeta_1 = \sup_{\{(a,b):a \in A, b \in B\}} \| a \|_X \text{ and } \zeta_2 = \sup_{\{(a,b):a \in A, b \in B\}} \| b \|_X .$$

It is clear to see that $\| a \| + \| b \| \leq \zeta_1 + \zeta_2$ *for all* $a \in A$ *and* $b \in B$, *which implies*
$$\sup_{\{(a,b):a \in A, b \in B\}} (\| a \|_X + \| b \|_X) \leq \zeta_1 + \zeta_2 = \sup_{\{(a,b):a \in A, b \in B\}} \| a \|_X + \sup_{\{(a,b):a \in A, b \in B\}} \| b \|_X .$$

Then, we obtain
$$\begin{aligned} \| A \oplus B \| &= \sup_{c \in A \oplus B} \| c \|_X = \sup_{\{(a,b):a \in A, b \in B\}} \| a + b \|_X \\ &\leq \sup_{\{(a,b):a \in A, b \in B\}} (\| a \|_X + \| b \|_X) \\ &\leq \sup_{\{(a,b):a \in A, b \in B\}} \| a \|_X + \sup_{\{(a,b):a \in A, b \in B\}} \| b \|_X \\ &= \sup_{a \in A} \| a \|_X + \sup_{b \in B} \| b \|_X = \| A \| + \| B \| . \end{aligned}$$

Therefore, we conclude that $(\mathcal{P}(X), \| \cdot \|)$ *is indeed an informal normed hyperspace. Given any* $\omega \in \Omega$, *there exists* $B \in \mathcal{P}(X)$ *satisfying* $\omega = B \ominus B$. *Therefore, we obtain*
$$\| \omega \| = \| B \ominus B \| = \sup_{\{(b_1,b_2):b_1,b_2 \in B\}} \| b_1 - b_2 \|_X .$$

Since $\| \omega \|$ *is not equal to zero in general, it means that the null condition is not satisfied.*

Proposition 4. *Let* $(\mathcal{P}(X), \| \cdot \|)$ *be an informal pseudo-seminormed hyperspace. Suppose that the informal norm* $\| \cdot \|$ *satisfies the null super-inequality. For any* $A, C, B_1, \cdots, B_m \in \mathcal{P}(X)$, *we have*
$$\| A \ominus C \| \leq \| A \ominus B_1 \| + \| B_1 \ominus B_2 \| + \cdots + \| B_j \ominus B_{j+1} \| + \cdots + \| B_m \ominus C \| .$$

Proof. We have
$$\begin{aligned} \| A \ominus C \| &\leq \| A \oplus (-C) \oplus B_1 \oplus \cdots \oplus B_m \oplus (-B_1) \oplus \cdots \oplus (-B_m) \| \\ &\quad \text{(using the null super-inequality for } m \text{ times)} \\ &= \| [A \oplus (-B_1)] \oplus [B_1 \oplus (-B_2)] + \cdots + [B_j \oplus (-B_{j+1})] + \cdots + [B_m \oplus (-C)] \| \\ &\leq \| A \ominus B_1 \| + \| B_1 \ominus B_2 \| + \cdots + \| B_j \ominus B_{j+1} \| + \cdots + \| B_m \ominus C \| \\ &\quad \text{(using the triangle inequality).} \end{aligned}$$

This completes the proof. □

4. Open Balls

If $(X, \| \cdot \|)$ is a (conventional) seminormed space, then we see that

$$\{y : \| x - y \| < \epsilon\} = \{x + z : \| z \| < \epsilon\}$$

by taking $y = x + z$. Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal seminormed hyperspace. Then the following equality

$$\{B : \| A \ominus B \| < \epsilon\} = \{A \oplus C : \| C \| < \epsilon\}$$

does not hold. The reason is that, by taking $B = A \oplus C$, we can just have

$$\| A \ominus B \| = \| A \ominus (A \oplus C) \| = \| -C \oplus \omega \| \neq \| C \|,$$

where $\omega = A \ominus A \in \Omega$. Therefore we can define two types of open ball.

Definition 3. *Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal pseudo-seminormed hyperspace. Two types of open balls with radius ϵ are defined by*

$$\mathcal{B}^{\diamond}(A; \epsilon) = \{A \oplus C : \| C \| < \epsilon\}$$

and

$$\mathcal{B}(A; \epsilon) = \{B : \| A \ominus B \| = \| B \ominus A \| < \epsilon\}.$$

Example 2. *Continued from Example 1, for any $A \in \mathcal{P}(X)$, we define*

$$\| A \| = \sup_{a \in A} \| a \|_X .$$

The open balls $\mathcal{B}(A; \epsilon)$ and $\mathcal{B}^{\diamond}(A; \epsilon)$ with radius ϵ are given by

$$\mathcal{B}(A; \epsilon) = \{B \in \mathcal{P}(X) : \| A \ominus B \| < \epsilon\} = \left\{ B \in \mathcal{P}(X) : \sup_{a \in A \ominus B} \| a \|_X < \epsilon \right\}$$

and

$$\mathcal{B}^{\diamond}(A; \epsilon) = \{A \oplus C \in \mathcal{P}(X) : \| C \| < \epsilon\} = \left\{ A \oplus C \in \mathcal{P}(X) : \sup_{c \in C} \| c \|_X < \epsilon \right\}.$$

Remark 1. *Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal pseudo-seminormed hyperspace. Then we have the following observations.*

- *For any $A \in \mathcal{P}(X)$, the equality $\| A \ominus A \| = 0$ does not necessarily hold true, unless $\| \cdot \|$ satisfies the null condition. In other words, the properties $A \in \mathcal{B}(A; \epsilon)$ can only hold true when $\| \cdot \|$ satisfies the null condition.*
- *Suppose that $\| \theta_{\mathcal{P}(X)} \| = \| \{\theta_X\} \| = 0$. Then $A \in \mathcal{B}^{\diamond}(A; \epsilon)$, since $A = A \oplus \theta_{\mathcal{P}(X)}$.*

Proposition 5. *Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal pseudo-seminormed hyperspace.*

(i) *For $A \in \mathcal{P}(X)$ with $\omega_A = A \ominus A \in \Omega$, we have $\mathcal{B}(A; \epsilon) \oplus \omega_A \subseteq \mathcal{B}^{\diamond}(A; \epsilon)$.*
(ii) *If $\| \cdot \|$ satisfies the null sub-inequality, then $\mathcal{B}^{\diamond}(A; \epsilon) \subseteq \mathcal{B}(A; \epsilon)$.*
(iii) *If $\| \cdot \|$ satisfies the null sub-inequality, for any $A \in \mathcal{P}(X)$ with $\omega_A = A \ominus A \in \Omega$, then $\mathcal{B}(A; \epsilon) \oplus \omega_A \subseteq \mathcal{B}(A; \epsilon)$ and $\mathcal{B}^{\diamond}(A; \epsilon) \oplus \omega_A \subseteq \mathcal{B}^{\diamond}(A; \epsilon)$.*

Proof. To prove part (i), for any $B \in \mathcal{B}(A; \epsilon)$, i.e., $\| B \ominus A \| < \epsilon$, if we take $C = B \ominus A$, then $\| C \| < \epsilon$ and $B \oplus \omega_A = A \oplus C$. This shows the inclusion

$$\mathcal{B}(A; \epsilon) \oplus \omega_A \subseteq \{A \oplus C : \| C \| < \epsilon\} = \mathcal{B}^{\diamond}(A; \epsilon).$$

To prove part (ii), for $C \in \mathcal{P}(X)$ with $\| C \| < \epsilon$, since $\| \cdot \|$ satisfies the null sub-inequality, it follows that

$$\| (A \oplus C) \ominus A \| = \| \omega_A \oplus C \| \leq \| C \| < \epsilon,$$

which says that $A \oplus C \in \mathcal{B}(A;\epsilon)$ and shows the inclusion

$$\mathcal{B}^{\diamond}(A;\epsilon) = \{A \oplus C : \| C \| < \epsilon\} \subseteq \mathcal{B}(A;\epsilon).$$

Part (iii) follows from parts (i) and (ii) immediately. This completes the proof. □

Proposition 6. *Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal pseudo-seminormed hyperspace.*

(i) *If $\| \cdot \|$ satisfies the null super-inequality, then $\mathcal{B}(A \oplus \omega;\epsilon) \subseteq \mathcal{B}(A;\epsilon)$ for any $\omega \in \Omega$.*
(ii) *If $\| \cdot \|$ satisfies the null sub-inequality, then we have the following inclusions:*
  - *$\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A \oplus \omega;\epsilon)$ for any $\omega \in \Omega$.*
  - *$\mathcal{B}^{\diamond}(A \oplus \omega;\epsilon) \subseteq \mathcal{B}^{\diamond}(A;\epsilon)$ for any $\omega \in \Omega$.*

(iii) *If $\| \cdot \|$ satisfies the null equality, then $\mathcal{B}(A \oplus \omega;\epsilon) = \mathcal{B}(A;\epsilon)$ for any $\omega \in \Omega$.*

Proof. To prove part (i), the inclusion $\mathcal{B}(A \oplus \omega;\epsilon) \subseteq \mathcal{B}(A;\epsilon)$ follows from the following expression

$$\epsilon > \| (A \oplus \omega) \ominus B \| = \| (A \ominus B) \oplus \omega \| \geq \| A \ominus B \|,$$

and the inclusion $\mathcal{B}(A \oplus \omega;\epsilon) \subseteq \mathcal{B}(A;\epsilon)$ follows from the following expression

$$\epsilon > \| B \ominus (A \oplus \omega) \| = \| (B \ominus A) \oplus \omega \| \geq \| B \ominus A \| .$$

To prove the first case of part (ii), the inclusion $\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A \oplus \omega;\epsilon)$ follows from the following expression

$$\epsilon > \| A \ominus B \| \geq \| (A \ominus B) \oplus \omega \| = \| (A \oplus \omega) \ominus B \| .$$

To prove the second case of part (ii), for $B = A \oplus \omega \oplus C \in \mathcal{B}^{\diamond}(A \oplus \omega;\epsilon)$ with $\| C \| < \epsilon$, let $\bar{C} = \omega \oplus C$. Then, using the null sub-inequality, we have

$$\| \bar{C} \| = \| \omega \oplus C \| \leq \| C \| < \epsilon, \tag{2}$$

which says that $B = A \oplus \bar{C} \in \mathcal{B}^{\diamond}(A;\epsilon)$. Therefore we obtain the inclusion $\mathcal{B}^{\diamond}(A \oplus \omega;\epsilon) \subseteq \mathcal{B}^{\diamond}(A;\epsilon)$. Part (iii) follows from parts (i) and (ii) immediately. This completes the proof. □

In the (conventional) normed hyperspace $(X, \| \cdot \|)$, we have the equality

$$\mathcal{B}(x;\epsilon) \oplus \hat{x} = \mathcal{B}(x \oplus \hat{x};\epsilon). \tag{3}$$

However, in the informal normed hyperspace $(\mathcal{P}(X), \| \cdot \|)$, the intuitive observation (3) will not hold true in general. The following proposition presents the exact relationship.

Proposition 7. *Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal pseudo-seminormed hyperspace.*

(i) *We have the equality $\mathcal{B}^{\diamond}(A;\epsilon) \oplus \widehat{A} = \mathcal{B}^{\diamond}(A \oplus \widehat{A};\epsilon)$. In particular, for any $\omega \in \Omega$, we also have $\mathcal{B}^{\diamond}(A;\epsilon) \oplus \omega = \mathcal{B}^{\diamond}(A \oplus \omega;\epsilon)$.*
(ii) *Suppose that $\| \cdot \|$ satisfies the null sub-inequality. Then we have the inclusion $\mathcal{B}(A;\epsilon) \oplus \widehat{A} \subseteq \mathcal{B}(A \oplus \widehat{A};\epsilon)$. We further assume that $\| \cdot \|$ satisfies the null equality. Then, for any $\omega \in \Omega$, we also have the inclusions $\mathcal{B}(A;\epsilon) \oplus \omega \subseteq \mathcal{B}(A;\epsilon)$ and $\mathcal{B}(\omega;\epsilon) \oplus \widehat{A} \subseteq \mathcal{B}(\widehat{A};\epsilon)$.*
(iii) *Suppose that $\| \cdot \|$ satisfies the null sub-inequality. For any $A \in \mathcal{P}(X)$ with $\omega_A = A \ominus A$, we have the inclusion $\mathcal{B}(A;\epsilon) \oplus \omega_A \subseteq A \oplus \mathcal{B}(\omega_A;\epsilon)$.*

(iv) *For any $\widehat{A} \in \mathcal{P}(X)$ with $\omega_{\widehat{A}} = \widehat{A} \ominus \widehat{A}$, we have the inclusion*

$$\mathcal{B}(A \oplus \widehat{A}; \epsilon) \oplus \omega_{\widehat{A}} \subseteq \mathcal{B}(A; \epsilon) \oplus \widehat{A}.$$

Proof. Part (i) follows from the following equality

$$(A \oplus C) \oplus \widehat{A} = (A \oplus \widehat{A}) \oplus C \text{ for } \parallel C \parallel < \epsilon.$$

To prove part (ii), for $B \in \mathcal{B}(A; \epsilon) \oplus \widehat{A}$, we have $B = \widehat{B} \oplus \widehat{A}$ with $\parallel A \ominus \widehat{B} \parallel < \epsilon$. Then, by the null sub-inequality, we can obtain

$$\parallel (A \oplus \widehat{A}) \ominus B \parallel = \parallel (A \oplus \widehat{A}) \ominus (\widehat{B} \oplus \widehat{A}) \parallel = \parallel (A \ominus \widehat{B}) \oplus (\widehat{A} \ominus \widehat{A}) \parallel \leq \parallel A \ominus \widehat{B} \parallel < \epsilon,$$

which says that $B \in \mathcal{B}(A \oplus \widehat{A}; \epsilon)$. Therefore we obtain the inclusion $\mathcal{B}(A; \epsilon) \oplus \widehat{A} \subseteq \mathcal{B}(A \oplus \widehat{A}; \epsilon)$. Now we take $\widehat{A} = \omega$. By part (iii) of Proposition 6, we have

$$\mathcal{B}(A; \epsilon) \oplus \omega \subseteq \mathcal{B}(A \oplus \omega; \epsilon) = \mathcal{B}(A; \epsilon).$$

Similarly, if we take $A = \omega$, then we have

$$\mathcal{B}(\omega; \epsilon) \oplus \widehat{A} \subseteq \mathcal{B}(\omega \oplus \widehat{A}; \epsilon) = \mathcal{B}(\widehat{A}; \epsilon).$$

To prove part (iii), for $\widehat{A} \in \mathcal{B}(A; \epsilon)$, we have $\widehat{A} \oplus \omega_A = A \oplus (\widehat{A} \ominus A)$. The null sub-inequality gives

$$\parallel \omega_A \ominus (\widehat{A} \ominus A) \parallel \leq \parallel \widehat{A} \ominus A \parallel < \epsilon,$$

which says that $\widehat{A} \ominus A \in \mathcal{B}(\omega; \epsilon)$, i.e.,

$$\widehat{A} \oplus \omega_A = A \oplus (\widehat{A} \ominus A) \in A \oplus \mathcal{B}(\omega_A; \epsilon).$$

To prove part (iv), for $B \in \mathcal{B}(A \oplus \widehat{A}; \epsilon)$, we have $\parallel B \ominus (A \oplus \widehat{A}) \parallel < \epsilon$. We also have

$$\epsilon > \parallel B \ominus (A \oplus \widehat{A}) \parallel = \parallel (B \ominus \widehat{A}) \ominus A \parallel .$$

This shows that $B \ominus \widehat{A} \in \mathcal{B}(A; \epsilon)$. Let $\omega_{\widehat{A}} = \widehat{A} \ominus \widehat{A} \in \Omega$. Since $B \oplus \omega_{\widehat{A}} = (B \ominus \widehat{A}) \oplus \widehat{A}$, it says that $B \oplus \omega_{\widehat{A}} \in \mathcal{B}(A; \epsilon) \oplus \widehat{A}$. In other words, we have the inclusion

$$\mathcal{B}(A \oplus \widehat{A}; \epsilon) \oplus \omega_{\widehat{A}} \subseteq \mathcal{B}(A; \epsilon) \oplus \widehat{A}.$$

This completes the proof. □

Proposition 8. *Let $(\mathcal{P}(X), \parallel \cdot \parallel)$ be an informal pseudo-seminormed hyperspace.*

(i) *The following statements hold true:*

- *Suppose that $\parallel \cdot \parallel$ satisfies the null super-inequality. For any $\omega \in \Omega$, if $A \oplus \omega \in \mathcal{B}(A_0; \epsilon)$, then $A \in \mathcal{B}(A_0; \epsilon)$.*
- *Suppose that $\parallel \cdot \parallel$ satisfies the null sub-inequality. For any $\omega \in \Omega$, if $A \in \mathcal{B}(A_0; \epsilon)$, then $A \oplus \omega \in \mathcal{B}(A_0; \epsilon)$, and if $A \in \mathcal{B}^{\diamond}(A_0; \epsilon)$, then $A \oplus \omega \in \mathcal{B}^{\diamond}(A_0; \epsilon)$.*
- *Suppose that $\parallel \cdot \parallel$ satisfies the null equality. Then, for any $\omega \in \Omega$, $A \oplus \omega \in \mathcal{B}(A_0; \epsilon)$ if and only if $A \in \mathcal{B}(A_0; \epsilon)$.*

(ii) *We have the inclusions*

$$\mathcal{B}(A; \epsilon) \subseteq \mathcal{B}(A; \epsilon) \oplus \Omega \text{ and } \mathcal{B}^{\diamond}(A; \epsilon) \subseteq \mathcal{B}^{\diamond}(A; \epsilon) \oplus \Omega.$$

If we further assume that $\parallel \cdot \parallel$ *satisfies the null sub-inequality, then*

$$\mathcal{B}(A;\epsilon) \oplus \Omega = \mathcal{B}(A;\epsilon) \text{ and } \mathcal{B}^{\diamond}(A;\epsilon) \oplus \Omega = \mathcal{B}^{\diamond}(A;\epsilon).$$

(iii) *Suppose that* $\parallel \cdot \parallel$ *satisfies the null condition. Given a fixed* $\omega \in \Omega$*, we have*

$$\Omega \oplus \omega \subseteq \mathcal{B}^{\diamond}(\omega;\epsilon) \text{ and } \Omega \subseteq \mathcal{B}(\omega;\epsilon).$$

(iv) *Suppose that* $\parallel \cdot \parallel$ *satisfies the null equality. Given any fixed* $\omega \in \Omega$ *and* $\alpha \neq 0$*, we have* $\alpha\mathcal{B}(\omega;\epsilon) \subseteq \mathcal{B}(\omega;|\alpha|\epsilon)$.

(v) *Given any fixed* $\omega \in \Omega$ *and* $\alpha \neq 0$*, we have*

$$\alpha\mathcal{B}^{\diamond}(\omega;\epsilon) \subseteq \mathcal{B}^{\diamond}(\alpha\omega;|\alpha|\epsilon) \text{ and } \mathcal{B}^{\diamond}(\alpha\omega;|\alpha|\epsilon) \subseteq \alpha\mathcal{B}^{\diamond}(\omega;\epsilon).$$

Proof. The first case of part (i) follows from the following expression

$$\parallel A \ominus A_0 \parallel \leq \parallel (A \oplus \omega) \ominus A_0 \parallel < \epsilon.$$

The second case of part (i) regarding the open ball $\mathcal{B}(A_0;\epsilon)$ follows from the following expression

$$\parallel (A \oplus \omega) \ominus A_0 \parallel \leq \parallel A \ominus A_0 \parallel < \epsilon. \tag{4}$$

For the open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$, if $A \in \mathcal{B}^{\diamond}(A_0;\epsilon)$, then $A = A_0 \oplus C$ with $\parallel C \parallel < \epsilon$. Given an $\omega \in \Omega$, let $\bar{C} = C \oplus \omega$. Therefore we have $A \oplus \omega = A_0 \oplus \bar{C}$, where

$$\parallel \bar{C} \parallel = \parallel C \oplus \omega \parallel \leq \parallel C \parallel < \epsilon, \tag{5}$$

which says that $A \oplus \omega \in \mathcal{B}^{\diamond}(A_0;\epsilon)$. The third case of part (i) follows from the previous two cases.

To prove part (ii), since $\theta_{\mathcal{P}(X)} \in \Omega$ is the zero element of $\mathcal{P}(X)$, it follows that $B = B \oplus \theta_{\mathcal{P}(X)}$. Therefore we have $\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A;\epsilon) \oplus \Omega$ and $\mathcal{B}^{\diamond}(A;\epsilon) \subseteq \mathcal{B}^{\diamond}(A;\epsilon) \oplus \Omega$. On the other hand, for $A \in \mathcal{B}(A_0;\epsilon)$ and $\omega \in \Omega$, from (4), we see that $A \oplus \omega \in \mathcal{B}(A_0;\epsilon)$, which shows the inclusion $\mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{B}(A_0;\epsilon)$. Also, for $B = A \oplus C \in \mathcal{B}^{\diamond}(A;\epsilon)$ with $\parallel C \parallel < \epsilon$, let $\bar{C} = \omega \oplus C$. By (5), we have $B \oplus \omega = A \oplus \bar{C} \in \mathcal{B}^{\diamond}(A;\epsilon)$, which shows the inclusion $\mathcal{B}^{\diamond}(A;\epsilon) \oplus \Omega \subseteq \mathcal{B}^{\diamond}(A;\epsilon)$. This proves part (ii).

To prove part (iii), for any $\omega' \in \Omega$, we have $\parallel \omega' \parallel = 0$, which says that $\omega \oplus \omega' \in \mathcal{B}^{\diamond}(\omega;\epsilon)$. Therefore we obtain the inclusion $\Omega \oplus \omega \subseteq \mathcal{B}^{\diamond}(\omega;\epsilon)$. On the other hand, we also have

$$\parallel \omega' \ominus \omega \parallel = \parallel \omega' \oplus (-\omega) \parallel \leq \parallel \omega' \parallel + \parallel -\omega \parallel = \parallel \omega' \parallel + \parallel \omega \parallel = 0,$$

which shows that $\omega' \in \mathcal{B}(\omega;\epsilon)$, i.e., $\Omega \subseteq \mathcal{B}(\omega;\epsilon)$.

To prove part (iv), for $A \in \mathcal{B}(\omega;\epsilon)$, since $\alpha\omega \in \Omega$, we have

$$\parallel \omega \ominus \alpha A \parallel = \parallel (\omega \oplus \alpha\omega) \ominus \alpha A \parallel = \parallel \alpha\omega \ominus \alpha A \parallel = \parallel \alpha(A \ominus \omega) \parallel = |\alpha| \parallel A \ominus \omega \parallel < |\alpha|\epsilon,$$

i.e., $\alpha A \in \mathcal{B}(\omega;|\alpha|\epsilon)$. This shows the inclusion $\alpha\mathcal{B}(\omega;\epsilon) \subseteq \mathcal{B}(\omega;|\alpha|\epsilon)$.

To prove the first inclusion of part (v), for $A \in \mathcal{B}(\omega;\epsilon)$, we have $A = \omega \oplus C$ with $\parallel C \parallel < \epsilon$. It follows that $\alpha A = \alpha\omega \oplus \alpha C$. Let $\bar{C} = \alpha C$. Then $\parallel \bar{C} \parallel < |\alpha|\epsilon$, which shows the inclusion $\alpha\mathcal{B}^{\diamond}(\omega;\epsilon) \subseteq \mathcal{B}^{\diamond}(\alpha\omega;|\alpha|\epsilon)$. To prove the second inclusion of part (v), for $A \in \mathcal{B}^{\diamond}(\alpha\omega;|\alpha|\epsilon)$, we have $A = \alpha\omega \oplus C$ with $\parallel C \parallel < |\alpha|\epsilon$. Let $\widehat{C} = C/\alpha$. Then

$$A = \alpha\omega \oplus C = \alpha\omega \oplus \alpha(C/\alpha) = \alpha\omega \oplus \alpha\widehat{C} = \alpha(\omega \oplus \widehat{C}) \text{ with } \parallel \widehat{C} \parallel < \epsilon,$$

which says that $A \in \alpha\mathcal{B}^{\diamond}(\omega;\epsilon)$. This completes the proof. □

5. Informal Open Sets

Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace. We are going to consider the open subsets of $\mathcal{P}(X)$.

Definition 4. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace, and let $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(X)$.*

- *A point $A_0 \in \mathcal{A}$ is said to be an informal interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \subseteq \mathcal{A}$. The collection of all informal interior points of $\mathcal{A}$ is called the informal interior of $\mathcal{A}$ and is denoted by $int(\mathcal{A})$.*
- *A point $A_0 \in \mathcal{A}$ is said to be an informal type-I-interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{A}$. The collection of all informal type-I-interior points of $\mathcal{A}$ is called the informal type-I-interior of $\mathcal{A}$ and is denoted by $int^{(I)}(\mathcal{A})$.*
- *A point $A_0 \in \mathcal{A}$ is said to be an informal type-II-interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \subseteq \mathcal{A} \oplus \Omega$. The collection of all informal type-II-interior points of $\mathcal{A}$ is called the informal type-II-interior of $\mathcal{A}$ and is denoted by $int^{(II)}(\mathcal{A})$.*
- *A point $A_0 \in \mathcal{A}$ is said to be an informal type-III-interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$. The collection of all informal type-III-interior points of $\mathcal{A}$ is called the informal type-III-interior of $\mathcal{A}$ and is denoted by $int^{(III)}(\mathcal{A})$.*

The different types of informal $\diamond$-interior points based on the open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$ can be similarly defined. For example, $int^{(\diamond III)}(\mathcal{A})$ denotes the informal $\diamond$-type-III-interior of $\mathcal{A}$.

Remark 2. *Recall that we cannot have the property $A \in \mathcal{B}(A;\epsilon)$ in general by Remark 1, unless $\|\cdot\|$ satisfies the null condition. Given any $A \in \mathcal{I}$ with $\| A \ominus A \| \neq 0$, it follows that $A \notin \mathcal{B}(A;\epsilon^*)$ for $\epsilon^* < \| A \ominus A \|$. Now, given $\epsilon < \epsilon^*$, it is clear that $\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A;\epsilon^*)$. Let us take $\mathcal{A} = \mathcal{B}(A;\epsilon^*)$. It means that the open ball $\mathcal{B}(A;\epsilon)$ is contained in $\mathcal{A}$ even though the center A is not in $\mathcal{A}$.*

Remark 3. *From Remark 2, it can happen that there exists an open ball such that $\mathcal{B}(A;\epsilon)$ is contained in $\mathcal{A}$ even though the center A is not in $\mathcal{A}$. In this situation, we will not say that A is an informal interior point, since A is not in $\mathcal{A}$. Also, the sets $\mathcal{B}(A;\epsilon) \oplus \Omega$ and $\mathcal{B}^{\diamond}(A;\epsilon) \oplus \Omega$ will not necessarily contain the center A. In other words, it can happen that there exists an open ball such that $\mathcal{B}(A;\epsilon) \oplus \Omega$ is contained in $\mathcal{A}$ even though the center A is not in $\mathcal{A}$. In this situation, we will not say that A is an informal type-I-interior point, since A is not in $\mathcal{A}$. We also have the following observations.*

- *Suppose that $\|\cdot\|$ satisfies the null condition. Then $A \in \mathcal{B}(A;\epsilon)$. Since $A = A \oplus \theta_{\mathcal{P}(X)}$, we also have $A \in \mathcal{B}(A;\epsilon) \oplus \Omega$.*
- *Suppose that $\| \theta_{\mathcal{P}(X)} \| = 0$. The second observation of Remark 1 says that $A \in \mathcal{B}^{\diamond}(A;\epsilon)$. Since $A = A \oplus \theta_{\mathcal{P}(X)}$, it follows that $A \in \mathcal{B}^{\diamond}(A;\epsilon) \oplus \Omega$.*

According to Remark 3, we can define the different concepts of informal pseudo-interior point.

Definition 5. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace, and let $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(X)$.*

- *A point $A_0 \in \mathcal{P}(X)$ is said to be an informal pseudo-interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \subseteq \mathcal{A}$. The collection of all informal pseudo-interior points of $\mathcal{A}$ is called the informal pseudo-interior of $\mathcal{A}$ and is denoted by $pint(\mathcal{A})$.*
- *A point $A_0 \in \mathcal{P}(X)$ is said to be an informal type-I-pseudo-interior point of $\mathcal{A}$ if, and only if, there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{A}$. The collection of all informal type-I-pseudo-interior points of $\mathcal{A}$ is called the informal type-I-pseudo-interior of $\mathcal{A}$ and is denoted by $pint^{(I)}(\mathcal{A})$.*

- *A point $A_0 \in \mathcal{P}(X)$ is said to be an informal type-II-pseudo-interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \subseteq \mathcal{A} \oplus \Omega$. The collection of all informal type-II-pseudo-interior points of $\mathcal{A}$ is called the informal type-II-pseudo-interior of $\mathcal{A}$ and is denoted by $pint^{(II)}(\mathcal{A})$.*
- *A point $A_0 \in \mathcal{P}(X)$ is said to be an informal type-III-pseudo-interior point of $\mathcal{A}$ if there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$. The collection of all informal type-III-pseudo-interior points of $\mathcal{A}$ is called the informal type-III-pseudo-interior of $\mathcal{A}$ and is denoted by $pint^{(III)}(\mathcal{A})$.*

The different types of informal $\diamond$-pseudo-interior point based on the open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$ can be similarly defined.

Remark 4. *We have to remark that the difference between Definitions 4 and 5 is that we consider $A_0 \in \mathcal{A}$ in Definition 4, and consider $A_0 \in \mathcal{P}(X)$ in Definition 5. From Remark 2, if $\epsilon^* < \| A \ominus A \|$, then A is a pseudo-interior point of $\mathcal{B}(A;\epsilon^*)$. We also have the following observations.*

- *It is clear that $int(\mathcal{A}) \subseteq pint(\mathcal{A})$, $int^{(I)}(\mathcal{A}) \subseteq pint^{(I)}(\mathcal{A})$, $int^{(II)}(\mathcal{A}) \subseteq pint^{(II)}(\mathcal{A})$ and $int^{(III)}(\mathcal{A}) \subseteq pint^{(III)}(\mathcal{A})$. The same inclusions can also apply to the different types of informal $\diamond$-interior and $\diamond$-pseudo-interior.*
- *It is clear that $int(\mathcal{A}) \subseteq \mathcal{A}$, $int^{(I)}(\mathcal{A}) \subseteq \mathcal{A}$, $int^{(II)}(\mathcal{A}) \subseteq \mathcal{A}$ and $int^{(III)}(\mathcal{A}) \subseteq \mathcal{A}$. However, the above kinds of inclusions cannot hold true for the informal pseudo-interior.*
- *From Remark 1, we have the following observations.*
  - *Suppose that $\| \cdot \|$ satisfies the null condition. Then these concepts of informal interior point and informal pseudo-interior point are equivalent, since A_0 is in the open ball $\mathcal{B}(A_0;\epsilon)$.*
  - *Suppose that $\| \theta \| = 0$. Then these concepts of informal $\diamond$-type of interior point and informal $\diamond$-type of pseudo-interior point are equivalent, since A_0 is in the open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$.*

Remark 5. *From part (ii) of Proposition 8, if $\| \cdot \|$ satisfies the null sub-inequality, then these concepts of informal interior point and informal type-I-interior point are equivalent, and these concepts of informal type-II-interior point and informal type-III-interior point are equivalent. The same situation also applies to the cases of informal pseudo-interior points. We also remark that if $\| \cdot \|$ satisfies the null condition, then $\| \cdot \|$ satisfies the null sub-inequality, since we have $\| A \oplus \omega \| \leq \| A \| + \| \omega \| = \| A \|$ for any $\omega \in \Omega$.*

Remark 6. *Suppose that $\| \cdot \|$ satisfies the null sub-inequality. From part (ii) of Proposition 5, we see that if A_0 is an informal interior (respectively type-I-interior, type-II-interior, type-III-interior) point then it is also an informal $\diamond$-interior (resp. $\diamond$-type-I-interior, $\diamond$-type-II-interior, $\diamond$-type-III-interior) point. In other words, from Remark 5, we have*

$$int(\mathcal{A}) = int^{(I)}(\mathcal{A}) \subseteq int^{(\diamond I)}(\mathcal{A}) = int^{\diamond}(\mathcal{A})$$

and

$$int^{(II)}(\mathcal{A}) = int^{(III)}(\mathcal{A}) \subseteq int^{(\diamond III)}(\mathcal{A}) = int^{(\diamond II)}(\mathcal{A}).$$

Regarding the different concepts of pseudo-interior point, we also have

$$pint(\mathcal{A}) = pint^{(I)}(\mathcal{A}) \subseteq pint^{(\diamond I)}(\mathcal{A}) = pint^{\diamond}(\mathcal{A})$$

and

$$pint^{(II)}(\mathcal{A}) = pint^{(III)}(\mathcal{A}) \subseteq pint^{(\diamond III)}(\mathcal{A}) = pint^{(\diamond II)}(\mathcal{A}).$$

Remark 7. *Let $(\mathcal{P}(X), \| \cdot \|)$ be an informal pseudo-seminormed hyperspace.*

- *Suppose that the center A_0 is in the open ball $\mathcal{B}(A_0;\epsilon)$. Then the concepts of informal interior point and informal pseudo-interior point are equivalent. It follows that $pint(\mathcal{A}) = int(\mathcal{A}) \subseteq \mathcal{A}$. Similarly, if the center A_0 is in the open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$, then $pint^{\diamond}(\mathcal{A}) = int^{\diamond}(\mathcal{A}) \subseteq \mathcal{A}$.*

- *From part (ii) of Proposition 8, we have* $\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A;\epsilon) \oplus \Omega$ *and* $\mathcal{B}^\diamond(A;\epsilon) \subseteq \mathcal{B}^\diamond(A;\epsilon) \oplus \Omega$*. Suppose that the center* A_0 *is in the open ball* $\mathcal{B}(A_0;\epsilon)$*. Let* A_0 *be an informal type-I-pseudo-interior point of* $\mathcal{A}$*. Since*
$$A_0 \in \mathcal{B}(A_0;\epsilon) \subseteq \mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{A},$$
using Remark 4, we obtain
$$\text{pint}^{(I)}(\mathcal{A}) \subseteq \text{int}(\mathcal{A}) \subseteq \mathcal{A} \text{ and } \text{pint}^{(I)}(\mathcal{A}) \subseteq \text{int}^{(I)}(\mathcal{A}) \subseteq \text{pint}^{(I)}(\mathcal{A}),$$
which also implies $\text{pint}^{(I)}(\mathcal{A}) = \text{int}^{(I)}(\mathcal{A})$*. Similarly, if the center* A_0 *is in the open ball* $\mathcal{B}^\diamond(A_0;\epsilon)$*, then* $\text{pint}^{(\diamond I)}(\mathcal{A}) = \text{int}^{(\diamond I)}(\mathcal{A})$*.*
- *Suppose that* $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$*. We have the following observations. Assume that the center* A_0 *is in the open ball* $\mathcal{B}(A_0;\epsilon)$*. Let* A_0 *be an informal type-II-pseudo-interior point of* $\mathcal{A}$*. Since*
$$A_0 \in \mathcal{B}(A_0;\epsilon) \subseteq \mathcal{A} \oplus \Omega \subseteq \mathcal{A},$$
we obtain
$$\text{pint}^{(II)}(\mathcal{A}) \subseteq \text{int}(\mathcal{A}) \subseteq \mathcal{A} \text{ and } \text{pint}^{(II)}(\mathcal{A}) \subseteq \text{int}^{(II)}(\mathcal{A}) \subseteq \text{pint}^{(II)}(\mathcal{A}),$$
which also implies $\text{pint}^{(II)}(\mathcal{A}) = \text{int}^{(II)}(\mathcal{A})$*. Similarly, if the center* A_0 *is in the open ball* $\mathcal{B}^\diamond(A_0;\epsilon)$*, then* $\text{pint}^{(\diamond II)}(\mathcal{A}) = \text{int}^{(\diamond II)}(\mathcal{A})$*.*
- *Suppose that* $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$*. We have the following observations. From part (ii) of Proposition 8, we have* $\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A;\epsilon) \oplus \Omega$ *and* $\mathcal{B}^\diamond(A;\epsilon) \subseteq \mathcal{B}^\diamond(A;\epsilon) \oplus \Omega$*. Assume that the center* A_0 *is in the open ball* $\mathcal{B}(A_0;\epsilon)$*. Let* A_0 *be an informal type-III-pseudo-interior point of* $\mathcal{A}$*. Since*
$$A_0 \in \mathcal{B}(A_0;\epsilon) \subseteq \mathcal{B}(A_0;\epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega \subseteq \mathcal{A},$$
we obtain
$$\text{pint}^{(III)}(\mathcal{A}) \subseteq \text{int}(\mathcal{A}) \subseteq \mathcal{A} \text{ and } \text{pint}^{(III)}(\mathcal{A}) \subseteq \text{int}^{(III)}(\mathcal{A}) \subseteq \text{pint}^{(III)}(\mathcal{A}),$$
which also implies $\text{pint}^{(III)}(\mathcal{A}) = \text{int}^{(III)}(\mathcal{A})$*. Similarly, if the center* A_0 *is in the open ball* $\mathcal{B}^\diamond(A_0;\epsilon)$*, then* $\text{pint}^{(\diamond III)}(\mathcal{A}) = \text{int}^{(\diamond III)}(\mathcal{A})$*.*

Definition 6. *Let* $(\mathcal{I}, \|\cdot\|)$ *be an informal pseudo-seminormed hyperspace, and let* $\mathcal{A}$ *be a nonempty subset of* $\mathcal{I}$*. The set* $\mathcal{A}$ *is said to be informally open if* $\mathcal{A} = \text{int}(\mathcal{A})$*. The set* $\mathcal{A}$ *is said to be informally type-I-open if* $\mathcal{A} = \text{int}^{(I)}(\mathcal{A})$*. The set* $\mathcal{A}$ *is said to be informally type-II-open if* $\mathcal{A} = \text{int}^{(II)}(\mathcal{A})$*. The set* $\mathcal{A}$ *is said to be informally type-III-open if* $\mathcal{A} = \text{int}^{(III)}(\mathcal{A})$*. We can similarly define the informal* $\diamond$*-open set based on the informal* $\diamond$*-interior. Also, the informal pseudo-openness can be similarly defined.*

We adopt the convention $\emptyset \oplus \Omega = \emptyset$.

Remark 8. *Let* $(\mathcal{P}(X), \|\cdot\|)$ *be an informal pseudo-seminormed hyperspace, and let* $\mathcal{A}$ *be a nonempty subset of* $\mathcal{P}(X)$*. We consider the extreme cases of the empty set* $\emptyset$ *and whole set* $\mathcal{P}(X)$*.*

- *Since the empty set* $\emptyset$ *contains no elements, it means that* $\emptyset$ *is informally open and pseudo-open (we can regard the empty set as an open ball). It is clear that* $\mathcal{P}(X)$ *is also informally open and pseudo-open, since* $A \in \mathcal{B} \subseteq X$ *for any open ball* $\mathcal{B}$*, i.e.,* $\mathcal{P}(X) \subseteq \text{int}(\mathcal{P}(X))$ *and* $\mathcal{P}(X) \subseteq \text{pint}(\mathcal{P}(X))$*.*
- *Since* $\emptyset \oplus \Omega = \emptyset \subseteq \emptyset$*, the emptyset* $\emptyset$ *is informally type-I-open and type-I-pseudo-open. It is clear that* $\mathcal{P}(X)$ *is also informally type-I-open and type-I-pseudo-open, since* $A \in \mathcal{B} \oplus \Omega \subseteq X$ *for any open ball* $\mathcal{B}$*, i.e.,* $\mathcal{P}(X) \subseteq \text{int}^{(I)}(\mathcal{P}(X))$ *and* $\mathcal{P}(X) \subseteq \text{pint}^{(I)}(\mathcal{P}(X))$*.*
- *Since* $\emptyset \subseteq \emptyset = \Omega \oplus \emptyset$*, it means that* $\emptyset$ *is informally type-II-open and type-II-pseudo-open. We also see that* $\mathcal{P}(X)$ *is an informal type-II-open and type-II-pseudo-open set, since, for any* $A \in \mathcal{P}(X)$ *and any open*

ball $\mathcal{B}$, we have $A \in \mathcal{B} \subseteq \mathcal{P}(X) \subseteq \mathcal{P}(X) \oplus \Omega$ by part (i) of Proposition 2, i.e., $\mathcal{P}(X) \subseteq int^{(II)}(\mathcal{P}(X))$ and $\mathcal{P}(X) \subseteq pint^{(II)}(\mathcal{P}(X))$.

- *Since $\emptyset \oplus \Omega \subseteq \Omega \oplus \emptyset$, it means that $\emptyset$ is informally type-III-open and type-III-pseudo-open. Now for any $A \in \mathcal{P}(X)$ and any open ball $\mathcal{B}$, we have $A \in \mathcal{B} \subseteq X$, which says that $\mathcal{B} \oplus \Omega \subseteq X \oplus \Omega$, i.e., $\mathcal{P}(X) \subseteq int^{(III)}(\mathcal{P}(X))$ and $\mathcal{P}(X) \subseteq pint^{(III)}(\mathcal{P}(X))$. This shows that $\mathcal{P}(X)$ is informally type-III-open and type-III-pseudo-open.*

We have the above similar results for the different types of informal $\diamond$-open sets.

Proposition 9. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace, and let $\mathcal{A}$ be a nonempty subset of $\mathcal{I}$.*

- *If $\mathcal{A}$ is informally pseudo-open, i.e., $\mathcal{A} = \mathrm{pint}(\mathcal{A})$, then $\mathcal{A}$ is also informally open, i.e., $\mathcal{A} = \mathrm{pint}(\mathcal{A}) = \mathrm{int}(\mathcal{A})$. If $\mathcal{A} = \mathrm{pint}^{\diamond}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{\diamond}(\mathcal{A}) = \mathrm{int}^{\diamond}(\mathcal{A})$.*
- *If $\mathcal{A} = \mathrm{pint}^{(I)}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{(I)}(\mathcal{A}) = \mathrm{int}^{(I)}(\mathcal{A})$. If $\mathcal{A} = \mathrm{pint}^{(\diamond I)}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{(\diamond I)}(\mathcal{A}) = \mathrm{int}^{(\diamond I)}(\mathcal{A})$.*
- *If $\mathcal{A} = \mathrm{pint}^{(II)}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{(II)}(\mathcal{A}) = \mathrm{int}^{(II)}(\mathcal{A})$. If $\mathcal{A} = \mathrm{pint}^{(\diamond II)}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{(\diamond II)}(\mathcal{A}) = \mathrm{int}^{(\diamond II)}(\mathcal{A})$.*
- *If $\mathcal{A} = \mathrm{pint}^{(III)}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{(III)}(\mathcal{A}) = \mathrm{int}^{(III)}(\mathcal{A})$. If $\mathcal{A} = \mathrm{pint}^{(\diamond III)}(\mathcal{A})$, then $\mathcal{A} = \mathrm{pint}^{(\diamond III)}(\mathcal{A}) = \mathrm{int}^{(\diamond III)}(\mathcal{A})$.*

Proof. If A is an informal pseudo-interior point, i.e., $A \in \mathrm{pint}(\mathcal{A}) = \mathcal{A}$, then there exists $\epsilon > 0$ such that $\mathcal{B}(A_0;\epsilon) \subseteq \mathcal{A}$. Since $A \in \mathcal{A}$, it follows that A is also an informal interior point, i.e., $\mathrm{pint}(\mathcal{A}) \subseteq \mathrm{int}(\mathcal{A})$. From the first observation of Remark 4, we obtain the desired result. The remaining cases can be similarly realized, and the proof is complete. □

Proposition 10. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *Suppose that $\|\cdot\|$ satisfies the null super-inequality.*

- *If $\mathcal{A}$ is any type of informally pseudo-open, then $A \in \mathcal{A}$ implies $A \oplus \omega \in \mathcal{A}$ for any $\omega \in \Omega$.*
- *If $\mathcal{A}$ is informally open, then $A \in \mathcal{A}$ implies $A \oplus \omega \in \mathrm{pint}(\mathcal{A})$ for any $\omega \in \Omega$.*
- *If $\mathcal{A}$ is informally type-I-open, then $A \in \mathcal{A}$ implies $A \oplus \omega \in \mathrm{pint}^{(I)}(\mathcal{A})$ for any $\omega \in \Omega$.*
- *If $\mathcal{A}$ is informally type-II-open, then $A \in \mathcal{A}$ implies $A \oplus \omega \in \mathrm{pint}^{(II)}(\mathcal{A})$ for any $\omega \in \Omega$.*
- *If $\mathcal{A}$ is informally type-III-open, then $A \in \mathcal{A}$ implies $A \oplus \omega \in \mathrm{pint}^{(III)}(\mathcal{A})$ for any $\omega \in \Omega$.*

(ii) *Suppose that $\|\cdot\|$ satisfies the null sub-inequality, and that $\mathcal{A}$ is any type of informally pseudo-open. Then the following statements hold true.*

- *$A \oplus \omega \in \mathcal{A}$ implies $A \in \mathcal{A}$ for any $\omega \in \Omega$.*
- *$\mathcal{A} \oplus \omega \subseteq \mathcal{A}$ for any $\omega \in \Omega$ and $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$.*
- *$A \oplus \omega \in \mathcal{A} \oplus \omega$ implies $A \in \mathcal{A}$ for any $\omega \in \Omega$.*
- *We have $\mathcal{A} = \mathcal{A} \oplus \Omega$.*

(iii) *Suppose that $\|\cdot\|$ satisfies the null sub-inequality, and that $\mathcal{A}$ is any type of informal $\diamond$-pseudo-open. Then $A \in \mathcal{A}$ implies $A \oplus \omega \in \mathcal{A}$ for any $\omega \in \Omega$.*

Proof. To prove part (i), suppose that $\mathcal{A}$ is informally type-III-pseudo-open. For $A \in \mathcal{A} = \mathrm{pint}^{(III)}(\mathcal{A})$, by definition, there exists $\epsilon > 0$ such that $\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$. From part (i) of Proposition 6, we also have $\mathcal{B}(A \oplus \omega;\epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$, which says that $A \oplus \omega \in \mathrm{pint}^{(III)}(\mathcal{A}) = \mathcal{A}$. Now we assume that $\mathcal{A}$ is informally type-III-open. Then $A \in \mathcal{A} = \mathrm{int}^{(III)}(\mathcal{A}) \subseteq \mathrm{pint}^{(III)}(\mathcal{A})$. We can also obtain $A \oplus \omega \in \mathrm{pint}^{(III)}(\mathcal{A})$. The other openness can be similarly obtained.

To prove the first case of part (ii), we consider the informal type-III-pseudo-open sets. If $A \oplus \omega \in \mathcal{A} = \mathrm{pint}^{(\mathrm{III})}(\mathcal{A})$, there exists $\epsilon > 0$ such that $\mathcal{B}(A \oplus \omega; \epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$. From part (ii) of Proposition 6, we also have $\mathcal{B}(A; \epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$, which shows that $A \in \mathrm{pint}^{(\mathrm{III})}(\mathcal{A}) = \mathcal{A}$.

To prove the second case of part (ii), we consider the informal type-III-pseudo-open sets. If $A \in \mathcal{A} \oplus \omega$, then $A = \widehat{A} \oplus \omega$ for some $\widehat{A} \in \mathcal{A} = \mathrm{pint}^{(\mathrm{III})}(\mathcal{A})$. Therefore there exists $\epsilon > 0$ such that $\mathcal{B}(\widehat{A}; \epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$. Since $\mathcal{B}(A; \epsilon) \subseteq \mathcal{B}(A \oplus \omega; \epsilon) = \mathcal{B}(\widehat{A}; \epsilon)$ by part (ii) of Proposition 6, we see that $\mathcal{B}(A; \epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$, i.e., $A \in \mathrm{pint}^{(\mathrm{III})}(\mathcal{A}) = \mathcal{A}$. Now, for $A \in \mathcal{A} \oplus \Omega$, we see that $A \in \mathcal{A} \oplus \omega$ for some $\omega \in \Omega$, which implies $A \in \mathcal{A}$. Therefore we obtain $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$.

To prove the third case of part (ii), using the second case of part (ii), we have

$$A \oplus \omega \in \mathcal{A} \oplus \omega \subseteq \mathcal{A} \oplus \Omega \subseteq \mathcal{A}.$$

Using the first case of part (ii), we obtain $A \in \mathcal{A}$.

To prove the fourth case of part (ii), since $A = A \oplus \{\theta_X\}$ and $\{\theta_X\} \in \Omega$, it follows that $\mathcal{A} \subseteq \mathcal{A} \oplus \Omega$. By the second case of part (ii), we obtain the desired result.

To prove part (iii), from part (ii) of Proposition 6, we have $\mathcal{B}^\diamond(A \oplus \omega; \epsilon) \subseteq \mathcal{B}^\diamond(A; \epsilon)$. Therefore, using the similar argument in the proof of part (i), we can obtain the desired results. This completes the proof. □

We remark that the results in Proposition 10 will not be true for any types of informal open sets. For example, in the proof of part (i), the inclusion $\mathcal{B}(A \oplus \omega; \epsilon) \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$ can just say that $A \oplus \omega \in \mathrm{pint}^{(\mathrm{III})}(\mathcal{A})$, since we do not know whether $A \oplus \omega$ is in $\mathcal{A}$ or not.

Proposition 11. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *Suppose that $\|\cdot\|$ satisfies the null condition.*

- *We have $\mathrm{int}(\mathcal{A}) = \mathrm{int}^{(\mathrm{I})}(\mathcal{A}) \oplus \Omega \subseteq \mathcal{A}$. In particular, if $\mathcal{A}$ is informally open or type-I-open, then $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$.*
- *We have $\mathrm{int}^{(\mathrm{II})}(\mathcal{A}) = \mathrm{int}^{(\mathrm{III})}(\mathcal{A}) \subseteq \mathcal{A} \oplus \Omega$.*

Moreover the concept of informal (resp. type-I, type-II, type-III) open set is equivalent to the concept of informal (resp. type-I, type-II, type-III) pseudo-open set.

(ii) *Suppose that $\|\cdot\|$ satisfies the null sub-inequality. Then*

$$(\mathrm{pint}^{(\mathrm{III})}(\mathcal{A}))^c \oplus \Omega = (\mathrm{pint}^{(\mathrm{II})}(\mathcal{A}))^c \oplus \Omega \subseteq (\mathrm{pint}^{(\mathrm{II})}(\mathcal{A}))^c = (\mathrm{pint}^{(\mathrm{III})}(\mathcal{A}))^c.$$

In particular, if $\mathcal{A}$ is informally type-II-pseudo-open or type-III-pseudo-open, then $\mathcal{A}^c \oplus \Omega \subseteq \mathcal{A}^c$.

Proof. To prove the first case of part (i), for any $A \in \mathrm{int}^{(\mathrm{I})}(\mathcal{A})$, there exists an open ball $\mathcal{B}(A; \epsilon)$ such that $\mathcal{B}(A; \epsilon) \oplus \Omega \subseteq \mathcal{A}$. Since $A \in \mathcal{B}(A; \epsilon)$ by the first observation of Remark 1, we have $A \oplus \Omega \subseteq \mathcal{B}(A; \epsilon) \oplus \Omega \subseteq \mathcal{A}$. This shows $\mathrm{int}^{(\mathrm{I})}(\mathcal{A}) \oplus \Omega \subseteq \mathcal{A}$. Using Remark 5, we obtain the desired results.

To prove the second case of part (i), for any $A \in \mathrm{int}^{(\mathrm{III})}(\mathcal{A})$, there exists an open ball $\mathcal{B}(A; \epsilon)$ such that $\mathcal{B}(A; \epsilon) \subseteq \mathcal{A} \oplus \Omega$. Then we have $A \in \mathcal{A} \oplus \Omega$, since $A \in \mathcal{B}(A; \epsilon)$. This shows $\mathrm{int}^{(\mathrm{II})}(\mathcal{A}) \subseteq \mathcal{A} \oplus \Omega$. Using Remark 5, we obtain the desired results. From Remark 4, we see that the concept of informal (resp. type-I, type-II, type-III) open set is equivalent to the concept of informal (resp. type-I, type-II, type-III) pseudo-open set.

To prove part (ii), for any $A \in (\mathrm{pint}^{(\mathrm{II})}(\mathcal{A}))^c \oplus \Omega$, we have $A = \widehat{A} \oplus \widehat{\omega}$ for some $\widehat{A} \in (\mathrm{pint}^{(\mathrm{II})}(\mathcal{A}))^c$ and $\widehat{\omega} \in \Omega$. By definition, we see that $\mathcal{B}(\widehat{A}; \epsilon) \not\subseteq \mathcal{A} \oplus \Omega$ for every $\epsilon > 0$. By part (ii) of Proposition 6, we also have $\mathcal{B}(A; \epsilon) \not\subseteq \mathcal{A} \oplus \Omega$ for every $\epsilon > 0$. This says that A is not an informal type-II-pseudo-interior point of $\mathcal{A}$, i.e., $A \notin \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$. This completes the proof. □

Proposition 12. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *$\mathcal{B}^{\diamond}(A_0;\epsilon)$ is informally $\diamond$-open, $\diamond$-type-II-open and $\diamond$-type-III-open. We also have the inclusions $\mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \text{pint}(\mathcal{B}^{\diamond}(A_0;\epsilon))$, $\mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \text{pint}^{(\diamond\text{II})}(\mathcal{B}^{\diamond}(A_0;\epsilon))$ and $\mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \text{pint}^{(\diamond\text{III})}(\mathcal{B}^{\diamond}(A_0;\epsilon))$.*

(ii) *$\mathcal{B}(A_0;\epsilon)$ is informally open, type-II-open and type-III-open. We also have the inclusions $\mathcal{B}(A_0;\epsilon) \subseteq \text{pint}(\mathcal{B}(A_0;\epsilon))$, $\mathcal{B}(A_0;\epsilon) \subseteq \text{pint}^{(\text{II})}(\mathcal{B}(A_0;\epsilon))$ and $\mathcal{B}(A_0;\epsilon) \subseteq \text{pint}^{(\text{III})}(\mathcal{B}(A_0;\epsilon))$.*

(iii) *Suppose that $\|\cdot\|$ satisfies the null sub-inequality. Then $\mathcal{B}^{\diamond}(A_0;\epsilon)$ is informally $\diamond$-type-I-open, and $\mathcal{B}(A_0;\epsilon)$ is informally type-I-open. We also have the inclusions $\mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \text{pint}^{(\diamond\text{I})}(\mathcal{B}^{\diamond}(A_0;\epsilon))$ and $\mathcal{B}(A_0;\epsilon) \subseteq \text{pint}^{(\text{I})}(\mathcal{B}(A_0;\epsilon))$.*

Proof. To prove part (i), for any $A \in \mathcal{B}^{\diamond}(A_0;\epsilon)$, we have $A = A_0 \oplus C$ with $\| C \| < \epsilon$. Let $\widehat{\epsilon} = \epsilon - \| C \| > 0$. For any $\widehat{A} \in \mathcal{B}^{\diamond}(A;\widehat{\epsilon})$, i.e., $\widehat{A} = A \oplus D$ with $\| D \| < \widehat{\epsilon}$, we obtain $\widehat{A} = A_0 \oplus C \oplus D$ and

$$\| C \oplus D \| \leq \| C \| + \| D \| = \epsilon - \widehat{\epsilon} + \| D \| < \epsilon - \widehat{\epsilon} + \widehat{\epsilon} = \epsilon,$$

which means that $\widehat{A} \in \mathcal{B}^{\diamond}(A_0;\epsilon)$, i.e.,

$$\mathcal{B}^{\diamond}(A;\widehat{\epsilon}) \subseteq \mathcal{B}^{\diamond}(A_0;\epsilon). \tag{6}$$

This shows that $\mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \text{int}(\mathcal{B}^{\diamond}(A_0;\epsilon))$. Therefore we obtain $\mathcal{B}^{\diamond}(A_0;\epsilon) = \text{int}(\mathcal{B}^{\diamond}(A_0;\epsilon))$. We can similarly obtain the inclusion $\mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \text{pint}(\mathcal{B}^{\diamond}(A_0;\epsilon))$. However, we cannot have the equality $\mathcal{B}^{\diamond}(A_0;\epsilon) = \text{pint}(\mathcal{B}^{\diamond}(A_0;\epsilon))$, since $\text{pint}(\mathcal{B}^{\diamond}(A_0;\epsilon))$ is not necessarily contained in $\mathcal{B}^{\diamond}(A_0;\epsilon)$. From (6), we have $\mathcal{B}^{\diamond}(x;\widehat{\epsilon}) \oplus \Omega \subseteq \mathcal{B}^{\diamond}(A_0;\epsilon) \oplus \Omega$. This says that $\mathcal{B}^{\diamond}(A_0;\epsilon)$ is informally $\diamond$-type-III-open. On the other hand, from (6) and part (ii) of Proposition 8, we also have

$$\mathcal{B}^{\diamond}(A;\widehat{\epsilon}) \subseteq \mathcal{B}^{\diamond}(A_0;\epsilon) \subseteq \mathcal{B}^{\diamond}(A_0;\epsilon) \oplus \Omega.$$

This shows that $\mathcal{B}^{\diamond}(A_0;\epsilon)$ is informally $\diamond$-type-II-open.

To prove part (ii), for any $A \in \mathcal{B}(A_0;\epsilon)$, we have $\| A \ominus A_0 \| < \epsilon$. Let $\widehat{\epsilon} = \| A \ominus A_0 \|$. For any $\widehat{A} \in \mathcal{B}(A;\epsilon - \widehat{\epsilon})$, we have $\| \widehat{A} \ominus A \| < \epsilon - \widehat{\epsilon}$. Therefore, by Proposition 4, we obtain

$$\| \widehat{A} \ominus A_0 \| \leq \| \widehat{A} \ominus A \| + \| A \ominus A_0 \| = \widehat{\epsilon} + \| \widehat{A} \ominus A \| < \widehat{\epsilon} + \epsilon - \widehat{\epsilon} = \epsilon,$$

which means that $\widehat{A} \in \mathcal{B}(A_0;\epsilon)$, i.e.,

$$\mathcal{B}(A;\epsilon - \widehat{\epsilon}) \subseteq \mathcal{B}(A_0;\epsilon). \tag{7}$$

This shows that $\mathcal{B}(A_0;\epsilon) \subseteq \text{int}(\mathcal{B}(A_0;\epsilon))$.

Therefore we obtain $\mathcal{B}(A_0;\epsilon) = \text{int}(\mathcal{B}(A_0;\epsilon))$. We can similarly obtain the inclusion $\mathcal{B}(A_0;\epsilon) \subseteq \text{pint}(\mathcal{B}(A_0;\epsilon))$. From (7), we have $\mathcal{B}(A;\epsilon - \widehat{\epsilon}) \oplus \Omega \subseteq \mathcal{B}(A_0;\epsilon) \oplus \Omega$. This says that $\mathcal{B}(A_0;\epsilon)$ is informally type-III-open. On the other hand, from (7) and part (ii) of Proposition 8, we also have

$$\mathcal{B}(A;\epsilon - \widehat{\epsilon}) \subseteq \mathcal{B}(A_0;\epsilon) \subseteq \mathcal{B}(A_0;\epsilon) \oplus \Omega.$$

This shows that $\mathcal{B}(A_0;\epsilon)$ is informally type-II-open.

To prove part (iii), from (6), (7) and part (ii) of Proposition 8, we have

$$\mathcal{B}^{\diamond}(A;\widehat{\epsilon}) \oplus \Omega \subseteq \mathcal{B}^{\diamond}(A_0;\epsilon) \oplus \Omega = \mathcal{B}^{\diamond}(A_0;\epsilon)$$

and

$$\mathcal{B}(A;\epsilon - \widehat{\epsilon}) \oplus \Omega \subseteq \mathcal{B}(A_0;\epsilon) \oplus \Omega = \mathcal{B}(A_0;\epsilon).$$

This shows that $\mathcal{B}^{\diamond}(A_0;\epsilon)$ is informally $\diamond$-type-I-open, and that $\mathcal{B}(A_0;\epsilon)$ is informally type-I-open. We complete the proof. $\square$

Proposition 13. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace. Suppose that the center A_0 is in the open balls $\mathcal{B}(A_0;\epsilon)$ and $\mathcal{B}^{\diamond}(A_0;\epsilon)$. The following statements hold true:*

(i) *$\mathcal{B}(A_0;\epsilon)$ is informally pseudo-open and $\diamond$-pseudo-open.*
(ii) *Suppose that $\|\cdot\|$ satisfies the null sub-inequality. Then $\mathcal{B}(A_0;\epsilon)$ is informally type-I-pseudo-open, type-II-pseudo-open and type-III-pseudo-open.*
(iii) *Suppose that $\|\cdot\|$ satisfies the null sub-inequality. Then $\mathcal{B}(A_0;\epsilon)$ is informally $\diamond$-type-I-pseudo-open, $\diamond$-type-II-pseudo-open and $\diamond$-type-III-pseudo-open.*

Proof. The results follow from Proposition 12, Remark 7 and part (ii) of Proposition 8 immediately. □

6. Topoloigcal Spaces

Now we are in a position to investigate the topological structure generated by the informal pseudo-seminormed hyperspace $(\mathcal{P}(X), \|\cdot\|)$ based on the different kinds of openness. We denote by τ_0 and $\tau_0^{(\diamond)}$ the set of all informal open and informal $\diamond$-open subsets of $\mathcal{P}(X)$, respectively, and by $\mathrm{p}\tau_0$ and $\mathrm{p}\tau_0^{(\diamond)}$ the set of all informal pseudo-open and informal $\diamond$-pseudo-open subsets of $\mathcal{P}(X)$, respectively. We denote by $\tau^{(\mathrm{I})}$ and $\tau^{(\diamond\mathrm{I})}$ the set of all informal type-I-open and informal $\diamond$-type-I-open subsets of $\mathcal{P}(X)$, respectively, and by $\mathrm{p}\tau^{(\mathrm{I})}$ and $\mathrm{p}\tau^{(\diamond\mathrm{I})}$ the set of all informal type-I-pseudo-open and informal $\diamond$-type-I-pseudo-open subsets of $\mathcal{P}(X)$, respectively. We can similarly define the families $\tau^{(\mathrm{II})}$, $\tau^{(\mathrm{III})}$, $\tau^{(\diamond\mathrm{II})}$, $\tau^{(\diamond\mathrm{III})}$, $\mathrm{p}\tau^{(\mathrm{II})}$, $\mathrm{p}\tau^{(\mathrm{III})}$, $\mathrm{p}\tau^{(\diamond\mathrm{II})}$ and $\mathrm{p}\tau^{(\diamond\mathrm{III})}$.

Proposition 14. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *$(\mathcal{P}(X), \tau^{(\mathrm{I})})$ and $(\mathcal{P}(X), \tau^{(\diamond\mathrm{I})})$ are topological spaces.*
(ii) *Suppose that each open ball $\mathcal{B}(A_0;\epsilon)$ contains the center A_0. Then $(\mathcal{P}(X), p\tau^{(\mathrm{I})}) = (\mathcal{P}(X), \tau^{(\mathrm{I})})$ is a topological space.*
(iii) *Suppose that each open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$ contains the center A_0. Then $(\mathcal{P}(X), p\tau^{(\diamond\mathrm{I})}) = (\mathcal{P}(X), \tau^{(\diamond\mathrm{I})})$ is a topological space.*

Proof. To prove part (i), by the second observation of Remark 8, we see that $\emptyset \in \tau^{(\mathrm{I})}$ and $\mathcal{P}(X) \in \tau^{(\mathrm{I})}$. Let $\mathcal{A} = \bigcap_{i=1}^{n} \mathcal{A}_i$, where $\mathcal{A}_i$ are informal type-I-open sets for all $i = 1, \cdots, n$. For $A \in \mathcal{A}$, we have $A \in \mathcal{A}_i$ for all $i = 1, \cdots, n$. Then there exist ϵ_i such that $\mathcal{B}(A;\epsilon_i) \oplus \Omega \subseteq \mathcal{A}_i$ for all $i = 1, \cdots, n$. Let $\epsilon = \min\{\epsilon_1, \cdots, \epsilon_n\}$. Then $\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq \mathcal{B}(A;\epsilon_i) \oplus \Omega \subseteq \mathcal{A}_i$ for all $i = 1, \cdots, n$, which says that $\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq \bigcap_{i=1}^{n} \mathcal{A}_i = \mathcal{A}$, i.e., $\mathcal{A} \subseteq \mathrm{int}^{(\mathrm{I})}(\mathcal{A})$. Therefore the intersection $\mathcal{A}$ is informally type-I-open by Remark 4. On the other hand, let $\mathcal{A} = \bigcup_{\delta} \mathcal{A}_{\delta}$. Then $A \in \mathcal{A}$ implies that $A \in \mathcal{A}_{\delta}$ for some δ. This indicates that $\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq \mathcal{A}_{\delta} \subseteq \mathcal{A}$ for some $\epsilon > 0$, i.e., $\mathcal{A} \subseteq \mathrm{int}^{(\mathrm{I})}(\mathcal{A})$. Therefore the union $\mathcal{A}$ is informally type-I-open. This shows that $(\mathcal{P}(X), \tau^{(\mathrm{I})})$ is a topological space. For the case of informal $\diamond$-type-I-open subsets of $\mathcal{P}(X)$, we can similarly obtain the desired result. Parts (ii) and (iii) follow from Remark 7 and part (i) immediately. This completes the proof. □

Remark 1 shows the sufficient conditions for the open ball $\mathcal{B}(A;\epsilon)$ containing the center A.

Proposition 15. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *$(\mathcal{P}(X), \tau_0)$ and $(\mathcal{P}(X), \tau_0^{(\diamond)})$ are topological spaces.*
(ii) *Suppose that each open ball $\mathcal{B}(A_0;\epsilon)$ contains the center A_0. Then $(\mathcal{P}(X), \tau_0) = (\mathcal{P}(X), p\tau_0)$ is a topological space.*
(iii) *Suppose that each open ball $\mathcal{B}^{\diamond}(A_0;\epsilon)$ contains the center A_0. Then $(\mathcal{P}(X), \tau_0^{(\diamond)}) = (\mathcal{P}(X), p\tau_0^{(\diamond)})$ is a topological space.*

Proof. The empty set $\emptyset$ and $\mathcal{P}(X)$ are informal open by the first observation of Remark 8. The remaining proof follows from the similar argument of Proposition 14 without considering the null set Ω. □

Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace. We consider the following families:

$$\widetilde{\tau}^{(\mathrm{II})} = \{\mathcal{A} \in \tau^{(\mathrm{II})} : \mathcal{A} \oplus \Omega \subseteq \mathcal{A}\}$$

and

$$\widetilde{\tau}^{(\mathrm{III})} = \{\mathcal{A} \in \tau^{(\mathrm{III})} : \mathcal{A} \oplus \Omega \subseteq \mathcal{A}\}.$$

We can similarly define $\widetilde{\tau}^{(\diamond\mathrm{II})}$ and $\widetilde{\tau}^{(\diamond\mathrm{III})}$. Then $\widetilde{\tau}^{(\mathrm{II})} \subseteq \tau^{(\mathrm{II})}$, $\widetilde{\tau}^{(\mathrm{III})} \subseteq \tau^{(\mathrm{III})}$, $\widetilde{\tau}^{(\diamond\mathrm{II})} \subseteq \tau^{(\diamond\mathrm{II})}$ and $\widetilde{\tau}^{(\diamond\mathrm{III})} \subseteq \tau^{(\diamond\mathrm{III})}$. We can also similarly define $\widetilde{\mathrm{p}\tau}^{(\mathrm{II})}$, $\widetilde{\mathrm{p}\tau}^{(\mathrm{III})}$, $\widetilde{\mathrm{p}\tau}^{(\diamond\mathrm{II})}$ and $\widetilde{\mathrm{p}\tau}^{(\diamond\mathrm{III})}$ regarding the informal pseudo-openness. Then $\widetilde{\mathrm{p}\tau}^{(\mathrm{II})} \subseteq \mathrm{p}\tau^{(\mathrm{II})}$, $\widetilde{\mathrm{p}\tau}^{(\mathrm{III})} \subseteq \mathrm{p}\tau^{(\mathrm{III})}$, $\widetilde{\mathrm{p}\tau}^{(\diamond\mathrm{II})} \subseteq \mathrm{p}\tau^{(\diamond\mathrm{II})}$ and $\widetilde{\mathrm{p}\tau}^{(\diamond\mathrm{III})} \subseteq \mathrm{p}\tau^{(\diamond\mathrm{III})}$.

Proposition 16. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace. Suppose that $\|\cdot\|$ satisfies the null sub-inequality. Then*

$$\widetilde{\mathrm{p}\tau}^{(\mathrm{II})} = \mathrm{p}\tau^{(\mathrm{II})} = \mathrm{p}\tau^{(\mathrm{III})} = \widetilde{\mathrm{p}\tau}^{(\mathrm{III})} \text{ and } \widetilde{\tau}^{(\mathrm{II})} = \tau^{(\mathrm{II})} = \tau^{(\mathrm{III})} = \widetilde{\tau}^{(\mathrm{III})}.$$

Proof. The results follow from Remark 5 and part (ii) of Proposition 10 immediately. □

Proposition 17. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *$(\mathcal{P}(X), \widetilde{\tau}^{(\mathrm{II})})$ and $(\mathcal{P}(X), \widetilde{\tau}^{(\diamond\mathrm{II})})$ are topological spaces.*

(ii) *The following statements hold true.*

- *Suppose that each open ball $\mathcal{B}(A;\epsilon)$ contains the center A. Then $(\mathcal{P}(X), \widetilde{\mathrm{p}\tau}^{(\mathrm{II})}) = (\mathcal{P}(X), \widetilde{\tau}^{(\mathrm{II})})$ is a topological space.*
- *Suppose that each open ball $\mathcal{B}^{\diamond}(A;\epsilon)$ contains the center A. Then $(\mathcal{P}(X), \widetilde{\mathrm{p}\tau}^{(\diamond\mathrm{II})}) = (\mathcal{P}(X), \widetilde{\tau}^{(\diamond\mathrm{II})})$ is a topological space.*

Proof. To prove part (i), given $\mathcal{A}_1, \mathcal{A}_2 \in \widetilde{\tau}^{(\mathrm{II})}$, let $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. For $A \in \mathcal{A}$, we have $A \in \mathcal{A}_i$ for $i = 1, 2$. Then there exist ϵ_i such that $\mathcal{B}(A;\epsilon_i) \subseteq \mathcal{A}_i \oplus \Omega$ for all $i = 1, 2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then

$$\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A;\epsilon_i) \subseteq \mathcal{A}_i \oplus \Omega$$

for all $i = 1, 2$, which says that

$$\mathcal{B}(A;\epsilon) \subseteq [(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)] = (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega = \mathcal{A} \oplus \Omega$$

by Proposition 3. This shows that $\mathcal{A}$ is informally type-II-open. For $A \in \mathcal{A} \oplus \Omega$, we have $A = \bar{A} \oplus \omega$ for some $\bar{A} \in \mathcal{A}$ and $\omega \in \Omega$. Since $\bar{A} \in \mathcal{A}_1 \cap \mathcal{A}_2$, it follows that $A \in \mathcal{A}_1 \oplus \Omega \subseteq \mathcal{A}_1$ and $A \in \mathcal{A}_2 \oplus \Omega \subseteq \mathcal{A}_2$, which says that $A \in \mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}$, i.e., $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$. This shows that $\mathcal{A}$ is indeed in $\widetilde{\tau}^{(\mathrm{II})}$. Therefore, the intersection of finitely many members of $\widetilde{\tau}^{(\mathrm{II})}$ is a member of $\widetilde{\tau}^{(\mathrm{II})}$.

Now, given a family $\{\mathcal{A}_\delta\}_{\delta \in \Lambda} \subset \tau^{(\mathrm{II})}$, let $\mathcal{A} = \bigcup_{\delta \in \Lambda} \mathcal{A}_\delta$. Then $A \in \mathcal{A}$ implies that $A \in \mathcal{A}_\delta$ for some $\delta \in \Lambda$. This says that

$$\mathcal{B}(A;\epsilon) \subseteq \mathcal{A}_\delta \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$$

for some $\epsilon > 0$. Therefore, the union A is informally type-II-open. For $A \in \mathcal{A} \oplus \Omega$, we have $A = \bar{A} \oplus \omega$, where $\bar{A} \in \mathcal{A}$, i.e., $\bar{A} \in \mathcal{A}_\delta$ for some $\delta \in \Lambda$. It also says that $A \in \mathcal{A}_\delta \oplus \Omega \subseteq \mathcal{A}_\delta \subseteq \mathcal{A}$, i.e., $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$. This shows that $\mathcal{A}$ is indeed in $\widetilde{\tau}^{(\mathrm{II})}$. By the third observation of Remark 8, we see that $\emptyset$ and $\mathcal{P}(X)$ are also informal type-II-open. It is not hard to see that $\emptyset \oplus \Omega = \emptyset$ and $\mathcal{P}(X) \oplus \Omega \subseteq \mathcal{P}(X)$, which shows that $\emptyset, X \in \widetilde{\tau}^{(\mathrm{II})}$. Therefore, $(\mathcal{P}(X), \widetilde{\tau}^{(\mathrm{II})})$ is indeed a topological space. The above arguments are also valid for $\widetilde{\tau}^{(\diamond\mathrm{II})}$.

Part (ii) follows immediately from the third observation of Remark 7 and part (i). This completes the proof. □

Proposition 18. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace.*

(i) *$(\mathcal{P}(X), \widetilde{\tau}^{(\mathrm{III})})$ and $(\mathcal{P}(X), \widetilde{\tau}^{(\diamond\mathrm{III})})$ are topological spaces.*

(ii) *The following statements hold true.*

- *Suppose that each open ball $\mathcal{B}(A;\epsilon)$ contains the center A. Then $(\mathcal{P}(X), \widetilde{p\tau}^{(\mathrm{III})}) = (\mathcal{P}(X), \widetilde{\tau}^{(\mathrm{III})})$ is a topological space.*
- *Suppose that each open ball $\mathcal{B}^{\diamond}(A;\epsilon)$ contains the center A. Then $(\mathcal{P}(X), \widetilde{p\tau}^{(\diamond\mathrm{III})}) = (\mathcal{P}(X), \widetilde{\tau}^{(\diamond\mathrm{III})})$ is a topological space.*

Proof. To prove part (i), by the fourth observation of Remark 8, it is clear to see that $\emptyset, \mathcal{P}(X) \in \tau^{(\mathrm{III})}$. Since $\emptyset \oplus \Omega = \emptyset$ and $\mathcal{P}(X) \oplus \Omega \subseteq \mathcal{P}(X)$, it follows that $\emptyset, \mathcal{P}(X) \in \widetilde{\tau}^{(\mathrm{III})}$. Given $\mathcal{A}_1, \mathcal{A}_2 \in \widetilde{\tau}^{(\mathrm{III})}$, let $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. For $A \in \mathcal{A}$, there exist ϵ_i such that $\mathcal{B}(A;\epsilon_i) \oplus \Omega \subseteq \mathcal{A}_i \oplus \Omega$ for all $i = 1,2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then

$$\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq \mathcal{B}(A;\epsilon_i) \oplus \Omega \subseteq \mathcal{A}_i \oplus \Omega$$

for all $i = 1,2$, which says that

$$\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq [(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)] = (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega = \mathcal{A} \oplus \Omega$$

by Proposition 3. This shows that $\mathcal{A}$ is informally type-III-open. From the proof of Proposition 17, we also see that $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$. Therefore, the intersection of finitely many members of $\widetilde{\tau}^{(\mathrm{III})}$ is a member of $\widetilde{\tau}^{(\mathrm{III})}$.

Now, given a family $\{\mathcal{A}_\delta\}_{\delta \in \Lambda} \subset \widetilde{\tau}^{(\mathrm{III})}$, let $\mathcal{A} = \bigcup_{\delta \in \Lambda} \mathcal{A}_\delta$. Then $A \in \mathcal{A}$ implies that $A \in \mathcal{A}_\delta$ for some $\delta \in \Lambda$. This says that

$$\mathcal{B}(A;\epsilon) \oplus \Omega \subseteq \mathcal{A}_\delta \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$$

for some $\epsilon > 0$. Therefore, the union $\mathcal{A}$ is informally type-III-open. From the proof of Proposition 17, we also see that $\mathcal{A} \oplus \Omega \subseteq \mathcal{A}$, i.e., $\mathcal{A} \in \widetilde{\tau}^{(\mathrm{III})}$. This shows that $(\mathcal{P}(X), \widetilde{\tau}^{(\mathrm{III})})$ is indeed a topological space. The above arguments are also valid for $\widetilde{\tau}^{(\diamond\mathrm{III})}$.

Part (ii) follows immediately from the fourth observation of Remark 7 and part (i). This completes the proof. □

Proposition 19. *Let $(\mathcal{P}(X), \|\cdot\|)$ be an informal pseudo-seminormed hyperspace. Suppose that $\|\cdot\|$ satisfies the null sub-inequality. If each open ball $\mathcal{B}(A;\epsilon)$ contains the center A, then $(\mathcal{P}(X), \mathrm{p}\tau^{(\mathrm{II})}) = (\mathcal{P}(X), \mathrm{p}\tau^{(\mathrm{III})})$ is a topological space.*

Proof. By the third observation of Remark 8, we see that $\emptyset, \mathcal{P}(X) \in \mathrm{p}\tau^{(\mathrm{II})}$. Given $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{p}\tau^{(\mathrm{II})}$, let $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. We want to show $\mathcal{A} = \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$. For $A \in \mathcal{A}$, we have $A \in \mathcal{A}_i$ for $i = 1,2$. There exist ϵ_i such that $\mathcal{B}(A;\epsilon_i) \subseteq \mathcal{A}_i \oplus \Omega$ for all $i = 1,2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $\mathcal{B}(A;\epsilon) \subseteq \mathcal{B}(A;\epsilon_i) \subseteq \mathcal{A}_i \oplus \Omega$ for $i = 1,2$, which says that, using part (ii) of Proposition 10,

$$\mathcal{B}(A;\epsilon) \subseteq [(\mathcal{A}_1 \oplus \Omega) \cap (\mathcal{A}_2 \oplus \Omega)] = \mathcal{A}_1 \cap \mathcal{A}_2 = (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega = \mathcal{A} \oplus \Omega.$$

This shows that $A \in \mathrm{int}^{(\mathrm{II})}(\mathcal{A})$, i.e., $\mathcal{A} \subseteq \mathrm{int}^{(\mathrm{II})}(\mathcal{A}) \subseteq \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$ by Remark 4. On the other hand, for $A \in \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$, using part (ii) of Proposition 10, we have

$$A \in \mathcal{B}(A;\epsilon) \subseteq \mathcal{A} \oplus \Omega = (\mathcal{A}_1 \cap \mathcal{A}_2) \oplus \Omega \subseteq \mathcal{A}_1 \oplus \Omega = \mathcal{A}_1.$$

We can similarly obtain $A \in \mathcal{A}_2$, i.e., $A \in \mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}$. This shows that $\mathrm{pint}^{(\mathrm{II})}(\mathcal{A}) \subseteq \mathcal{A}$. Therefore, we conclude that the intersection of finitely many members of $\mathrm{p}\tau^{(\mathrm{II})}$ is a member of $\mathrm{p}\tau^{(\mathrm{II})}$.

Now, given a family $\{\mathcal{A}_\delta\}_{\delta\in\Lambda} \subset \mathrm{p}\tau^{(\mathrm{II})}$, let $\mathcal{A} = \bigcup_{\delta\in\Lambda} \mathcal{A}_\delta$. Then $A \in \mathcal{A}$ implies that $A \in \mathcal{A}_\delta$ for some $\delta \in \Lambda$. This says that

$$\mathcal{B}(A;\epsilon) \subseteq \mathcal{A}_\delta \oplus \Omega \subseteq \mathcal{A} \oplus \Omega$$

for some $\epsilon > 0$. Therefore we obtain $\mathcal{A} \subseteq \mathrm{int}^{(\mathrm{II})}(\mathcal{A}) \subseteq \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$. On the other hand, for $A \in \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$, we have

$$A \in \mathcal{B}(A;\epsilon) \subseteq \mathcal{A} \oplus \Omega = \mathcal{A}$$

by part (ii) of Proposition 10. This shows that $\mathrm{pint}^{(\mathrm{II})}(\mathcal{A}) \subseteq \mathcal{A}$, i.e., $\mathcal{A} = \mathrm{pint}^{(\mathrm{II})}(\mathcal{A})$. Therefore, by Remark 5, we conclude that $(\mathcal{P}(X), \mathrm{p}\tau^{(\mathrm{II})}) = (\mathcal{P}(X), \mathrm{p}\tau^{(\mathrm{III})})$ is a topological space. This completes the proof. □

7. Conclusions

The hyperspace denoted by $\mathcal{P}(X)$ is the collection of all subsets of a vector space X. Under the set addition

$$A \oplus B = \{a + b : a \in A \text{ and } b \in B\}$$

and the scalar multiplication

$$\lambda A = \{\lambda a : a \in A\},$$

the hyperspace $\mathcal{P}(X)$ cannot form a vector space. The reason is that each $A \in \mathcal{P}(X)$ cannot have the additive inverse element. In this paper, the null set defined by

$$\Omega = \{A \ominus A : A \in \mathcal{P}(X)\}$$

can be treated as a kind of "zero element" of $\mathcal{P}(X)$. Although $\mathcal{P}(X)$ is not a vector space, a so-called informal norm is introduced to $\mathcal{P}(X)$, which will mimic the conventional norm. Using this informal norm, two different concepts of open balls are proposed, which are used to define many types of open sets. Therefore, we can generate many types of topologies based on these different concepts of open sets.

As we mentioned before, the theory of set-valued analysis has been applied to nonlinear analysis, differential inclusion, fixed point theory and set-valued optimization, which treats each element in $\mathcal{P}(X)$ as a subset of X. In this paper, each element of $\mathcal{P}(X)$ is treated as a "point", and the family $\mathcal{P}(X)$ is treated as a universal set. The topological structures studied in this paper may provide the potential applications in nonlinear analysis, differential inclusion, fixed point theory and set-valued optimization (or set optimization) based on the different point of view regarding the elements of $\mathcal{P}(X)$, which will be for future research.

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