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An Investigation of the Common Solutions for Coupled Systems of Functional Equations Arising in Dynamic Programming

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Abstract: The purpose of this paper is to introduce the new notion of a specific point in the space of the bounded real-valued functions on a given non-empty set and present a result based on the existence and uniqueness of such points. As a consequence of our results, we discuss the existence of a unique common solution to coupled systems of functional equations arising in dynamic programming.

Keywords: ϑ-contraction mapping; α-coupled fixed point; functional equation

1. Introduction and Preliminaries

Banach fixed-point theorem [1], considered to be the source of metrical fixed-point theory, has been generalized by many researchers; see [2–5]. One of the most interesting generalizations of this theorem was given by Jleli and Samet [6] by introducing the notion of ϑ-contraction.

Definition 1 ([6])**.** *A self-mapping Υ on a metric space (Λ, d) is said to be a ϑ-contraction, if there exist $\vartheta \in \Theta$, and $k \in (0,1)$ such that*

$$\eta_1, \eta_2 \in \Lambda, \quad d(\Upsilon\eta_1, \Upsilon\eta_2) \neq 0 \Rightarrow \vartheta(d(\Upsilon\eta_1, \Upsilon\eta_2)) \leq [\vartheta(d(\eta_1, \eta_2))]^k,$$

where Θ is the set of functions $\vartheta : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

(ϑ_1) *ϑ is non-decreasing;*
(ϑ_2) *for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to+\infty} \vartheta(t_n) = 1$ iff $\lim_{n\to+\infty} t_n = 0$;*
(ϑ_3) *there exist $r \in (0,1)$ and $\lambda \in (0, +\infty]$ such that $\lim_{t\to 0^+} \frac{\vartheta(t)-1}{t^r} = \lambda$.*

Then, Jleli and Samet [6] proved that every ϑ-contraction on a complete metric space has a unique fixed point.

Here we give an example which illustrates the functions in Θ.

Example 1. *Let $\vartheta_1, \vartheta_2 : (0, +\infty) \to (1, +\infty)$ defined by*

$$\vartheta_1(t) = e^{\sqrt{t}}, \quad \vartheta_2(t) = 1 + \sqrt{t}.$$

Then $\vartheta_1, \vartheta_2 \in \Theta$.

Throughout this paper, for a fixed non-empty set Ω, we use the notation $B(\Omega)$ which stands for the set of all bounded real-valued functions on Ω. Also, unless otherwise specified, d is the sup metric on $B(\Omega)$ defined by

$$d(h,k) = \sup_{\eta\in\Omega} |h\eta - k\eta|,$$

for all $h,k \in B(\Omega)$. It is well known that $B(\Omega)$, endowed with the sup metric, is a complete metric space.

Recently, Harjani et al. [7] introduced the notion of α-coupled fixed point in the space of the bounded functions on a non-empty set as follows.

Definition 2 ([7])**.** *Let Ω be a non-empty set and $\alpha : \Omega \to \Omega$ be a given mapping. An element $(\eta,\xi) \in B(\Omega) \times B(\Omega)$ is called an α-coupled fixed point of mapping $\Gamma : B(\Omega) \times B(\Omega) \to B(\Omega)$ if $\Gamma(\eta,\xi) = \eta$ and $\Gamma(\eta(\alpha),\xi(\alpha)) = \xi$.*

They also used the above concept to prove the existence and uniqueness of solutions for a coupled system of functional equations arising in dynamic programming. The purpose of this paper is to introduce the notion of α-coupled common fixed points and present a result based on the existence and uniqueness of such points. As a consequence of our results, we discuss the existence of a unique common solution of coupled systems of functional equations arising in dynamic programming.

2. Main Theoretical Results

First, we introduce the notion of α-coupled common fixed points as follows.

Definition 3. *Let Ω be a non-empty set and $\alpha : \Omega \to \Omega$ be a given mapping. An element $(\eta,\xi) \in B(\Omega) \times B(\Omega)$ is called an α-coupled common fixed point of mappings $\Gamma, \Upsilon : B(\Omega) \times B(\Omega) \to B(\Omega)$ if $\Gamma(\eta,\xi) = \Upsilon(\eta,\xi) = \eta$ and $\Gamma(\eta(\alpha),\xi(\alpha)) = \Upsilon(\eta(\alpha),\xi(\alpha)) = \xi$.*

Now, we give the main theorem of this paper.

Theorem 1. *Let Ω be a non-empty set, $\alpha : \Omega \to \Omega$ and $\Gamma, \Upsilon : B(\Omega) \times B(\Omega) \to B(\Omega)$ be given mappings. If there exist $\vartheta \in \Theta$ and $k \in (0,1)$ such that*

$$\vartheta(d(\Gamma(\eta_1,\eta_2), \Upsilon(\xi_1,\xi_2))) \leq [\vartheta(\max\{d(\eta_1,\xi_1), d(\eta_2,\xi_2)\})]^k, \tag{1}$$

for all $\eta_1,\eta_2,\xi_1,\xi_2 \in B(\Omega)$ with $d(\Gamma(\eta_1,\eta_2), \Upsilon(\xi_1,\xi_2)) > 0$, then Γ and Υ have a unique α-coupled common fixed point.

Before going to the proof, we give the following lemma which will be used efficiently in the proof of Theorem 1.

Lemma 1. *Let (Λ, d) be a complete metric space and, σ and ϱ be self-mappings on Λ such that*

$$\eta_1,\eta_2 \in \Lambda, \quad d(\sigma\eta_1, \varrho\eta_2) > 0 \Rightarrow \vartheta(d(\sigma\eta_1,\varrho\eta_2)) \leq [\vartheta(d(\eta_1,\eta_2))]^k, \tag{2}$$

where $\vartheta \in \Theta$ and $k \in (0,1)$. Then σ and ϱ have a unique common fixed point.

Proof. Notice that by (2), we deduce

$$\ln[\vartheta(d(\sigma\eta_1,\varrho\eta_2))] \leq k\ln[\vartheta(d(\eta_1,\eta_2))] < \ln[\vartheta(d(\eta_1,\eta_2))],$$

which implies from (ϑ_1) that

$$d(\sigma\eta_1,\varrho\eta_2) \leq d(\eta_1,\eta_2), \quad \text{for all } \eta_1,\eta_2 \in \Lambda. \tag{3}$$

First, we prove that ξ is a fixed point of σ if and only if ξ is a fixed point of ϱ. Suppose that ξ is a fixed point of ϱ. Also, assume that ξ is not a fixed point of σ. Then, considering (3), we have

$$0 < d(\sigma\xi, \xi) = d(\sigma\xi, \varrho\xi) \leq d(\xi, \xi) = 0$$

which is a contradiction, and this implies that $\sigma\xi = \xi$. Similarly, it is easy to show that if ξ is a fixed point of σ, then ξ is a fixed point of ϱ.

Let $\eta_0 \in \Lambda$. Define the sequence $\{\eta_n\}$ in Λ by $\eta_{2n+1} = \sigma\eta_{2n}$ and $\eta_{2n+2} = \varrho\eta_{2n+1}$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $\eta_{2n} = \eta_{2n+1}$ for some $n \in \mathbb{N}$, then $\eta_{2n} = \sigma\eta_{2n}$. Thus, η_{2n} is a fixed point of σ and so x_{2n} is a fixed point of ϱ, that is, $\eta_{2n} = \sigma\eta_{2n} = \varrho\eta_{2n}$. Similarly, if $\eta_{2n+1} = \eta_{2n+2}$ for some $n \in \mathbb{N}$, then it is easy to see that $\eta_{2n+1} = \sigma\eta_{2n+1} = \varrho\eta_{2n+1}$. Hence we can assume that $\eta_n \neq \eta_{n+1}$ for all $n \in \mathbb{N}$. Then, for $n = 2m+1$, where $m \in \mathbb{N} \cup \{0\}$, using (2) we get

$$\begin{aligned}
\vartheta(d(\eta_n, \eta_{n+1})) &= \vartheta(d(\eta_{2m+1}, \eta_{2m+2})) \\
&= \vartheta(d(\sigma\eta_{2m}, \varrho\eta_{2m+1})) \\
&\leq [\vartheta(d(\eta_{2m}, \eta_{2m+1}))]^k \\
&\leq [\vartheta(d(\eta_{2m-1}, \eta_{2m}))]^{k^2} \\
&\vdots \\
&\leq [\vartheta(d(\eta_0, \eta_1))]^{k^{2m+1}} \\
&= [\vartheta(d(\eta_0, \eta_1))]^{k^n}.
\end{aligned}$$

By a similar method to above, for $n = 2m,$, where $m \in \mathbb{N} \cup \{0\}$, we can again obtain

$$\vartheta(d(\eta_n, \eta_{n+1})) \leq [\vartheta(d(\eta_0, \eta_1))]^{k^n}.$$

Thus, for all $n \in \mathbb{N}$, we have

$$1 < \vartheta(d(\eta_n, \eta_{n+1})) \leq [\vartheta(d(\eta_0, \eta_1))]^{k^n}. \tag{4}$$

Letting $n \to +\infty$ in the above equation, we get

$$\lim_{n\to+\infty} \vartheta(d(\eta_n, \eta_{n+1})) = 1,$$

which implies by (ϑ_2) that

$$\lim_{n\to+\infty} d(\eta_n, \eta_{n+1}) = 0. \tag{5}$$

Let $d_n := d(\eta_n, \eta_{n+1})$ for all $n \in \mathbb{N}$. To prove that $\{\eta_n\}$ is a Cauchy sequence, let us consider condition (ϑ_3). Then there exist $r \in (0,1)$ and $\lambda \in (0, +\infty]$ such that

$$\lim_{n\to+\infty} \frac{\vartheta(d_n) - 1}{(d_n)^r} = \lambda. \tag{6}$$

Let $\delta \in (0, \lambda)$. By the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$[d_n]^r \leq \delta^{-1}[\vartheta(d_n) - 1], \quad \text{for all } n > n_0.$$

Using (4) and the above inequality, we infer

$$n[d_n]^r \leq \delta^{-1} n([\vartheta(d_0)]^{k^n} - 1), \quad \text{for all } n > n_0.$$

This implies that

$$\lim_{n\to+\infty} n[d_n]^r = \lim_{n\to+\infty} n[d(\eta_n,\eta_{n+1})]^r = 0.$$

Then, there exists $n_1 \in \mathbb{N}$ such that

$$d(\eta_n,\eta_{n+1}) \le \frac{1}{n^{1/r}}, \quad \text{for all } n > n_1. \tag{7}$$

Let $m > n > n_1$. Then, using the triangular inequality and (7), we have

$$d(\eta_n,\eta_m) \le \sum_{k=n}^{m-1} d(\eta_k,\eta_{k+1}) \le \sum_{k=n}^{m-1} \frac{1}{k^{1/r}} \le \sum_{k=n}^{\infty} \frac{1}{k^{1/r}}$$

and hence $\{\eta_n\}$ is a Cauchy sequence in Λ. From the completeness of (Λ, d), there exists $\xi \in \Lambda$ such that $\eta_n \to \xi$ as $n \to +\infty$.

Now, we show that ξ is a common fixed point of σ and ϱ. By considering (3), we deduce

$$d(\eta_{2n+1},\varrho\xi) = d(\sigma\eta_{2n},\varrho\xi) \le d(\eta_{2n},\xi).$$

Passing to limit as $n \to +\infty$ in the above inequality, we obtain $d(\xi,\varrho\xi) = 0$ and so $\xi = \varrho\xi$. That is, ξ is a fixed point of ϱ. Taking into account the fact that ξ is a fixed point of σ iff ξ is a fixed point of ϱ, we conclude that ξ is also a fixed point of σ.

To show the uniqueness of common fixed point of σ and ϱ, suppose that there exist $\eta_1,\eta_2 \in \Lambda$ with $\eta_1 \ne \eta_2$ such that $\eta_1 = \sigma\eta_1 = \varrho\eta_1$ and $\eta_2 = \sigma\eta_2 = \varrho\eta_2$. Then, from (2), we get

$$\vartheta(d(\eta_1,\eta_2)) = \vartheta(d(\sigma\eta_1,\varrho\eta_2)) \le [\vartheta(d(\eta_1,\eta_2))]^k < \vartheta(d(\eta_1,\eta_2)),$$

which is a contradiction. Then σ and ϱ have one and only one common fixed point. □

Now, we are ready to present the proof of Theorem 1.

Proof. Define $\delta : B(\Omega) \times B(\Omega) \to [0,+\infty)$ by

$$\delta((\eta_1,\eta_2),(\xi_1,\xi_2)) = \max\{d(\eta_1,\xi_1), d(\eta_2,\xi_2)\}, \quad \text{for all } \eta_1,\eta_2,\xi_1,\xi_2 \in B(\Omega).$$

Then, $(B(\Omega)\times B(\Omega),\delta)$ is a complete metric space, since $(B(\Omega),d)$ is complete.

Consider the mappings $\Sigma_\Gamma, \Sigma_\Upsilon : B(\Omega)\times B(\Omega) \to B(\Omega)\times B(\Omega)$ defined by

$$\Sigma_\Gamma(U) = (\Gamma(\eta_1,\eta_2), \Gamma(\eta_1(\alpha),\eta_2(\alpha)))$$

and

$$\Sigma_\Upsilon(U) = (\Upsilon(\eta_1,\eta_2), \Upsilon(\eta_1(\alpha),\eta_2(\alpha))),$$

where $U = (\eta_1,\eta_2)$. Then, Σ_Γ and Σ_Υ satisfy all assumptions of Lemma 1. Indeed, taking account of (ϑ_1) and (1), for all $U = (\eta_1,\eta_2), V = (\xi_1,\xi_2) \in B(\Omega)\times B(\Omega)$, we deduce

$$\begin{aligned}
&\vartheta(\delta(\Sigma_\Gamma(U),\Sigma_\Upsilon(V))) \\
&= \vartheta(\delta((\Gamma(\eta_1,\eta_2),\Gamma(\eta_1(\alpha),\eta_2(\alpha))),(\Upsilon(\xi_1,\xi_2),\Upsilon(\xi_1(\alpha),\xi_2(\alpha))))) \\
&= \vartheta(\max\{d(\Gamma(\eta_1,\eta_2),\Upsilon(\xi_1,\xi_2)), d(\Gamma(\eta_1(\alpha),\eta_2(\alpha)),\Upsilon(\xi_1(\alpha),\xi_2(\alpha)))\}) \\
&= \max\{\vartheta(d(\Gamma(\eta_1,\eta_2),\Upsilon(\xi_1,\xi_2))), \vartheta(d(\Gamma(\eta_1(\alpha),\eta_2(\alpha)),\Upsilon(\xi_1(\alpha),\xi_2(\alpha))))\} \\
&\le \max\{[\vartheta(\max\{d(\eta_1,\xi_1),d(\eta_2,\xi_2)\})]^k, \\
&\qquad [\vartheta(\max\{d(\eta_1(\alpha),\xi_1(\alpha)),d(\eta_2(\alpha),\xi_2(\alpha))\})]^k\}.
\end{aligned}$$

Since

$$\begin{aligned} d(\eta_1(\alpha), \xi_1(\alpha)) &= \sup_{\omega \in \Omega} |\eta_1(\alpha)(\omega), \xi_1(\alpha)(\omega)| \\ &\leq \sup_{\omega \in \Omega} |\eta_1(\omega), \xi_1(\omega)| = d(\eta_1, \xi_1), \end{aligned}$$

and similarly $d(\eta_2(\alpha), \xi_2(\alpha)) \leq d(\eta_2, \xi_2)$, we infer that

$$\begin{aligned} \vartheta(\delta(\Sigma_\Gamma(U), \Sigma_\Upsilon(V))) &\leq [\vartheta(\max\{d(\eta_1, \xi_1), d(\eta_2, \xi_2)\})]^k \\ &= [\vartheta(\delta(U, V))]^k. \end{aligned}$$

That is, Σ_Γ and Σ_Υ satisfy the inequality (2). Therefore, by Lemma 1, there exists a unique $U^* = (\eta_1^*, \eta_2^*) \in B(\Omega) \times B(\Omega)$ such that $\Sigma_\Gamma(U^*) = \Sigma_\Upsilon(U^*) = U^*$. This means that

$$\Gamma(\eta_1^*, \eta_2^*) = \Upsilon(\eta_1^*, \eta_2^*) = \eta_1^*$$

and

$$\Gamma(\eta_1^*(\alpha), \eta_2^*(\alpha)) = \Upsilon(\eta_1^*(\alpha), \eta_2^*(\alpha)) = \eta_2^*.$$

This finishes the proof. □

3. Application to Dynamic Programming

Consider the following coupled systems of functional equations

$$\begin{aligned} r_1(\eta) &= \sup_{\xi \in \Delta} \{p(\eta, \xi) + P(\eta, \xi, r_1(\kappa(\eta, \xi)), s_1(\kappa(\eta, \xi)))\} \\ s_1(\eta) &= \sup_{\xi \in \Delta} \{p(\eta, \xi) + P(\eta, \xi, r_1(\alpha(\kappa(\eta, \xi))), s_1(\alpha(\kappa(\eta, \xi))))\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} r_2(\eta) &= \sup_{\xi \in \Delta} \{p(\eta, \xi) + Q(\eta, \xi, r_2(\kappa(\eta, \xi)), s_2(\kappa(\eta, \xi)))\} \\ s_2(\eta) &= \sup_{\xi \in \Delta} \{p(\eta, \xi) + Q(\eta, \xi, r_2(\alpha(\kappa(\eta, \xi))), s_2(\alpha(\kappa(\eta, \xi))))\} \end{aligned} \tag{9}$$

for all $\eta \in \Omega$, which appear in the study of dynamic programming (see [8]), where Ω is a state space, Δ is a decision space, $\kappa : \Omega \times \Delta \to \Omega$, $p : \Omega \times \Delta \to \mathbb{R}$, $P, Q : \Omega \times \Delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha : \Omega \to \Omega$ are given mappings.

In this section, we discuss the existence of a unique common solution to the systems of functional Equations (8) and (9) by using the obtained results in the previous section.

Theorem 2. *Consider the systems of functional Equations (8) and (9). Assume that the following conditions are satisfied:*

(*i*) $p : \Omega \times \Delta \to \mathbb{R}$ *and* $P, Q : \Omega \times \Delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are bounded functions;*
(*ii*) *there exists* $\beta \in (0, 1)$ *such that for arbitrary points* $\eta \in \Omega$, $\xi \in \Delta$ *and* $h, k, h_1, k_1 \in \mathbb{R}$,

$$|P(\eta, \xi, h, k) - Q(\eta, \xi, h_1, k_1)| \leq \left[\left[1 + \sqrt{\sup\{|h - h_1|, |k - k_1|\}}\right]^\beta - 1\right]^2.$$

Then Equations (8) and (9) have a unique common solution in $B(\Omega) \times B(\Omega)$.

Proof. First, we consider the operators Γ and Υ defined on $B(\Omega)\times B(\Omega)$ as

$$\begin{aligned}(\Gamma(r,s))(\eta)&=\sup_{\xi\in\Delta}\{p(\eta,\xi)+P(\eta,\xi,r(\kappa(\eta,\xi)),s(\kappa(\eta,\xi)))\},\\(\Upsilon(r,s))(\eta)&=\sup_{\xi\in\Delta}\{p(\eta,\xi)+Q(\eta,\xi,r(\kappa(\eta,\xi)),s(\kappa(\eta,\xi)))\}\end{aligned}\tag{10}$$

for all $(r,s)\in B(\Omega)\times B(\Omega)$ and $\eta\in\Omega$. Since functions p, P and Q are bounded, then Γ and Υ are well-defined.

Now we will show that Γ and Υ satisfy the condition (1) in Theorem 1 with the sub metric d. Let $(r_1,s_1),(r_2,s_2)\in B(\Omega)\times B(\Omega)$. Then, by (ii), we get

$$\begin{aligned}d(\Gamma(&r_1,s_1),\Upsilon(r_2,s_2))\\&=\sup_{\eta\in\Omega}|\Gamma(r_1,s_1)(\eta)-\Upsilon(r_2,s_2)(\eta)|\\&=\sup_{\eta\in\Omega}\Bigg|\sup_{\xi\in\Delta}\{p(\eta,\xi)+P(\eta,\xi,r_1(\kappa(\eta,\xi)),s_1(\kappa(\eta,\xi)))\}\\&\qquad-\sup_{\xi\in\Delta}\{p(\eta,\xi)+Q(\eta,\xi,r_2(\kappa(\eta,\xi)),s_2(\kappa(\eta,\xi)))\}\Bigg|\\&\le\sup_{\eta\in\Omega}\Bigg\{\sup_{\xi\in\Delta}|P(\eta,\xi,r_1(\kappa(\eta,\xi)),s_1(\kappa(\eta,\xi)))\\&\qquad-Q(\eta,\xi,r_2(\kappa(\eta,\xi)),s_2(\kappa(\eta,\xi)))|\Bigg\}\\&\le\sup_{\eta\in\Omega}\left\{\sup_{\xi\in\Delta}\left\{\left[\left[1+\sqrt{\sup\{A,B\}}\right]^\beta-1\right]^2\right\}\right\},\end{aligned}\tag{11}$$

where

$$A:=|r_1(\kappa(\eta,\xi))-r_2(\kappa(\eta,\xi))|\quad\text{and}\quad B:=|s_1(\kappa(\eta,\xi))-s_2(\kappa(\eta,\xi))|.$$

It yields that

$$d(\Gamma(r_1,s_1),\Upsilon(r_2,s_2))\le\left[\left[1+\sqrt{\sup\{d(r_1,r_2),d(s_1,s_2)\}}\right]^\beta-1\right]^2.\tag{12}$$

From the above inequality, we obtain

$$1+\sqrt{d(\Gamma(r_1,s_1),\Upsilon(r_2,s_2))}\le\left[1+\sqrt{\sup\{d(r_1,r_2),d(s_1,s_2)\}}\right]^\beta.\tag{13}$$

By setting $\vartheta\in\Theta$ by $\vartheta(t)=1+\sqrt{t}$ for all $t>0$ and using (13), we infer

$$\vartheta(d(\Gamma(r_1,s_1),\Upsilon(r_2,s_2)))\le[\vartheta(\max\{d(r_1,r_2),d(s_1,s_2)\})]^\beta$$

for all $(r_1,s_1),(r_2,s_2)\in B(\Omega)\times B(\Omega)$. This means that the condition (1) of Theorem 1 holds and consequently, Γ and Υ have a unique α-coupled common fixed point, i.e., Equations (8) and (9) have a unique common solution in $B(\Omega)\times B(\Omega)$. □

By using the same method in the proof of Theorem 2 together Theorem 1 with the function $\vartheta\in\Theta$ defined by $\vartheta(t)=e^{\sqrt{t}}$, we get the following result.

Theorem 3. *Consider the systems of functional Equations (8) and (9). Assume that the following conditions are satisfied:*

(i) *$p : \Omega \times \Delta \to \mathbb{R}$ and $P, Q : \Omega \times \Delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are bounded functions;*
(ii) *there exists $\beta \in (0, +\infty)$ such that for arbitrary points $\eta \in \Omega$, $\xi \in \Delta$ and $h, k, h_1, k_1 \in \mathbb{R}$,*

$$|P(\eta, \xi, h, k) - Q(\eta, \xi, h_1, k_1)| \leq e^{-\beta} \sup\{|h - h_1|, |k - k_1|\}.$$

Then Equations (8) and (9) have a unique common solution in $B(\Omega) \times B(\Omega)$.

4. Conclusions

In this paper, we introduced the notion of α-coupled common fixed points and established the existence and uniqueness of such points. We applied our results to ensure the existence of a unique common solution of coupled systems of functional equations arising in dynamic programming. We think that this new concept will be a powerful tool in searching for the existence of solutions for coupled systems of integral equations, differential equations, and also fractional integro-differential equations.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Ansari, A.H.; Işık, H.; Radenović, S. Coupled fixed point theorems for contractive mappings involving new function classes and applications. *Filomat* **2017**, *31*, 1893–1907. [CrossRef]
3. Işık, H. Solvability to coupled systems of functional equations via fixed point theory. *TWMS J. App. Eng. Math.* **2018**, *8*, 230–237. [CrossRef]
4. Işık, H.; Radenović, S. A new version of coupled fixed point results in ordered metric spaces with applications. *UPB Sci. Bull. Ser. A* **2017**, *79*, 131–138.
5. Işık, H.; Turkoglu, D. Coupled fixed point theorems for new contractive mixed monotone mappings and applications to integral equations. *Filomat* **2014**, *28*, 1253–1264. [CrossRef]
6. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequal. Appl.* **2014**, *2014*, 38. [CrossRef]
7. Harjani, J.; Rocha, J.; Sadarangani, K. α-Coupled fixed points and their application in dynamic programming. *Abstr. Appl. Anal.* **2014**, *2014*, 593645. [CrossRef]
8. Bellman, R.; Lee, E.S. Functional equations in dynamic programming. *Aequ. Math.* **1978**, *17*, 1–18. [CrossRef]

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