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Coincidence Point Results for Multivalued Suzuki Type Mappings Using θ -Contraction in b -Metric Spaces

Naeem Saleem ¹, Jelena Vujaković ^{2,*}, Wali Ullah Baloch ¹ and Stojan Radenović ^{3,4,*}

¹ Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan; naeem.saleem2@gmail.com (N.S.); waliullah7861@gmail.com (W.U.B.)

² Faculty of Sciences and Mathematics, University of Priština, Lole Ribara 29, 38220 Kosovska Mitrovica, Serbia

³ Nonlinear Analysis research Group, Ton Duc Thang University, Ho Chi Minh City 700000, Vietnam

⁴ Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 700000, Vietnam

* Correspondence: jelena.vujakovic@pr.ac.rs (J.V.); stojan.radenovic@tdtu.edu.vn (S.R.)

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Abstract: In this paper, we introduce the concept of coincidence best proximity point for multivalued Suzuki-type α -admissible mapping using θ -contraction in b -metric space. Some examples are presented here to understand the use of the main results and to support the results proved herein. The obtained results extend and generalize various existing results in literature.

Keywords: b -metric space; ψ -contraction; θ -contraction; α -admissible; best proximity points.

MSC: 47H10; 47H04; 47H07

1. Introduction and Preliminaries

In 1922, Stefan Banach [1] proved his famous result “Banach contraction principle”, which states that “let (X, d) be a complete metric space and $T : X \rightarrow X$ be a contraction, then T has a unique fixed point”. The constructive proof of theorem helps the researchers working in Computer Sciences to develop algorithm based upon the proof of theorem, and it able them to solve complex networking problem by relating it with “fixed point problem”. This is one of its application in Computer Sciences. Later, researchers found its applications in several branches of sciences, specially, Economics, Data Science, Physics, Medical Science, Game Theory, etc. Due to several application of “fixed point theory”, researchers was motivated to further generalize it in different directions, by generalizing the contractive conditions, underlying space and concept of completeness. Among the several generalizations of “Banach fixed point theorem”, weak contractive conditions were introduced for finding unique “fixed point”. Often these weak conditions are related with metric spaces and some time are related with contractive conditions. In case of self-mappings, the solution u^* of the operator equation $Tu = u$ is the “fixed point” of mapping T (such that $d(u, Tu) = 0$, if mapping T is nonself, then “fixed point” of T will not exist. In this case, if T is nonself-mapping, then we cannot find any such u^* that satisfy the “fixed point” problem $u \neq Tu$ (or $d(u, Tu) \neq 0$), then it is evident to minimize the $d(u, Tu)$; any such u^* that minimize the given optimization problem:

$$\min_{u \in X} d(u, Tu) \tag{1}$$

is known as the “approximate fixed point” of T .

Further, for nonself mappings $T : U \rightarrow V$, where sets U and V are nonempty subsets of metric space (X, d) , also $U \cap V = \emptyset$. In this case, $u \in U$, then $Tu \in V$, where $U \cap V = \emptyset$, in this scenario, is the minimization/optimization problem (1) that reduces to best proximity point problem, and any point u^* that satisfies

$$d(u, Tu) = d(U, V) \tag{2}$$

is called “best proximity point” of T . Note that if condition $U \cap V = \emptyset$ is removed then $d(U, V) = 0$, in this case, every best proximity point can be reduced to “fixed point” of T .

Finding the “best proximity points” for two mappings is another kind of generalization of “best proximity point”; any $u^* \in X$ that satisfies $d(g(u^*), Tu^*) = d(U, V)$; here, U and V are nonempty subsets of (X, d) and $T : U \rightarrow V$ and let $g : U \rightarrow U$ be any mapping. Point u^* is called “coincidence best proximity point” of mappings g and T . If $g = I_U$ (identity over U) then every “coincidence best proximity point” will reduced to “best proximity point” of mapping T .

Extreme values are the largest and smallest values a function attains in specific interval. These extreme values of functions peaked our interest by observing how it knew the highest/lowest values of a stock or the fastest/slowest a body is moving. All these kinds of problems are related (to lower the risk and increase the benefit/profit) with optimization problem. The best proximity points are actually approximate fixed points with least error; we model the given optimization problem with a functional equation or operator, then we optimize the given model using best approximation technique. Now, these functions observe some very specific properties that would be hard to find in real-world problems, so as to relate these functions with specific constraints.

In 1989 and 1993, Bakhtin [2] and Czerwik [3], respectively, introduced the concept of b -metric space. As an application, Equation (2) is used in several iterative schemes, and the homotopy perturbation method (see , for details, in [4,5]. After the revolution in mathematics due to L. Zadeh ([6]), by presenting the concept of fuzzy sets, Kramosil and Veeramani [7–9] introduced the revolutionary idea of fuzzy metric spaces. Several authors around the globe studied fixed point theory in a new and different environment of fuzzy metric space. It gets more exposure due to the vast applications of fuzzy metric spaces in controlling the noise in data, smoothing the data, and decision-making, but the authors did not pay attention to study the best proximity point theory in fuzzy metric spaces. In 2012, N. Saleem et al. investigated best proximity and coincidence point results in fuzzy metric spaces [10–15].

Among the several generalization of fixed and best proximity point theory, one is to generalize the contractive conditions and generalize the underlying spaces. Also, researcher try to study the best proximity point results for multivalued mapping (this was not an easy task). Several authors obtained best proximity points for multivalued mapping, for details, see [13]).

In generalization of contractive conditions, the existence and convergence of best proximity points were discussed by various author (for details, see [16–19]).

T. Suzuki [20,21] generalized the Banach contraction principle; later, A. Akbar and M. Gabeleh [22] studied the best proximity point for Suzuki-type contraction.

We will use the following notions in our main results.

$$\begin{aligned} U_0 &= \{u \in U : d(u, v) = d(U, V) \text{ for some } v \in V\}, \\ V_0 &= \{v \in V : d(u, v) = d(U, V) \text{ for some } u \in U\} \text{ and} \\ d(U, V) &= \inf\{d(u, v), u \in U, v \in V\}. \end{aligned}$$

Definition 1 ([3]). Let X be a nonempty set and the mapping $d_b : X \times X \rightarrow [0, \infty)$ satisfies

- (b₁) $d_b(u, v) = 0$ iff $u = v$,
- (b₂) $d_b(u, v) = d_b(v, u)$,
- (b₃) $d_b(u, v) \leq s [d_b(u, w) + d_b(w, v)]$, for all u, v, w in X ,

where s is any real number such that $s \geq 1$, then (X, d) is known as b -metric space.

For more details, see [23–29].

Note that, henceforth, X will represent a complete b -metric space instead of (X, d_b) , and U_0 and V_0 are nonempty subsets of complete b -metric space X until otherwise stated.

Definition 2 ([2]). Let X be a b -metric space and $u \in X$, then

- A sequence $\{u_n\}$ is convergent and converges to u in X if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_b(u_n, u) < \varepsilon$, for all $n > n_0$, is represented as $\lim_{n \rightarrow \infty} u_n = u$ or $u_n \rightarrow u$ as $n \rightarrow \infty$.
- A sequence $\{u_n\}$ is Cauchy sequence in X , if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$d_b(u_n, u_m) < \varepsilon, \text{ for all } n, m > n_0$$

or equivalently, if

$$\lim_{n, m \rightarrow \infty} d_b(u_n, u_m) = 0.$$

- A b -metric space X is a complete b -metric space if every Cauchy sequence in (X, d_b) is convergent in X .

In 2012, Samet et al. [30] introduced the concept of α - ψ -contraction and α -admissible mapping and proved various fixed point theorems. Further, Samet introduced the concept of α -admissible mapping, defined as follows.

Definition 3 ([30]). Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping, then T is an α -admissible mapping if

$$\alpha(u, v) \geq 1 \text{ implies } \alpha(Tu, Tv) \geq 1, \text{ for all } u, v \in X.$$

Definition 4 ([31]). If U and V are two nonempty subsets of metric space X and $\alpha : U \times U \rightarrow [0, \infty)$, then $T : U \rightarrow V$ is known as α -proximal admissible mapping, if

$$\left. \begin{array}{l} \alpha(u_1, u_2) \geq 1 \\ d(v_1, Tu_1) = d(U, V) \\ d(v_2, Tu_2) = d(U, V) \end{array} \right\} \text{ implies } \alpha(v_1, v_2) \geq 1,$$

for all $u_1, u_2, v_1, v_2 \in U$.

Remark 1 ([31]). If we take $U = V$ in above definition, then α -proximal admissible mapping becomes α -admissible mapping.

Definition 5. Let (X, d) be a metric space, a mapping $g : X \rightarrow X$ is said to be isometry mapping if

$$d(gu, gv) = d(u, v),$$

for all $u, v \in X$.

Proposition 1 ([32]). A self-mapping $g : U \rightarrow U$ is said to satisfy α_R -property if there exist a mapping $\alpha : U \times U \rightarrow [0, \infty)$ such that

$$\alpha(gu, gv) \geq 1 \text{ implies that } \alpha(u, v) \geq 1.$$

Definition 6 ([20]). Let U and V be two nonempty subsets of metric space (X, d) with $U_0 \neq \emptyset$, then the pair (U, V) satisfies weak P -property if

$$\left. \begin{array}{l} d(u_1, v_1) = d(U, V) \\ d(u_2, v_2) = d(U, V) \end{array} \right\} \text{ implies } d(u_1, u_2) \leq d(v_1, v_2)$$

for all $u_1, u_2 \in U_0$ and $v_1, v_2 \in V_0$.

Now, we are going to define a Pompeiu–Hausdroff metric [33] on $\mathcal{CB}(X)$ as

$$\mathcal{H}(U, V) = \max\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(v, U)\},$$

for $U, V \in \mathcal{CB}(X)$, where $\mathcal{CB}(X)$ represents the closed and bounded subsets of X .

Definition 7 ([30]). Let Δ_θ represent the family of all functions $\theta : (0, \infty) \rightarrow [1, \infty)$, satisfying the following.

- (Θ_1) θ is continuous and increasing function;
- (Θ_2) $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} \theta(\alpha_n) = 1$; here, α_n is a sequence from the domain of θ ,
- (Θ_3) if for all $l \in [0, \infty)$, there exists r , such that $r \in (0, 1)$ then $\lim_{r \rightarrow 0^+} \frac{\theta(u) - 1}{u^r} = l$.
- (Θ^*) $\theta(u) \geq u$, for all $u > 0$.

A function $\theta \in \Delta_\theta$ if it satisfies the properties $\Theta_1 - \Theta_3$ and a function $\theta \in \Delta_{\theta^*}$ if θ satisfies all the conditions of Δ_θ and additional property Θ^* .

Now, we are going to define some classes of comparison functions which carry some particular properties as follows.

Definition 8 ([34–36]).

- (a) Consider Ψ_1 as a class of increasing functions and $\lim \psi^n(u) = 0$, for any $u \geq 0$.
A function $\psi \in \Psi_1$ is called comparison function, which is continuous at $u = 0$, and for any $p \geq 1$, p^{th} -iteration of a comparison function ψ is also a comparison function, further for any positive u $\psi(u) < u$.
- (b) Ψ_2 is class of functions, consisting upon the nondecreasing functions ψ , and $\sum_{n=1}^\infty \psi^n(u)$ is finite, for all $u > 0$.
Clearly, $\Psi_2 \subseteq \Psi_1$.
- (c) Ψ_3 is class of functions, consisting upon increasing functions, and there exists $n_0 \in \mathbb{N}$, $a \in (0, 1)$ and a series of non-negative numbers is convergent $\sum_{n=1}^\infty u_n$, such that for any $u \geq 0$,

$$\psi^{n+1}(u) \leq a\psi^n(u) + u_n \text{ for all } n \geq n_0.$$

The function $\psi \in \Psi_3$ is known as a c -comparison function.

- (d) Ψ_4 is class of function, consisting upon monotone increasing functions and there exists an $n_0 \in \mathbb{N}$, $a \in (0, 1)$, $s \in [1, \infty)$ and a convergent series of non-negative numbers $\sum_{n=1}^\infty u_n$ such that for any $u \geq 0$,

$$s^{n+1}\psi^{n+1}(u) \leq as^n\psi^n(u) + u_n \text{ for all } n \geq n_0.$$

The function $\psi \in \Psi_4$ is known as a b -comparison function.

Note that, if $s = 1$, then $\Psi_4 = \Psi_3$.

Lemma 1 ([34]). If ψ is a b -comparison function with $s \geq 1$, then the series $\sum_{n=0}^\infty s^n \psi^n(u)$ is convergent for $u > 0$ and the function $r_b(u) = \sum_{n=0}^\infty s^n \psi^n(u) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is increasing and continuous at $u = 0$.

Lemma 2 ([37]). If a sequence $\{u_n\}$ in a b -metric space, (X, d_b) satisfies

$$d(u_{n+1}, u_{n+2}) \leq rd(u_n, u_{n+1}) \text{ for all } n \in \mathbb{N}$$

for some $0 < r < 1$, then $\{u_n\}$ is a Cauchy sequence in X provided that $rs < 1$.

Note that through out this article, we assume that d_b (b -metric) is continuous.

2. Main Results

Now, we will introduce the Suzuki-type α - ψ_g -modified proximal contraction and Suzuki-type α - ψ -modified proximal contraction as follows.

Definition 9.

1. A pair of mappings (g, T) where $g : U \rightarrow U$ and $T : U \rightarrow \mathcal{CB}(V)$ is called Suzuki-type α - ψ_g -modified proximal contraction, if T is α -proximal admissible, and

$$\frac{1}{2s^2}D^*(gu, Tu) \leq d(u, v),$$

implies that

$$\alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}_g(u, v)),$$

where

$$\mathcal{M}_g(u, v) = \max \left\{ d(gu, gv), \frac{D(gu, Tu) + D(gv, Tv) - 2sd(U, V)}{2s}, \frac{D(gv, Tv) - s^2d(U, V)}{s^2}, \frac{D(gu, Tu) - sD(gv, Tu)}{s} \right\},$$

2. A mapping $T : U \rightarrow \mathcal{CB}(V)$ is called a Suzuki-type α - ψ -modified proximal contraction, if T is α -proximal admissible, and

$$\frac{1}{2s^2}D^*(u, Tu) \leq d(u, v),$$

implies that

$$\alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}(u, v)),$$

where

$$\mathcal{M}(u, v) = \max \left\{ d(u, v), \frac{D(u, Tu) + D(v, Tv) - 2sd(U, V)}{2s}, \frac{D(v, Tv) - s^2d(U, V)}{s^2}, \frac{D(u, Tu) - sD(v, Tu)}{s} \right\},$$

for $s \geq 1$, $\alpha : U \times U \rightarrow [0, \infty)$, $\psi \in \Psi_4$ (a b -comparison function)

Note that from now on, we will use

$$D^*(u, Tu) = D(u, Tu) - sd(U, V),$$

for all $u, v \in U$, and $\mathcal{CB}(V)$ denotes the closed and bounded subsets of V .

Our first result related with “coincidence best proximity point” for a pair of mappings (g, T) , which satisfy Suzuki-type α - ψ_g -modified proximal contraction is as follows.

Theorem 1. Let U and V be nonempty and closed subsets of a complete b -metric space (X, d_b) . Consider a pair of continuous mappings (g, T) that satisfy Suzuki-type α - ψ_g -modified proximal contractive condition with $T(U_0) \subseteq V_0$, $U_0 \subseteq g(U_0)$, where g is an isometry mapping satisfying α_R -property. Also, the pair of subsets (U, V) satisfies the weak P -property. Further suppose that there exist some $u_0, u_1 \in U_0$, such that

$$D_b(gu_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1,$$

then, mappings (g, T) has a unique coincidence best proximity point.

Proof. Let $u_0, u_1 \in U_0$ such that $D_b(gu_1, Tu_0) = d_b(U, V)$ and $\alpha(u_0, u_1) \geq 1$. As $Tu_1 \in T(U_0) \subseteq V_0$, there exist an element $gu_2 = u'_2 \in U_0 \subset g(U_0)$ such that $D_b(gu_2, Tu_1) = d_b(U, V)$. As T is α -proximal admissible, we have $\alpha(gu_1, gu_2) \geq 1$; also, g satisfies α_R -property, and therefore $\alpha(gu_1, gu_2) \geq 1$ implies $\alpha(u_1, u_2) \geq 1$. Further,

$$D_b(gu_1, Tu_0) = D_b(gu_2, Tu_1) = d_b(U, V), \alpha(u_2, u_1) \geq 1 \text{ and } \alpha(u_1, u_0) \geq 1. \tag{3}$$

As

$$\begin{aligned} \frac{1}{2s^2}D_b^*(gu_0, Tu_0) &= \frac{1}{2s^2}[D_b(gu_0, Tu_0) - sd(U, V)] \\ &\leq \frac{1}{2s^2}[s(d_b(gu_0, gu_1) + D_b(gu_1, Tu_0)) - sd(U, V)] \\ &\leq \frac{1}{2s}d_b(gu_0, gu_1) \leq \frac{1}{2s}d_b(u_0, u_1) < d_b(u_0, u_1), \end{aligned}$$

which further implies that

$$\frac{1}{2s^2}D_b^*(gu_0, Tu_0) < d_b(u_0, u_1).$$

As $\alpha(u_0, u_1) \geq 1$ and the pair of mappings (g, T) are Suzuki-type α - ψ_g -modified proximal contractions, we have

$$\mathcal{H}(Tu_0, Tu_1) \leq \alpha(u_0, u_1)\mathcal{H}(Tu_0, Tu_1) \leq \psi(\mathcal{M}_g(u_0, u_1)), \tag{4}$$

where

$$\begin{aligned} \mathcal{M}_g(u_0, u_1) &= \max \left\{ \begin{aligned} &d_b(gu_0, gu_1), \frac{D_b(gu_0, Tu_0) + D_b(gu_1, Tu_1)}{2s} - d_b(U, V), \\ &\frac{D_b(gu_1, Tu_1) - s^2d_b(U, V)}{s^2}, \frac{D_b(gu_0, Tu_0) - sD_b(gu_1, Tu_0)}{s} \end{aligned} \right\} \\ &\leq \max \left\{ \begin{aligned} &d_b(gu_0, gu_1), \\ &\frac{1}{2s} [s(d_b(gu_0, gu_1)D_b(gu_1, Tu_0)) + s(d_b(gu_1, gu_2) + D_b(gu_2, Tu_1))] \\ &- d_b(U, V), \frac{s[d_b(gu_1, gu_2) + D_b(gu_2, Tu_1)]}{s} - d_b(U, V), \\ &\frac{1}{s} (s[d_b(gu_0, gu_1) + D_b(gu_1, Tu_0)] - D_b(gu_1, Tu_0)) \end{aligned} \right\} \\ &= \max \left\{ \begin{aligned} &d_b(u_0, u_1), \frac{1}{2} [d_b(u_0, u_1) + 2d(U, V) + d_b(u_1, u_2)] - d_b(U, V), \\ &d_b(u_1, u_2), d_b(u_0, u_1) \end{aligned} \right\} \\ &= \max\{d_b(u_0, u_1), \frac{1}{2}[d_b(u_0, u_1) + d_b(u_1, u_2)], d_b(u_1, u_2)\} \\ &\leq \max\{d_b(u_0, u_1), d_b(u_1, u_2)\}. \end{aligned}$$

Therefore,

$$\mathcal{M}_g(u_0, u_1) \leq \max\{d_b(u_0, u_1), d_b(u_1, u_2)\}. \tag{5}$$

As the pair of sets (U, V) satisfies the weak P -property and the mapping g is an isometry mapping, we have

$$d_b(u_1, u_2) = d_b(gu_1, gu_2) \leq \mathcal{H}(Tu_0, Tu_1) \leq \psi(\max\{d_b(u_0, u_1), d_b(u_1, u_2)\}). \tag{6}$$

If $u_0 = u_1$, then from (3), we have

$$D_b(gu_0, Tu_0) = D_b(gu_1, Tu_0) = d_b(U, V),$$

which shows that u_0 is the coincidence best proximity point of pair (g, T) and the proof is complete.

Now, consider if $u_0 \neq u_1$, then $d_b(u_0, u_1) > 0$. Further, from inequality (6), suppose that

$$\max\{d_b(u_0, u_1), d_b(u_1, u_2)\} = d_b(u_1, u_2)$$

then inequality (6) implies that

$$d_b(u_1, u_2) \leq \psi(d_b(u_1, u_2)), \tag{7}$$

which holds true if $u_2 = u_1$, then proof is finished, and we will obtain u_1 as a “coincidence best proximity point” of the mappings g and T , so from (3), we have

$$D_b(gu_1, Tu_1) = D_b(gu_2, Tu_1) = d_b(U, V).$$

If $u_2 \neq u_1$, then from (7),

$$d_b(u_1, u_2) \leq \psi(d_b(u_1, u_2)) < d_b(u_1, u_2),$$

which is a contradiction, therefore

$$\max\{d_b(u_0, u_1), d_b(u_1, u_2)\} = d_b(u_0, u_1),$$

from (6), we have

$$d_b(u_1, u_2) \leq \psi(d_b(u_0, u_1)). \tag{8}$$

Thus, there exist some $q > 1$ such that

$$0 < d_b(u_1, u_2) < qd_b(u_1, u_2) \leq q\psi(d_b(u_0, u_1)) = q\psi(t_0), \tag{9}$$

where $t_0 = d(u_0, u_1)$. Now, consider two distinct elements, $u_1 \neq u_2 \in U_0$, such that $D_b(gu_2, Tu_1) = d_b(U, V)$ with $\alpha(u_1, u_2) \geq 1$. As $Tu_2 \in T(U_0) \subseteq V_0$, there exist an element $gu_3 = u'_3 \in U_0 \subseteq g(U_0)$ such that $D_b(gu_3, Tu_2) = d_b(U, V)$. As T is α -proximal admissible mapping, $\alpha(gu_2, gu_3) \geq 1$, which implies that $\alpha(u_2, u_3) \geq 1$ (as g satisfies the α_R -property), and we have

$$D_b(gu_2, Tu_1) = D_b(gu_3, Tu_2) = d_b(U, V), \alpha(u_3, u_2) \geq 1 \text{ and } \alpha(u_2, u_1) \geq 1, \tag{10}$$

from (9), we can write $\psi(d_b(u_1, u_2)) < \psi(q\psi(t_0))$ as $\psi \in \Psi_4$. If we set $q_1 = \frac{\psi(q\psi(t_0))}{\psi(d_b(u_1, u_2))}$, then $q_1 > 1$. If $u_3 = u_2$ then from (10), u_2 will be the coincidence best proximity point of mappings g and T , then the proof of theorem is finished. Now, consider $u_3 \neq u_2$, then we have

$$\begin{aligned} \frac{1}{2s^2} D_b^*(gu_1, Tu_1) &= \frac{1}{2s^2} [D_b(gu_1, Tu_1) - sd(U, V)] \\ &\leq \frac{1}{2s^2} [s(d_b(gu_1, gu_2) + D_b(gu_2, Tu_1)) - sd(U, V)] \\ &\leq \frac{1}{2s} d_b(gu_1, gu_2) \leq \frac{1}{2s} d_b(u_1, u_2) < d_b(u_1, u_2). \end{aligned}$$

After simplification, we have

$$\frac{1}{2s^2} D_b^*(gu_1, Tu_1) < d_b(u_1, u_2).$$

As $\alpha(u_1, u_2) \geq 1$ and mapping T is Suzuki-type α - ψ_g -modified proximal contraction, then we have

$$\mathcal{H}(Tu_1, Tu_2) \leq \alpha(u_1, u_2)\mathcal{H}(Tu_1, Tu_2) \leq \psi(\mathcal{M}_g(u_1, u_2)), \tag{11}$$

where

$$\begin{aligned}
 \mathcal{M}_g(u_1, u_2) &= \max \left\{ \begin{aligned} &d_b(gu_1, gu_2), \frac{D_b(gu_1, Tu_1) + D_b(gu_2, Tu_2)}{2s} - d_b(U, V), \\ &\frac{D_b(gu_2, Tu_2) - s^2 d_b(U, V)}{s^2}, \frac{D_b(gu_1, Tu_1) - s D_b(gu_2, Tu_1)}{s} \end{aligned} \right\} \\
 &\leq \max \left\{ \begin{aligned} &d_b(gu_1, gu_2), \\ &\frac{1}{2s} [s(d_b(gu_1, gu_2) + D_b(gu_2, Tu_1)) + s(d_b(gu_2, gu_3) + D_b(gu_3, Tu_2))] \\ &- d_b(U, V), \frac{s[d_b(gu_2, gu_3) + D_b(gu_3, Tu_2)]}{s} - d_b(U, V), \\ &\frac{1}{s} (s[d_b(gu_1, gu_2) + D_b(gu_2, Tu_1)] - D_b(gu_2, Tu_1)) \end{aligned} \right\} \\
 &\leq \max \left\{ \begin{aligned} &d_b(u_1, u_2), \frac{1}{2} [d_b(u_1, u_2) + 2d(U, V) + d_b(u_2, u_3)] - d_b(U, V), \\ &d_b(u_2, u_3), d_b(u_1, u_2) \end{aligned} \right\} \\
 &\leq \max\{d_b(u_1, u_2), \frac{1}{2} [d_b(u_1, u_2) + d_b(u_2, u_3)], d_b(u_2, u_3)\} \\
 &\leq \max\{d_b(u_1, u_2), d_b(u_2, u_3)\}.
 \end{aligned}$$

Therefore,

$$\mathcal{M}_g(u_1, u_2) \leq \max\{d_b(u_1, u_2), d_b(u_2, u_3)\}. \tag{12}$$

As the pair of sets (U, V) satisfies the weak P -property and mapping g is isometry, so we have

$$d_b(u_2, u_3) = d_b(gu_2, gu_3) \leq \mathcal{H}(Tu_1, Tu_2) \leq \psi(\max\{d_b(u_1, u_2), d_b(u_2, u_3)\}), \text{ for all } n \in \mathbb{N}. \tag{13}$$

Suppose $\max\{d_b(u_1, u_2), d_b(u_2, u_3)\} = d_b(u_2, u_3)$, then inequality (13) implies that

$$d_b(u_2, u_3) \leq \psi(d_b(u_2, u_3)), \tag{14}$$

which holds true if $u_2 = u_3$; in this case, u_2 becomes coincidence best proximity point for pair of mappings (g, T) and the proof is finished. If $u_2 \neq u_3$, then inequality (14) implies

$$d_b(u_2, u_3) \leq \psi(d_b(u_2, u_3)) < d_b(u_2, u_3)$$

which is a contradiction; therefore, $\max\{d_b(u_1, u_2), d_b(u_2, u_3)\} = d_b(u_1, u_2)$ from inequality (13), and we have

$$d_b(u_2, u_3) \leq \psi(d_b(u_1, u_2)). \tag{15}$$

Thus,

$$0 < d_b(u_2, u_3) < q_1 d_b(u_2, u_3) \leq q_1 \psi(d_b(u_1, u_2)) = \psi(q\psi(t_0)). \tag{16}$$

As $\psi \in \Psi_4$, then, from inequality (16), we have

$$\psi(d_b(u_2, u_3)) < \psi^2(q\psi(t_0)).$$

If we set $q_2 = \frac{\psi^2(q\psi(t_0))}{\psi(d_b(u_2, u_3))}$, then $q_2 > 1$. Continuing in this way, we can obtain a sequence $\{u_n\}$ in U_0 such that

$$D_b(gu_n, Tu_{n-1}) = D_b(gu_{n+1}, Tu_n) = d_b(U, V), \alpha(u_{n+1}, u_n) \geq 1 \text{ and } \alpha(u_n, u_{n-1}) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{17}$$

Now, we have

$$\begin{aligned} \frac{1}{2s^2} D_b^*(gu_{n-1}, Tu_{n-1}) &= \frac{1}{2s^2} [D_b(gu_{n-1}, Tu_{n-1}) - sd(U, V)] \\ &\leq \frac{1}{2s^2} [s(D_b(gu_{n-1}, gu_n) + D_b(gu_n, Tu_{n-1})) - sd(U, V)] \\ &\leq \frac{1}{2s} D_b(gu_{n-1}, gu_n) \leq \frac{1}{2s} D_b(u_{n-1}, u_n) < D_b(u_{n-1}, u_n). \end{aligned}$$

Then,

$$\frac{1}{2s^2} D_b^*(gu_{n-1}, Tu_{n-1}) < D_b(u_{n-1}, u_n).$$

As $\alpha(u_{n-1}, u_n) \geq 1$ and mapping T is Suzuki-type α - ψ $_g$ -modified proximal contractive condition, we can write

$$\mathcal{H}(Tu_{n-1}, Tu_n) \leq \alpha(u_{n-1}, u_n) \mathcal{H}(Tu_{n-1}, Tu_n) \leq \psi(\mathcal{M}_g(u_{n-1}, u_n)), \tag{18}$$

where

$$\begin{aligned} \mathcal{M}_g(u_{n-1}, u_n) &= \max \left\{ \begin{array}{l} d_b(gu_{n-1}, gu_n), \frac{D_b(gu_{n-1}, Tu_{n-1}) + D_b(gu_n, Tu_n)}{2s} - d_b(U, V), \\ \frac{D_b(gu_n, Tu_n) - s^2 d_b(U, V)}{s^2}, \frac{D_b(gu_{n-1}, Tu_{n-1}) - s D_b(gu_n, Tu_{n-1})}{s} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d_b(gu_{n-1}, gu_n), \\ \frac{1}{2s} [s(d_b(gu_{n-1}, gu_n) + D_b(gu_n, Tu_{n-1})) + s(d_b(gu_n, gu_{n+1}) + D_b(gu_{n+1}, Tu_n))] \\ -d_b(U, V), \frac{s[d_b(gu_n, gu_{n+1}) + D_b(gu_{n+1}, Tu_n)]}{s} - d_b(U, V), \\ \frac{1}{s} (s[d_b(gu_{n-1}, gu_n) + D_b(gu_n, Tu_{n-1}) - D_b(gu_n, Tu_{n-1})]) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d_b(u_{n-1}, u_n), \frac{1}{2} [d_b(u_{n-1}, u_n) + 2d(U, V) + d_b(u_n, u_{n+1})] - d_b(U, V), \\ d_b(u_n, u_{n+1}), d_b(u_{n-1}, u_n) \end{array} \right\} \\ &\leq \max\{d_b(u_{n-1}, u_n), \frac{1}{2} [d_b(u_{n-1}, u_n) + d_b(u_n, u_{n+1})], d_b(u_n, u_{n+1})\} \\ &\leq \max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\}. \end{aligned}$$

Therefore, we have

$$\mathcal{M}_g(u_{n-1}, u_n) \leq \max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\}. \tag{19}$$

As the pair of sets (U, V) satisfies the weak P -property and g is isometry mapping, we have

$$d_b(u_n, u_{n+1}) = d_b(gu_n, gu_{n+1}) \leq \mathcal{H}(Tu_{n-1}, Tu_n) \leq \psi(\max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\}), \text{ for all } n \in \mathbb{N}. \tag{20}$$

If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then, from (17), we have

$$D_b(gu_{n_0}, Tu_{n_0}) = D_b(gu_{n_0+1}, Tu_{n_0}) = d_b(U, V),$$

which shows that u_{n_0} is the coincidence best proximity point of pair (g, T) . Suppose $u_n \neq u_{n+1}$, then $d_b(u_n, u_{n+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\} = d_b(u_n, u_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, then inequality (20) can be written as

$$d_b(u_n, u_{n+1}) \leq \psi(d_b(u_n, u_{n+1})),$$

which is a contradiction, therefore $\max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\} = d_b(u_{n-1}, u_n)$; then, from inequality (20), we have

$$d_b(u_n, u_{n+1}) \leq \psi(d_b(u_{n-1}, u_n)), \tag{21}$$

and

$$d_b(u_n, u_{n+1}) \leq \psi^{n-1}(q\psi(t_0)), \tag{22}$$

where $t_0 = d(u_0, u_1)$.

Now, we have to prove that $\{u_n\}$ is a Cauchy sequence in U . Note that

$$\begin{aligned} d_b(u_n, u_m) &\leq sd_b(u_n, u_{n+1}) + s^2 d_b(u_{n+1}, u_{n+2}) + \dots + s^{m-n-1} d_b(u_{m-2}, u_{m-1}) + s^{m-n-1} d_b(u_{m-1}, u_m) \\ &\leq s\psi^{n-1}(q\psi(t_0)) + s^2\psi^n(q\psi(t_0)) + \dots + s^{m-n-1}\psi^{m-3}(q\psi(t_0)) + s^{m-n-1}\psi^{m-2}(q\psi(t_0)) \\ &= \frac{1}{s^{n-2}} [s^{n-1}\psi^{n-1}(q\psi(t_0)) + s^n\psi^n(q\psi(t_0)) + \dots + s^{m-3}\psi^{m-3}(q\psi(t_0)) + s^{m-2}\psi^{m-2}(q\psi(t_0))] \\ &= \frac{1}{s^{n-2}} \sum_{i=n-1}^{m-2} s^i \psi^i(q\psi(t_0)) \\ &= \frac{1}{s^{n-2}} \left(\sum_{i=1}^{m-2} s^i \psi^i(q\psi(t_0)) - \sum_{i=1}^{n-2} s^i \psi^i(q\psi(t_0)) \right). \end{aligned}$$

That is,

$$d_b(u_n, u_m) \leq \frac{1}{s^{n-2}} \left(\sum_{i=1}^{m-2} s^i \psi^i(q\psi(t_0)) - \sum_{i=1}^{n-2} s^i \psi^i(q\psi(t_0)) \right). \tag{23}$$

Assume $S_n = \sum_{i=0}^n s^i \psi^i(q\psi(t_0))$. Then, the above inequality can be written as

$$d_b(u_n, u_m) \leq \frac{1}{s^{n-2}} (S_{m-2} - S_{n-2}). \tag{24}$$

It follows from Lemma (1) that $\sum_{i=0}^\infty s^i \psi^i(t)$ converges for any $t \geq 0$. Thus, $\lim_{n \rightarrow \infty} S_{n-2} = S$, for some $S \in [0, \infty)$. If $s = 1$, then from inequality (24), we have

$$\lim_{n \rightarrow \infty} d_b(u_n, u_m) \leq \lim_{n \rightarrow \infty} (S_{m-2} - S_{n-2}) = 0.$$

If $s > 1$, then from inequality (24), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} d_b(u_n, u_m) &\leq \lim_{n \rightarrow \infty} \frac{(S_{m-2} - S_{n-2})}{s^{n-2}} \\ &\leq \lim_{n \rightarrow \infty} \frac{S_{m-2}}{s^{n-2}} = 0. \end{aligned}$$

Therefore, $\lim_{m,n \rightarrow \infty} d_b(u_n, u_m) = 0$ and $\{u_n\}$ is a Cauchy sequence in U_0 . As U_0 is a closed subset of complete b - metric space (X, d_b) , then there exist $z \in U_0 \subseteq X$, such that

$$d_b(u_n, z) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

As g, T are continuous mappings, we can deduce that $\mathcal{H}(Tu_n, Tz) \rightarrow 0$, as $n \rightarrow \infty$. Therefore,

$$d_b(U, V) = \lim_{n \rightarrow \infty} D_b(gu_{n+1}, Tu_n) = D_b(gz, Tz),$$

which shows that z is the coincidence best proximity point of pair (g, T) .

For the uniqueness of coincidence best proximity point of T , suppose to the contrary that $u, v \in U_0$ are two coincidence best proximity points of pair (g, T) with $u \neq v$, so we have

$$D_b(gu, Tu) = D_b(gv, Tv) = d(U, V).$$

As the pair (U, V) satisfies the weak P -property and mapping g is isometry, then we have

$$d_b(u, v) = d_b(gu, gv) \leq \mathcal{H}(Tu, Tv).$$

Here,

$$D_b^*(gu, Tu) = D_b(gu, Tu) - sd(U, V),$$

thus

$$\frac{1}{2s^2}D_b^*(gu, Tu) \leq d_b(gu, gv) \leq d_b(u, v)$$

implies that

$$d_b(u, v) \leq \mathcal{H}(Tu, Tv) \leq \alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}_g(u, v)).$$

After simple calculations, we have $\mathcal{M}_g(u, v) = d_b(gu, gv) \leq d_b(u, v)$ (as g is isometry mapping), then we have

$$d_b(u, v) \leq \psi(d_b(u, v)) < d_b(u, v),$$

which is a contradiction, and therefore the coincidence best proximity point is unique. \square

In our next result, we proved the existence and uniqueness of best proximity point for Suzuki-type α - ψ -modified proximal contraction T in complete b -metric space.

Theorem 2. *Let U and V be nonempty closed subsets of a complete b -metric space X . Consider a continuous mapping, T , that satisfies the Suzuki-type α - ψ -modified proximal contractive condition, and $T(U_0) \subseteq V_0$. Also, the pair of subsets (U, V) satisfy the weak P -property. Further, suppose that there exist some $u_0, u_1 \in U_0$, such that*

$$D_b(u_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1,$$

then mapping T has a unique best proximity point.

Proof. By taking mapping $g = I_U$ (identity mapping over U is isometry mapping), the remaining proof is in line with Theorem (1). \square

The following example is presented to elaborate the result presented in Theorem (2).

Example 1. Consider $X = \{(0,2), (1,2), (1,3), (2,2), (2,3), (3,3), (4,6), (5,6)\}$ be a complete b -metric space (X, d_b) , if

$$d_b(u, v) = \max\{|u_1 - v_1|^2, |u_2 - v_2|^2\}, \text{ where } u, v \in X.$$

Also, suppose that

$$\begin{aligned} U &= \{(0,2), (2,2), (2,3)\} \text{ and} \\ V &= \{(1,2), (1,3), (3,3)\} \end{aligned}$$

are the nonempty subsets of X . After simple calculation, we have $d_b(U, V) = 1$,

$$\begin{aligned} U_0 &= \{(0,2), (2,2), (2,3)\} = U, \text{ and} \\ V_0 &= \{(1,2), (1,3), (3,3)\} = V. \end{aligned}$$

For all $u_1, u_2 \in U_0 \subseteq U$ and $v_1, v_2 \in V_0 \subseteq V$; further, pair (U, V) satisfies weak P -property, as (X, d_b) is b -metric with $s = 2$. Now, consider a mapping, $T : U \rightarrow \mathcal{CB}(V)$, defined as

$$T(u) = \begin{cases} \{(1,2)\} & \text{if } u \in \{(0,2)\}, \\ \{(1,3), (3,3)\} & \text{if } u \in \{(2,2), (2,3)\}, \end{cases}$$

clearly $T(U_0) \subseteq V_0$. Now, we have to show that mapping T satisfy the Suzuki-type α - ψ -modified proximal contraction. The following part of Suzuki-type α - ψ -modified proximal contraction holds for all $u, v \in U_0$,

$$\frac{1}{2s^2}D_b^*(u, Tu) \leq d_b(u, v). \tag{25}$$

Now, we must show that the second part of Suzuki-type α - ψ -modified proximal contraction holds for all $u, v \in U_0$

$$\alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}(u, v)). \tag{26}$$

Now, consider if $u = (0, 2)$ and $v \in \{(2, t), t \in \{2, 3\}\}$, where $u \neq v$. Then, we have

$$\mathcal{M}(u, v) = 4, \mathcal{H}(Tu, Tv) = 4.$$

Further, if $u \in \{(2, t), t \in \{2, 3\}\}$ and $v \in \{(0, 2), (2, 2), (2, 3)\}$, then

$$\mathcal{M}(u, v) = 1, \mathcal{H}(Tu, Tv) = 0.$$

then, after simple calculation, inequality (25) holds true for all $u \neq v \in U_0$. By considering $s \geq 2$, $\alpha(u, v) = 1$ for all $u, v \in U$, and $\psi(t) = \frac{999}{1000}t \in \Psi_4$, then inequality (26) holds true for all $u, v \in U$, which shows that T satisfy the Suzuki-type α - ψ -modified proximal contractive condition; further, all conditions of Theorem (1) hold true, therefore T has best proximity points in U .

Corollary 1. Let U, V be two nonempty and closed subsets of a complete b -metric space X . Suppose $T : U \rightarrow V$ be a continuous Suzuki-type α - ψ -modified proximal contraction with $T(U_0) \subseteq V_0$ and pair (U, V) satisfies the weak P -property. Further, suppose that if there exist some $u_0, u_1 \in U_0$, such that

$$d_b(u_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1,$$

then mapping T has a unique best proximity point.

Corollary 2. Let U, V be nonempty and closed subsets of a complete b -metric space X and pair (U, V) satisfy the weak P -property. Suppose a continuous mapping $T : U \rightarrow \mathcal{CB}(V)$ satisfying

$$\frac{1}{2s^2}D^*(u, Tu) \leq d(u, v),$$

implies that

$$\alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(d(u, v)).$$

for all $u, v \in U$. Further, if there exist some $u_0, u_1 \in U_0$, such that

$$D_b(u_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1,$$

then mapping T has unique best proximity point.

Proof. After simple calculations, we have

$$\mathcal{M}(u, v) = \max \left\{ d(u, v), \frac{D(u,Tu)+D(v,Tv)-2sd(U,V)}{2s}, \frac{D(v,Tv)-s^2d(U,V)}{s^2}, \frac{D(u,Tu)-sD(v,Tu)}{s} \right\} = d(u, v),$$

and the rest proof of this corollary is on the same lines as Theorem (1). \square

Remark 2. It is clear that all the above results hold for complete metric space by taking $s = 1$.

3. Suzuki Type α - θ -Modified Proximal Contractive Mapping

This section is dedicated to stating and proving the coincidence best proximity point result for Suzuki-type α - θ -modified proximal contraction.

Definition 10. A pair of mappings (g, T) , where $g : U \rightarrow U$ and $T : U \rightarrow \mathcal{CB}(V)$, is said to satisfy the following.

1. Suzuki-type α - θ_g -modified proximal contraction, if T is α -proximal admissible

$$\frac{1}{2s^2} D^*(gu, Tu) \leq d(u, v) \text{ and } \mathcal{H}(Tu, Tv) > 0$$

implies that

$$\alpha(u, v)\theta(\mathcal{H}(Tu, Tv)) \leq \frac{r}{s} [\theta(\mathcal{M}_g(u, v))]^k,$$

where

$$\mathcal{M}_g(u, v) = \max \left\{ \begin{array}{l} d(gu, gv), \frac{D(gu, Tu) + D(gv, Tv) - 2sd(U, V)}{2s} \\ \frac{D(gv, Tv) - s^2 d(U, V)}{s^2}, \frac{D(gu, Tu) - sD(gv, Tu)}{s} \end{array} \right\}$$

2. Suzuki-type α - θ -modified proximal contraction, if T is α -proximal admissible,

$$\frac{1}{2s^2} D^*(u, Tu) \leq d(u, v) \text{ and } \mathcal{H}(Tu, Tv) > 0$$

implies that

$$\alpha(u, v)\theta(\mathcal{H}(Tu, Tv)) \leq \frac{r}{s} [\theta(\mathcal{M}(u, v))]^k,$$

where

$$\mathcal{M}(u, v) = \max \left\{ \begin{array}{l} d(u, v), \frac{D(u, Tu) + D(v, Tv) - 2sd(U, V)}{2s} \\ \frac{D(v, Tv) - s^2 d(U, V)}{s^2}, \frac{D(u, Tu) - sD(v, Tu)}{s} \end{array} \right\},$$

also $r, k \in [0, 1)$, $\alpha : U \times U \rightarrow [0, \infty)$, $\theta \in \Delta_{\theta^*}$, $s \geq 1$, and

$$D^*(u, Tu) = D(u, Tu) - sd(U, V).$$

In our next result, we will state and prove a coincidence best proximity point theorem for Suzuki-type α - θ_g -modified proximal contraction in complete b -metric space.

Theorem 3. Suppose U and V are nonempty closed subsets of a complete b -metric space (X, d_b) with $U_0 \neq \emptyset$. Suppose a pair of continuous mappings (g, T) of Suzuki-type α - θ_g -modified proximal contraction, where $T : U \rightarrow \mathcal{CB}(V)$ and $g : U \rightarrow U$. Moreover, g is isometry mapping satisfying α_R -property; further, $T(U_0) \subseteq V_0$, $U_0 \subseteq g(U_0)$ and (U, V) satisfy the weak P -property, and suppose that there exist $u_0, u_1 \in U_0$, such that

$$D_b(gu_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1.$$

Then, pair (g, T) has a unique coincidence best proximity point.

Proof. Let u_n be the n^{th} term of the sequence $\{u_n\}$ generated by following the same line of proof as in Theorem (1), we can construct a sequence $\{u_n\}$ in U_0 , satisfying the following,

$$D_b(gu_{n+1}, Tu_n) = d_b(U, V), \alpha(u_n, u_{n+1}) \geq 1, n \in \mathbb{N} \cup \{0\} \text{ and } u_n \neq u_{n+1}. \tag{27}$$

As

$$\begin{aligned} \frac{1}{2s^2} D_b^*(gu_{n-1}, Tu_{n-1}) &= \frac{1}{2s^2} [D_b(gu_{n-1}, Tu_{n-1}) - sd(U, V)] \\ &\leq \frac{1}{2s^2} [s(D_b(gu_{n-1}, gu_n) + D_b(gu_n, Tu_{n-1})) - sd(U, V)] \\ &\leq \frac{1}{2s} D_b(gu_{n-1}, gu_n) = \frac{1}{2s} D_b(u_{n-1}, u_n) < D_b(u_{n-1}, u_n), \end{aligned}$$

therefore

$$\frac{1}{2s^2} D_b^*(gu_{n-1}, Tu_{n-1}) \leq d_b(u_n, u_{n-1}) \text{ and } d_b(u_n, u_{n-1}) > 0.$$

As pair (g, T) is Suzuki-type α - θ_g -modified proximal contraction, then we have

$$\theta(\mathcal{H}(Tu_{n-1}, Tu_n)) \leq \alpha(u_{n-1}, u_n)\theta(\mathcal{H}(Tu_n, Tu_{n-1})) \leq \frac{r}{s}(\mathcal{M}_g(u_n, u_{n-1}))^k. \tag{28}$$

As $\alpha(u_{n-1}, u_n) \geq 1$, using (19) from Theorem (1), we have

$$\mathcal{M}_g(u_n, u_{n-1}) \leq \max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\}.$$

Choose a real number r_1 such that $0 \leq r < t < r_1 < 1$, with $\frac{1}{\sqrt{r_1}} > 1$; also, u_{n-1} and u_n are the given points in U_0 . As pair (U, V) satisfies the weak P -property, θ is increasing, and $\theta(t) \geq t$ if $t > 0$, we have

$$d_b(u_n, u_{n+1}) = d_b(gu_n, gu_{n+1}) \leq \theta(d_b(gu_n, gu_{n+1})) \leq \theta(\mathcal{H}(Tu_{n-1}, Tu_n)).$$

Also,

$$\begin{aligned} \theta(\mathcal{H}(Tu_{n-1}, Tu_n)) &\leq \left(\frac{1}{\sqrt{r_1}}\right) \theta(\mathcal{H}(Tu_{n-1}, Tu_n)) \\ &\leq \left(\frac{1}{\sqrt{r_1}}\right) \frac{r}{s} (\mathcal{M}_g(u_{n-1}, u_n))^k \\ &\leq \left(\frac{\sqrt{r_1}}{s}\right) (\max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\})^k, \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{29}$$

If

$$\max\{d_b(u_{n-1}, u_n), d_b(u_n, u_{n+1})\} = d_b(u_n, u_{n+1}),$$

then from above inequalities, we have

$$d_b(u_n, u_{n+1}) \leq \left(\frac{\sqrt{r_1}}{s}\right) d_b(u_n, u_{n+1})^k \leq \left(\frac{\sqrt{r_1}}{s}\right) d_b(u_n, u_{n+1}),$$

holds true if $u_n = u_{n+1}$, then u_n is a coincidence best proximity point of pair (g, T) and proof is finished; if $u_n \neq u_{n+1}$, then it is a contradiction, as $\sqrt{r_1} < 1$ and $s > 1$. Therefore, we have

$$d_b(u_n, u_{n+1}) \leq \frac{\sqrt{r_1}}{s} d_b(u_{n-1}, u_n), \text{ for all } n \in \mathbb{N}. \tag{30}$$

Set $r = \frac{\sqrt{r_1}}{s}$ as $r < 1$ and $rs = \sqrt{r_1} < 1$, it follows from Lemma (2) that $\{u_n\}$ is a Cauchy sequence in U_0 , where U_0 is closed subset of complete b -metric space (X, d_b) . Thus, there exists an element $z \in U_0 \subseteq U$, such that $u_n \rightarrow z$, as $n \rightarrow \infty$. As g and T are continuous mappings, $Tu_n \rightarrow Tz$ as $n \rightarrow \infty$, which implies that

$$d_b(U, V) = \lim_{n \rightarrow \infty} D_b(gu_{n+1}, Tu_n) = D_b(gz, Tz),$$

as required.

Uniqueness: On the contrary, suppose that pair of mappings (g, T) has more than one coincidence best proximity points, suppose u and v are two distinct coincidence best proximity points of mappings (g, T) , so we have

$$D_b(gu, Tu) = D_b(gv, Tv) = d_b(U, V).$$

As the pair (U, V) satisfy the weak P -property and g is an isometry mapping, we have

$$d_b(u, v) = d_b(gu, gv) \leq \mathcal{H}(Tu, Tv).$$

Here,

$$D_b^*(gu, Tu) = D_b(gu, Tu) - sd_b(U, V),$$

thus

$$\frac{1}{2s^2} D_b^*(gu, Tu) < d_b(u, v)$$

implies that,

$$d_b(u, v) = d_b(gu, gv) \leq \mathcal{H}(Tu, Tv) \leq \alpha(u, v)\mathcal{H}(Tu, Tv) \leq \frac{r}{s} [\mathcal{M}_g(u, v)]^k.$$

After simple calculations, we have $[\mathcal{M}_g(u, v)]^k = [d_b(gu, gv)]^k \leq (d_b(u, v))^k$, then we have

$$d_b(u, v) \leq \frac{r}{s} (d_b(u, v))^k < d_b(u, v),$$

a contradiction, therefore the coincidence best proximity point of (g, T) is unique. \square

Theorem 4. Suppose U and V are nonempty closed subsets of a complete b -metric space (X, d_b) with $U_0 \neq \emptyset$. Let $T : U \rightarrow \mathcal{CB}(V)$ be a continuous Suzuki-type α - θ -modified proximal contraction. Moreover, $T(U_0) \subseteq V_0$ and (U, V) satisfy the weak P -property, further suppose that there exist $u_0, u_1 \in U_0$ such that

$$D_b(u_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1.$$

Then, mapping T has a unique best proximity point.

Proof. If we take $g = I_U$ (mapping g as Identity on U), the remaining proof follows the same lines. \square

Example 2. Consider $U = \{3, 5, 6\}$ and $V = \{1, 2, 7\}$ as subsets of $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and consider a b -metric $d_b : X \times X \rightarrow [0, \infty)$, defined as

$$d_b(u, v) = |u - v|^2, \text{ for all } u \in U, v \in V.$$

Then, (X, d_b) is a complete b -metric space with $s \geq 2$. After simple calculation, we have $d_b(U, V) = 1$, $U_0 = \{3, 6\}$ and $V_0 = \{2, 7\}$, and a mapping T is defined as

$$Tu = \begin{cases} \{2\}, & \text{if } u = 3, \\ \{1, 7\}, & \text{if } u \in \{5, 6\}. \end{cases}$$

Clearly, $T(U_0) \subseteq V_0$ and pair (U, V) satisfy the weak P -property. Now, we will show that mapping T satisfy the Suzuki-type α - θ -modified proximal contractive condition:

$$\frac{1}{2s^2} D_b^*(u, Tu) \leq d_b(u, v) \text{ here } u, v \in U_0, \tag{31}$$

as $D_b^*(u, Tu) = d_b(u, Tu) - sd(U, V)$, so

$$\frac{1}{2s^2} [D_b(u, Tu) - sd(U, V)] \leq d_b(u, v) \tag{32}$$

implies that

$$\alpha(u, v)\theta(\mathcal{H}(Tu, Tv)) \leq \frac{r}{s} [\mathcal{M}(u, v)]^k. \tag{33}$$

Now, consider $u, v \in \{3, 5\}$ and $u \neq v$; then, after simple calculation, we have

$$\mathcal{M}(u, v) = 4, \mathcal{H}(Tu, Tv) = 1,$$

if $u, v \in \{3, 6\}$ and $u \neq v$, then after simple calculation we have

$$\mathcal{M}(u, v) = 9, \mathcal{H}(Tu, Tv) = 1,$$

if $u, v \in \{5, 6\}$ and $u \neq v$, then after simple calculation we have

$$\mathcal{M}(u, v) = 1, \mathcal{H}(Tu, Tv) = 0$$

for all $v \in U$ with $k \in [0, 1)$, thus if $\alpha(u, v) = \begin{cases} 1, & \text{if } u \geq v \\ 0, & \text{otherwise} \end{cases}$, for $\theta(t) = t + 1 \in \Delta_{\theta^*}$, therefore for $k = 0.9$ cases (33) and (31) hold. Therefore, $u = 2$ is the best proximity point of T in U .

Corollary 3. Let U, V be nonempty closed subsets of a complete b -metric space X . Let mapping $T : U \rightarrow V$ be a continuous Suzuki-type α - θ -modified proximal contraction with $T(U_0) \subseteq V_0$, also pair (U, V) satisfies the weak P -property, further suppose that there exist some $u_0, u_1 \in U_0$, such that

$$d_b(u_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1,$$

then the mapping T has a unique best proximity point.

Corollary 4. Let U, V be nonempty closed subsets of a complete b -metric space X and pair (U, V) satisfy weak P -property. Suppose $T : U \rightarrow \mathcal{CB}(V)$ be a continuous, satisfying

$$\frac{1}{2s^2} D_b^*(u, Tu) \leq d_b(u, v) \text{ and } \mathcal{H}(Tu, Tv) > 0,$$

which implies that,

$$\alpha(u, v)\theta(\mathcal{H}(Tu, Tv)) \leq \frac{r}{s} [d_b(u, v)]^k.$$

for all $u, v \in U$, $r, k \in (0, 1)$ and $s \geq 1$. Further, suppose that if there exist some $u_0, u_1 \in U_0$ such that

$$D_b(u_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1,$$

then mapping T has a unique best proximity point.

Proof. After simple calculations, as discussed in proof of Theorem (3), we have

$$\mathcal{M}(u, v) = \max \left\{ \begin{array}{l} d_b(u, v), \frac{D_b(u, Tu) + D_b(v, Tv) - 2sd_b(U, V)}{2s} \\ \frac{D_b(v, Tv) - s^2 d_b(U, V)}{s^2}, \frac{D_b(u, Tu) - sD_b(v, Tu)}{s} \end{array} \right\} = d_b(u, v),$$

remaining proof of this Corollary is on the same lines as Theorem (3). \square

Remark 3. All the above results holds for complete metric space with $s = 1$, as every b -metric space is a metric space for $s = 1$.

4. Results in Partially Ordered B -Metric Space

In this section, we will discuss coincidence best proximity point theorem for modified Suzuki-type contraction in partially ordered b -metric space. Henceforth, we will consider the following notion,

$$\Delta = \{(u, v) \in U_0 \times U_0 : u \preceq v \text{ or } v \preceq u\}.$$

Definition 11 ([38]). A mapping $T : U \rightarrow V$ is said to be order preserving if and only if

$$u_1 \preceq u_2 \text{ implies } Tu_1 \preceq Tu_2,$$

for all $u_1, u_2 \in U$.

Definition 12 ([38]). A mapping $T : U \rightarrow V$ is said to be partially order preserving if and only if

$$\left. \begin{array}{l} u_1 \preceq u_2 \\ d(u_1, Tu_1) = d(U, V) \\ d(u_2, Tu_2) = d(U, V) \end{array} \right\} \text{ implies } u_1 \preceq u_2,$$

for all $u_1, u_2 \in U$.

Definition 13. A pair of mappings (g, T) , where $g : U \rightarrow U$ and $T : U \rightarrow \mathcal{CB}(V)$ is ordered Suzuki-type ψ_g -modified proximal contraction, if for $u, v \in U$,

$$\frac{1}{2}D^*(gu, Tu) \leq d(u, v) \text{ implies } d(Tu, Tv) \leq \psi(\mathcal{M}_g(u, v)), \text{ for all } (u, v) \in \Delta.$$

Theorem 5. Let U and V be nonempty and closed subsets of a complete partially ordered b -metric space (X, d_V, \preceq) . Suppose a pair of continuous mappings (g, T) is an ordered Suzuki-type ψ_g -modified proximal contraction with $T(U_0) \subseteq V_0$ and $U_0 \subseteq g(U_0)$, where g is an isometry mapping satisfying α_R -property; also, T is proximal order preserving and pair (U, V) satisfies the weak P -property. Further, suppose that there exist some $u_0, u_1 \in U_0$, such that

$$D_b(u_1, Tu_0) = d(U, V) \text{ and } (u_0, u_1) \in \Delta,$$

then (g, T) has a unique coincidence best proximity point.

Proof. Define $\alpha : U \times U \rightarrow (0, \infty)$ as

$$\alpha(u, v) = \begin{cases} 1, & \text{if } (u, v) \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

As T is α -proximal admissible mapping, as defined below,

$$\begin{cases} \alpha(u_1, u_2) \geq 1 \\ D_b(gu_1, Tu_1) = d(U, V), \\ D_b(gu_2, Tu_2) = d(U, V), \end{cases}$$

equivalently, we have

$$\begin{cases} (u_1, u_2) \in \Delta \\ D_b(u_1, Tu_1) = d(U, V), \\ D_b(u_2, Tu_2) = d(U, V). \end{cases}$$

As T is proximally ordered preserving $(u_1, u_2) \in \Delta$, that is, $\alpha(u_1, u_2) \geq 1$. As T is proximally ordered preserving, we have

$$D_b(gu_1, Tu_0) = d_b(U, V) \text{ and } \alpha(u_0, u_1) \geq 1.$$

Note that if $(u, v) \in \Delta$, then $\alpha(u, v) = 1$; otherwise, $\alpha(u, v) = 0$. As mapping T is ordered Suzuki-type α - ψ_g -modified proximal mapping, we have

$$\frac{1}{2}D_b^*(gu, Tu) \leq d_b(u, v), \alpha(u, v) \geq 1 \text{ implies } \alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}_g(u, v)).$$

Let us consider $\{u_n\}$ as a sequence, then $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ with $u_n \rightarrow u$ as $n \rightarrow \infty$, then we can say that $(u_n, u_{n+1}) \in \Delta$, for all $n \in \mathbb{N} \cup \{0\}$, with $u_n \rightarrow u$ as $n \rightarrow \infty$. Therefore, all conditions of Theorem (1) hold and the coincidence best proximity point of mappings (g, T) exist. \square

Similarly, we can prove the following theorem.

Theorem 6. Suppose X, U, U_0 , and V are as in Theorem (5), let pair (g, T) be an ordered Suzuki-type α - θ_g -modified proximal contractive mappings, where $g : U \rightarrow U$ and $T : U \rightarrow \mathcal{CB}(V)$ with all assumptions of Theorem (5). Then unique coincidence best proximity point of mappings (g, T) exist.

5. Application to Fixed Point Theory

In this section, we will provide some results related fixed point theory for modified Suzuki contraction. Our result extends [21] and also generalize the main theorem of Suzuki [39].

Here, if we consider $U = V = X$, then we have the following definitions.

Definition 14. A mapping $T : X \rightarrow \mathcal{CB}(X)$ is Suzuki-type α - ψ -modified contraction if

$$\frac{1}{2s^2}d_b(u, Tu) \leq d_b(u, v) \text{ implies } \alpha(u, v)\mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}(u, v)), \text{ for all } u, v \in X.$$

Definition 15. A mapping $T : X \rightarrow \mathcal{CB}(X)$ is Suzuki-type α - θ -modified contraction if

$$\frac{1}{2s^2}d_b(u, Tu) \leq d_b(u, v) \text{ implies } \alpha(u, v)\theta(\mathcal{H}(Tu, Tv)) \leq \frac{r}{s}[\mathcal{M}(u, v)]^k,$$

for all $u, v \in X, \alpha : X \times X \rightarrow [0, \infty), r, k \in (0, 1), s \geq 1$ and $\theta \in \Delta_{\theta^*}$.

Now, from Theorems (2) and (4), we can deduce new results related with fixed point theorems.

Theorem 7. Let (X, d_b) be a complete b -metric space and consider a continuous mapping $T : X \rightarrow \mathcal{CB}(X)$ be a Suzuki-type α - ψ -modified contraction; further, if there exist u_0 with $\alpha(u_0, Tu_0) \geq 1$, then mapping T has a unique fixed point.

Proof. We take $U = V = X$ in Theorem (2), as for self-mapping every proximal Suzuki-type α - ψ -modified contraction becomes Suzuki-type α - ψ -modified contraction, and from (1), for self mapping, every proximal α -admissible mapping becomes α -admissible mapping, all conditions of Theorem (2) are satisfied; therefore, according to Theorem (2), we can find u as a best proximity point of mapping T , which implies that

$$d_b(u, Tu) = d_b(U, V)$$

but for $U = V = X$ then $d_b(U, V) = 0 = d_b(u, Tu)$, from above, we can say in case of self-mapping every Suzuki-type α - ψ -modified contraction mapping T has a unique fixed point. \square

Theorem 8. Suppose X be a complete b -metric space and $T : X \rightarrow \mathcal{CB}(X)$ is a Suzuki-type α - θ -modified contraction that satisfies all the conditions of Theorem (7). Then, T has a unique fixed point.

Proof. We take $U = V = X$ in Theorem (4), as for self-mapping every proximal Suzuki-type α - θ -modified contraction becomes Suzuki type α - θ -modified contraction, and from (1), for self mappings, every proximal α -admissible mapping becomes α -admissible mapping, all conditions of Theorem (4) are satisfied; therefore, according to Theorem (4), we can find u a best proximity point of mapping T , which implies

$$d_b(u, Tu) = d_b(U, V)$$

but if $U = V = X$, then $d_b(U, V) = 0 = d_b(u, Tu)$; therefore, for self-mapping, every Suzuki-type α - θ -modified contraction mapping T has a unique fixed point. \square

Definition 16. A mapping $T : X \rightarrow CB(X)$ is an ordered Suzuki-type ψ -modified contraction, if

$$\frac{1}{2s^2}d_b(u, Tu) \leq d_b(u, v) \text{ implies } \mathcal{H}(Tu, Tv) \leq \psi(\mathcal{M}(u, v)), \text{ for all } (u, v) \in \Delta, \psi \in \Psi \text{ and } s \geq 1.$$

Definition 17. A mapping $T : X \rightarrow CB(X)$ is an ordered Suzuki-type θ -modified contraction, if

$$\frac{1}{2s^2}d_b(u, Tu) \leq d_b(u, v) \text{ implies } \theta(\mathcal{H}(Tu, Tv)) \leq \frac{r}{s}[\mathcal{M}(u, v)]^k,$$

for all $(u, v) \in \Delta$, $\theta \in \Delta_\theta$, $r, k \in (0, 1)$ and $s \geq 1$.

Theorem 9. Let (X, d_b, \preceq) is a complete partially ordered b -metric space, consider an increasing continuous mapping $T : X \rightarrow CB(X)$ be an ordered Suzuki-type ψ -modified contraction with $u_0 \in X$, such that $(u_0, Tu_0) \in \Delta$, then T has a unique fixed point.

Proof. Following the same lines of proof of Theorem (5), and taking in account for self-mapping such that $(u_0, Tu_0) \in \Delta$, we have $\alpha(u_0, Tu_0) = 1$, then every ordered Suzuki-type α - ψ -modified contraction becomes ordered Suzuki-type ψ -modified contraction and the remaining conditions of Theorem (5) holds. Then, T has a unique fixed point. \square

Finally, we have a fixed point theorem for Suzuki-type ordered θ -modified contraction in complete partial ordered b -metric space:

Theorem 10. Let (X, d_b, \preceq) is a complete partially ordered b -metric space and $T : X \rightarrow CB(X)$ is Suzuki-type ordered θ -modified contraction satisfying the condition of Theorem (9), then T has a unique fixed point.

6. Conclusions

In this article, a multivalued Suzuki-type α - ψ -modified proximal contraction and Suzuki-type α - ψ -modified proximal contraction are introduced; further, some coincidence best proximity point and best proximity point results are proved, which generalized the main results in [40] in the sense of b -metric space. Some of the best proximity point results are also proved for multivalued Suzuki-type α - ψ -modified proximal contraction and Suzuki-type α - θ -modified proximal contraction. Further, some coincidence best proximity point theorem for multivalued modified Suzuki-type contraction in partially ordered b -metric space are proved. An application of the main results related to fixed point theorems for modified Suzuki contraction are presented. The obtained results extend from those in [21] and also generalized the main theorem of T. Suzuki ([39]). Some examples are presented to explain and support the obtained results.

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