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Modified Halpern Iterative Method for Solving Hierarchical Problem and Split Combination of Variational Inclusion Problem in Hilbert Space

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Abstract: The purpose of this paper is to introduce the split combination of variational inclusion problem which combines the concept of the modified variational inclusion problem introduced by Khuangsatung and Kangtunyakarn and the split variational inclusion problem introduced by Moudafi. Using a modified Halpern iterative method, we prove the strong convergence theorem for finding a common solution for the hierarchical fixed point problem and the split combination of variational inclusion problem. The result presented in this paper demonstrates the corresponding result for the split zero point problem and the split combination of variation inequality problem. Moreover, we discuss a numerical example for supporting our result and the numerical example shows that our result is not true if some conditions fail.

Keywords: split variational inclusion problem; hierarchical fixed point problem; fixed point problem

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1. Introduction

Throughout this article, we let H be a real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$ and let C be a nonempty closed convex subset of a real Hilbert spaces H.

Definition 1. *Let C be a nonempty subset of a real Hilbert spaces H and $Z : C \to C$ be a self mapping. Z is called a nonexpansive mapping if*

$$\|Zx - Zy\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

Z is called a firmly nonexpansive mapping if

$$\|Zx - Zy\|^2 \leq \langle x - y, Zx - Zy \rangle, \text{ for all } x, y \in C.$$

A mapping $W : C \to H$ is called *α-inverse strongly monotone* [1], if there exists a positive real number α such that

$$\langle x - y, Wx - Wy \rangle \geq \alpha \|Wx - Wy\|^2, \forall x, y \in C. \tag{1}$$

If $W : C \to H$ is α-inverse strongly monotone, then W is *monotone* mapping, that is,

$$\langle Wx - Wy, x - y \rangle \geq 0, \ \forall x, y \in H.$$

Remark 1. *(i) If $\alpha = 1$ in Equation (1), then W is firmly nonexpansive mapping.*

For $i = 1, 2, ..., N$, let $A_i : H \to H$ be a single-valued mapping and $M : H \to 2^H$ be a multi-valued mapping, from the concept of variational inclusion problems, Khuangsatung and Kangtunyakarn [2] introduced the problem of finding $x \in H$ such that

$$\theta \in \Sigma_{i=1}^{N} a_i A_i x + Mx, \tag{2}$$

for all $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$ and θ is a zero vector. This problem is called *the modified variational inclusion*. The set of solutions of Equation (2) is denoted by $VI(H, \sum_{i=1}^{N} a_i A_i, M)$. If we set $A_i = B$ for $i = 1, 2, ..., N$ then Equation (2) reduces to $\theta \in Bx + Mx$, which is the *variational inclusion* problem. The set of solution of variational inclusion problem is denoted by $VI(H, B, M)$.

The variational inclusion problems are extensively studied in mathematical programming, optimal control, mathematical economics, etc. In recent years, considerable interest has been shown in developing various extensions and generalization of the variational inclusion problem; for instance [3,4] and reference therein.

The operator M is called a maximal monotone [5], if M is monotone, i.e., $\langle u - v, x - y\rangle \geq 0$, wherever $u \in M(x)$, $v \in M(y)$ and the graph $G(M)$ of M (that is, $G(M) := \{(x, u) \in H \times H : u \in M(x)\}$) is not property contained in the graph of any other monotone operator.

Let *resolvent operator* $J_\lambda^M : H \to H$ be defined by $J_\lambda^M(x) = (I + \lambda M)^{-1}(x)$, for all $x \in H$, where M is a multi-valued maximal monotone mapping, $\lambda > 0$ and I is an identity mapping.

Let $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is denoted $Fix(T) = \{x \in C : Tx = x\}$. Fixed point problem is an important area of mathematical analysis. This problem applies about the solution in many problem in Hilbert space such as nonlinear operator equation, variational inclusion problem, etc.; for instance [2–18].

Khuangsatung and Kangtunyakarn [2] proposed the following iterative algorithm:

$$\begin{cases} w_1, \mu \in H, \\ \sum_{i=1}^{N} b_i \Psi_i(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - w_n\rangle \geq 0, \ \forall y \in C, \\ w_{n+1} = \alpha_n \mu + \beta_n w_n + \gamma_n J_\lambda^M (I - \lambda \sum_{i=1}^{N} a_i A_i) w_n + \eta_n (I - \rho_n (I - S)) w_n + \delta_n z_n, \ \forall n \geq 1, \end{cases}$$

where $S : H \to H$ is a κ-strictly pseudononspreading mapping (i.e., if there exists $\kappa \in [0, 1)$ such that $\|Su - Sv\|^2 \leq \|u - v\|^2 + \kappa\|(I - S)u - (I - S)v\|^2 + 2\langle u - Su, v - Sv\rangle$, $\forall u, v \in H$) and under certain assumptions of $\Psi_i : C \times C \to \mathbb{R}$ is a bifunction for all $i = 1, 2, ..., N$, they proved strong convergence theorem for solving the modified variational inclusion problem under some suitable conditions of $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ and $\{\rho_n\}$.

Over the decades, there are many mathematicians interested in studying the variational inequality problem, which is one of the important problems. The methods used to solve this problem can be applied for other solutions such as physics, economics, finance, optimization, network analysis, medical images, water resourced and structural analysis. The set of solution of the *variational inequality problem* is denoted by

$$VI(C, A) = \{u \in C : \langle v - u, Au\rangle \geq 0\},$$

for all $v \in C$ and $A : C \to H$ is a mapping.

Many iterative methods have been developed for solving variational inequality problem, see, for instance [7,8].

By using the concept of the variational inequality problem, Moudafi and Mainge [9] firstly introduced *hierarchical fixed point problem* for a nonexpansive mapping T with respect to another nonexpansive mapping S on H: Find $x^* \in Fix(T)$ such that

$$\langle Sx^* - x^*, x - x^*\rangle \leq 0, \ \forall x \in Fix(T), \tag{3}$$

where $S : H \to H$ is a nonexpansive mapping. It is easy to see that Equation (3) is equivalent to the following fixed point problem: Find $x^* \in H$ such that

$$x^* = P_{Fix(T)} \circ Sx^*, \tag{4}$$

where $P_{Fix(T)}$ is the metric projection of H onto $Fix(T)$. The solution set of Equation (3) is denoted by $\Phi = \{x^* \in H : \langle Sx^* - x^*, x - x^* \rangle \leq 0, \forall x \in Fix(T)\}$. It is obvious that $\Phi = VI(Fix(T), I - S)$. Note that Equation (3) covers monotone variational inequality on fixed point sets, minimization problem, etc. Many iterative methods have been developed for solving the hierarchical fixed point problem in Equation (3), see example [9–11].

By using the concept of Krasnoselski–Mann iterative algorithm, Moudafi [10] introduced iterative scheme (5) for nonexpansive mapping S, T on a subset C of Hilbert space:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\sigma_n P x_n + (1 - \sigma_n)T x_n), \ \forall n \geq 0. \end{cases} \tag{5}$$

He proved the weak convergence theorem of the sequence $\{x_n\}$, where $\{\alpha_n\}$,$\{\sigma_n\} \subset (0, 1)$ satisfies

(i) $\sum_{n=0}^{+\infty} \sigma_n < +\infty$,
(ii) $\sum_{n=0}^{+\infty} \alpha_n(1 - \alpha_n) = +\infty$,
(iii) $\lim_{n\to+\infty} \frac{\|x_{n+1} - x_n\|}{(1-\alpha_n)\sigma_n} = 0$.

Let H_1 and H_2 be two real Hilbert spaces and C, Q be a nonempty closed convex subset of a real Hilbert spaces H_1 and H_2 , respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Censor and Elfving [14] introduced *the split feasibility problem* (SEP) which is to find a point $x \in C$ and $Ax \in Q$. Many authors have studied this concept of SEP to modified their problem, see example [12–15].

In 2010, Censor, Gibali and Reich [13] introduced the split variational inequality problem which relies on the split feasibility problem and thus created the iterative algorithm for solving a strong convergence theorem of the split variational inclusion problem; more detail [13].

The split monotone variational inclusion problem, which consists of special cases, which is being used in practice as a model in the intensity-modulated radiation therapy treatment planning, the modeling of many inverse problems, and other problems; see for instance [11–15].

For every $i = 1, 2, ..., N$. Let $A_i : H_1 \to H_1$, $B_i : H_2 \to H_2$ be mappings and $M_A : H_1 \to 2^{H_1}$ and $M_B : H_2 \to 2^{H_2}$ be multi-value mappings. Inspired and motivated by Moudafi [12] and Khuangsatung and Kangtunyakarn [2], we define *the split combination of the variational inclusion problem (SCVIP)* which is find $x^* \in H_1$ such that

$$\theta_{H_1} \in \sum_{i=1}^{N} a_i A_i x^* + M_A x^*, \tag{6}$$

and

$$y^* = Ax^* \text{ such that } \theta_{H_2} \in \sum_{i=1}^{N} b_i B_i y^* + M_B y^*, \tag{7}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$.

The set of all the solutions for Equations (6) and (7) are denoted by $\Omega = \{x \in VI(H_1, \sum_{i=1}^{N} a_i A_i, M_A) : \ Ax \in VI(H_2, \sum_{i=1}^{N} b_i B_i, M_B)\}$.

If we set $A_i = A$ and $B_i = B$ for all $i = 1, 2, ..., N$ then SCVIP reduces to *the split monotone variational inclusion* problem (SMVI), which is,

$$\text{find } x^* \in H_1 \text{ such that } 0 \in A(x^*) + M_A(x^*), \tag{8}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B(y^*) + M_B(y^*), \tag{9}$$

introduced by Moudafi [12]. The set of all these solutions for Equations (8) and (9) are denoted by $\Theta = \{x^* \in VI(H_1, f, B_2) : \ Ax^* \in VI(H_2, g, B_2)\}$.

Very recently, Kazmi et al. [11] proved the strong convergence theorem under suitable condition of parameters for solving the hierarchical fixed point problem and SMVI by using hybrid iterative method as follows:

$$\begin{cases} x_0 \in C, \, C_0 = C; \\ u_n = (1-\alpha_n)x_n + \alpha_n P_C(\sigma_n S x_n + (1-\sigma_n)W_n x_n); \\ z_n = J_\lambda^{M_1}(I - \lambda f)(u_n); \\ w_n = J_\lambda^{M_2}(I - \lambda g)(Az_n); \\ y_n = z_n + \gamma A^*(w_n - Az_n); \\ C_n = \{z \in C : \|y_n - z\|^2 \leq (1-\alpha_n\sigma_n)\|x_n - z\|^2 + \alpha_n\sigma_n\|Sx_n - z\|^2\}; \\ Q_n \, \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}; \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, n \geq 0. \end{cases} \tag{10}$$

where $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone operators, $f : C \to H_1$ is θ_1-inverse strongly monotone mapping, $g : Q \to H_2$ is θ_2-inverse strongly monotone mapping, $\{T_i\}_{i=1}^N : C \to C$ is a finite family of nonexpansive mappings and W_n is a W-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,N}$ for all $n \in \mathbb{N} \cup \{0\}$.

Based on the results mentioned above, we give our theorem for SCVIP and some important results as follows:

(i) We first establish Lemma 8 which shows the equivalence between SCVIP and fixed point problem of nonexpansive mapping under suitable conditions on our parameters. Further, we give some example to support Lemma 8 and the example shows that Lemma 8 is not true if some condition fails.
(ii) We establish a strong convergence theorem of the sequences generated by the modified Halpern iterative method for finding a common solution of hierarchical fixed point problem for a nonexpansive mapping and SCVIP.
(iii) We apply our main result to obtain a strong convergence theorem of the sequences generated by the modified Halpern iterative method for finding a common solution of hierarchical fixed point problem for a nonexpansive mapping and split combination of variational inequality problem and a strong convergence theorem for finding a common solution of hierarchical fixed point problem for nonexpansive mapping and split zero point problem.
(iv) We give some illustrative numerical examples to support our main result and our examples show that our main result is not true if some conditions fail.

2. Preliminaries

In this paper, we denote weak and strong convergence by the notations '$\rightharpoonup$' and '$\rightarrow$', respectively. We recall some concepts and results needed in the sequel.

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C. \tag{11}$$

The mapping P_C is called the matric projection of H onto C. It is well known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \, \forall x \in H. \tag{12}$$

Moreover, $P_C x$ is characterized by the fact $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \forall y \in C, \tag{13}$$

which implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H, y \in C. \tag{14}$$

Lemma 1 ([4]). *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} = (1 - \alpha_n)a_n + b_n + c_n, \text{ for all } n \geq 0.$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(i) *if $b_n \leq \alpha_n C$ where $C \geq 0$, then $\{a_n\}$ is a bounded sequence,*
(ii) *if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2 ([19]). *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E, and $S : C \to C$ a nonexpansive mapping with $Fix(S) \neq \emptyset$. Then $I - S$ is demiclosed at zero.*

Lemma 3 ([4]). *Let $u \in H$ be a solution of variational inclusion if and only if $u = J_\lambda^M(u - \lambda Bu)$, $\forall \lambda > 0$, i.e.,*

$$VI(H, B, M) = Fix(J_\lambda^M(I - \lambda B)), \forall \lambda > 0.$$

where $B : H \to H$ is a single-valued mapping. Further, if $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is a closed convex subset in H.

Lemma 4 ([4]). *The resolvent operator J_λ^M associated with M is single-valued, nonexpansive for all $\lambda > 0$ and 1-inverse strongly monotone.*

Lemma 5 ([2]). *Let H be a real Hilbert space and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, ..., N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = \min_{i=1,2,...,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N VI(H, A_i, M) \neq \emptyset$. Then*

$$VI\left(H, \sum_{i=1}^{N} a_i A_i, M\right) = \bigcap_{i=1}^{N} VI(H, A_i, M),$$

where $\sum_{i=1}^N a_i = 1$ and $0 < a_i < 1$ for every $i = 1, 2, ..., N$. Moreover, $J_\lambda^M(I - \lambda \sum_{i=1}^N a_i A_i)$ is a nonexpansive mapping, for all $0 < \lambda < 2\eta$.

Example 1. *Let $H = \mathbb{R}$. For every $i = 1, 2, ..., N$, let $A_i : \mathbb{R} \to \mathbb{R}$ define by $A_i x = \frac{ix}{4} + (i+1)$ for all $x \in H$ and $M : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $Mx = \{\frac{x}{4}\}$ for all $x \in \mathbb{R}$. Let $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ for all $i = 1, 2, ..., N$. Then $VI(H, \sum_{i=1}^N a_i A_i, M) = \bigcap_{i=1}^N VI(H, A_i, M)$.*

Proof of Solution. Since $A_i x = \frac{ix}{4} + (i+1)$, we have A_i is $\frac{4}{i}$-inverse strongly monotone mapping. By definition of a_i and A_i, we have

$$\sum_{i=1}^{N} a_i A_i x = \sum_{i=1}^{N} (\frac{3}{4^i} + \frac{1}{N4^N}) A_i x = \sum_{i=1}^{N} (\frac{3}{4^i} + \frac{1}{N4^N})(\frac{ix}{4} + (i+1)).$$

From Lemma 5, we have $VI(H, \sum_{i=1}^N a_i A_i, M) = \bigcap_{i=1}^N VI(H, A_i, M) = \{-4\}$. $\square$

Example 2. *Let $H = \mathbb{R}$. For every $i = 1, 2, ..., N$, let $A_i : \mathbb{R} \to \mathbb{R}$ define by $A_i x = \frac{ix}{4} + (i+1)$ for all $x \in H$ and $M : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $Mx = \{\frac{x}{4}\}$ for all $x \in \mathbb{R}$. Let $a_i = \frac{3}{4^i} + \frac{1}{N}(\frac{1}{4^N} + 1)$ for all $i = 1, 2, ..., N$. Then $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \bigcap_{i=1}^N VI(H, A_i, M)$.*

Proof of Solution. Since $A_i x = \frac{ix}{4} + (i+1)$, we have A_i is $\frac{4}{i}$-inverse strongly monotone mapping. By definition of a_i and A_i, we have

$$\sum_{i=1}^N a_i A_i x = \sum_{i=1}^N (\frac{3}{4^i} + \frac{1}{N}(\frac{1}{4^N} + 1)) A_i x = \sum_{i=1}^N (\frac{3}{4^i} + \frac{1}{N}(\frac{1}{4^N} + 1))(\frac{ix}{4} + (i+1)).$$

Then $\bigcap_{i=1}^N VI(H, A_i, M) = \{-4\}$ and $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \{-4\}$. It implies that $VI(H, \sum_{i=1}^N a_i A_i, M) \neq \bigcap_{i=1}^N VI(H, A_i, M)$ because $\sum_{i=1}^N a_i = 2$. □

Remark 2. *Example 1 shows that Lemma 5 is true where $\sum_{i=1}^N a_i = 1$ and Example 2 shows that Lemma 5 is not true if a condition fails, that is $\sum_{i=1}^N a_i \neq 1$.*

Lemma 6 ([17]). *Let $C \subseteq H$ be a nonempty closed and convex set and let $T : C \to H$ be a nonexpansive mapping. Then $Fix(T)$ is closed and convex.*

Lemma 7. *Let H_1 and H_2 be Hilbert spaces. Let $M_A : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping and $M_B : H_2 \to 2^{H_2}$ be a multi-valued maximal monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $A_i : H_1 \to H_1$ be α_i-inverse strongly monotone with $\eta_A = \min_{i=1,2,...,N}\{\alpha_i\}$ and $B_i : H_2 \to H_2$ be β_i-inverse strongly monotone with $\eta_B = \min_{i=1,2,...,N}\{\beta_i\}$. For each $x, y \in H_1$, then*

$$\begin{aligned}
&\|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax) \\
&- J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\
&\leq \|x - y\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2,
\end{aligned}$$

where $\lambda_A \in (0, 2\eta_A)$, $\lambda_B \in (0, 2\eta_B)$, $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$ and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of A^*A.

Proof. Let $x, y \in H_1$. Consider

$$\begin{aligned}
&\|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax) \\
&- J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\
&\leq \|(x - y) - \gamma(A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay)\|^2 \\
&= \|x - y\|^2 - 2\gamma\langle x - y, A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\rangle \\
&\quad + \gamma^2\|A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax - A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ay\|^2
\end{aligned}$$

$$\leq \|x-y\|^2 + 2\gamma\langle Ay - Ax, (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\rangle$$
$$+ \gamma^2 L\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2$$
$$= \|x-y\|^2 + 2\gamma\langle Ay - Ax + J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ax - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ax$$
$$+ J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ay - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ay, (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax$$
$$- (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\rangle$$
$$+ \gamma^2 L\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2$$
$$= \|x-y\|^2 + 2\gamma\Big[\langle J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ay - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ax,$$
$$(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\rangle$$
$$- \langle (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay,$$
$$(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\rangle\Big]$$
$$+ \gamma^2 L\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2$$
$$\leq \|x-y\|^2 + 2\gamma\Big[\frac{1}{2}\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2$$
$$- \|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2\Big]$$
$$+ \gamma^2 L\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2$$
$$= \|x-y\|^2 - \gamma(1-\gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay\|^2.$$

Hence

$$\|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax)$$
$$- J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(y - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ay)\|^2$$
$$\leq \|x-y\|^2 - \gamma(1-\gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \textstyle\sum_{i=1}^{N} b_i B_i))Ax - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \textstyle\sum_{i=1}^{N} b_i B_i))Ay\|^2. \quad \square$$

We introduce Lemma 8 which shows an association between the SCVIP and the fixed point problem of nonexpansive mapping under suitable conditions on our parameters. Furthermore, we give examples for supporting Lemma 8 and the examples shows that Lemma 8 is not true if parameters are not satisfied.

Lemma 8. *Let H_1 and H_2 be Hilbert spaces. Let $M_A : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping and $M_B : H_2 \to 2^{H_2}$ be a multi-valued maximal monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $A_i : H_1 \to H_1$ be α_i-inverse strongly monotone with $\eta_A = \min_{i=1,2,...,N}\{\alpha_i\}$ and $B_i : H_2 \to H_2$ be β_i-inverse strongly monotone with $\eta_B = \min_{i=1,2,...,N}\{\beta_i\}$. Suppose that $\Omega \neq \emptyset$. Then the following are equivalent:*

(i) $x^* \in \Omega$
(ii) $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*)$,

where $\lambda_A \in (0, 2\eta_A)$, $\lambda_B \in (0, 2\eta_B)$, $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$ and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of A^*A.

Proof. Let the condition holds.

$(i) \Rightarrow (ii)$ Let $x^* \in \Omega$, we have $x^* \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A)$ and $Ax^* \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)$.

From Lemma 3, we have $x^* \in Fix(J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i))$ and $Ax^* \in Fix(J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))$, which implies that $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*$ and $Ax^* = J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*$.

By $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*$ and $Ax^* = J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*$, we have

$$
\begin{aligned}
J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\
&= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(Ax^* - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Ax^*) \\
&= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^* \\
&= x^*.
\end{aligned}
$$

It implies that

$$
J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*. \tag{15}
$$

$(ii) \Rightarrow (i)$ Let $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = x^*$ and let $w \in \Omega$.

We will show that $I - \lambda_A \sum_{i=1}^N a_i A_i$ and $I - \lambda_B \sum_{i=1}^N b_i B_i$ are nonexpansive, for all $i = 1, 2, ..., N$.

Since $A_i : C \to H$ be α_i-inverse strongly monotone mapping with $\eta_A = \min_{i=1,2,...,N}\{\alpha_i\}$ and $\lambda_A \in (0, 2\eta_A)$, we have

$$
\begin{aligned}
\|(I - \lambda_A \sum_{i=1}^N a_i A_i)x - (I - \lambda_A \sum_{i=1}^N a_i A_i)y\|^2 \\
&= \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \langle x - y, A_i x - A_i y \rangle + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_A \sum_{i=1}^N a_i \alpha_i \|A_i x - A_i y\|^2 + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_A \eta_A \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 + \lambda_A^2 \sum_{i=1}^N a_i \|A_i x - A_i y\|^2 \\
&= \|x - y\|^2 + \lambda_A \sum_{i=1}^N a_i (\lambda_A - 2\eta_A) \|A_i x - A_i y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}
$$

Thus $I - \lambda_A \sum_{i=1}^N a_i A_i$ is a nonexpansive mapping, for all $i = 1,2,...,N$. By using the same proof, we obtain that $I - \lambda_B \sum_{i=1}^N b_i B_i$, for all $i = 1,2,...,N$ is a nonexpansive mapping and $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)$ is nonexpansive mapping.

From $w \in \Omega$ and $(i) \Rightarrow (ii)$, we have $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Aw = Aw$ and $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(w - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw) = w$.

From Lemma 7 and $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)Aw = w$, we have

$$\begin{aligned}
\|x^* - w\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(w - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw)\|^2 \\
&\leq \|x^* - w\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^* \\
&\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Aw\|^2 \\
&= \|x^* - w\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*\|^2. \qquad (16)
\end{aligned}$$

Applying Equation (16), we have

$$Ax^* \in Fix(J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i)). \qquad (17)$$

From Lemma 5, we have

$$Ax^* \in VI\left(H_1, \sum_{i=1}^N b_i B_i, M_B\right). \qquad (18)$$

From the definition of x^* and Equation (17), we have

$$\begin{aligned}
x^* &= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) \\
&= J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)x^*.
\end{aligned}$$

From Lemma 5, we have

$$x^* \in VI\left(H_2, \sum_{i=1}^N a_i A_i, M_A\right). \qquad (19)$$

From Equations (18) and (19), we have $x^* \in \Omega$. □

Example 3. *Let $H_1 = H_2 = \mathbb{R}$. For every $i = 1,2,...,N$, let $A_i : \mathbb{R} \to \mathbb{R}$ define by $A_i x = \frac{ix}{4} + (i+1)$ for all $x \in H_1$ and $B_i : \mathbb{R} \to \mathbb{R}$ define by $B_i y = \frac{iy}{2} + (i+1)$ for all $y \in H_2$. Let $M_A : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_A x = \{\frac{x}{4}\}$ for all $x \in \mathbb{R}$ and $M_B : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_B x = \{\frac{y}{2}\}$ for all $y \in \mathbb{R}$. Let $Ax = x$, for all $x \in \mathbb{R}$. Let $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ and $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ for all $i = 1,2,...,N$. Then $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax^*) = -4$.*

Proof of Solution. It is easy to observe A_i is $\frac{4}{i}$-inverse strongly monotone mapping and B_i is $\frac{2}{i}$-inverse strongly monotone mapping. By definition of A_i, B_i and a_i, b_i, we have

$$\sum_{i=1}^{N} a_i A_i x = \sum_{i=1}^{N} (\frac{3}{4^i} + \frac{1}{N4^N}) A_i x = \sum_{i=1}^{N} (\frac{3}{4^i} + \frac{1}{N4^N})(\frac{ix}{4} + (i+1)),$$

and

$$\sum_{i=1}^{N} b_i B_i y = \sum_{i=1}^{N} (\frac{2}{3^i} + \frac{1}{N3^N}) B_i y = \sum_{i=1}^{N} (\frac{2}{3^i} + \frac{1}{N3^N})(\frac{iy}{2} + (i+1)).$$

Then $\Omega = \{-4\}$. From definition of A, we have $L = 1$. Choose $\lambda_A = \frac{1}{N}$, $\lambda_B = \frac{1}{N}$ and $\gamma = \frac{1}{10}$. From Lemma 8, we have $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*) = -4$. □

Example 4. *Let $H_1 = H_2 = \mathbb{R}$. For every $i = 1, 2, ..., N$, let $A_i : \mathbb{R} \to \mathbb{R}$ define by $A_i x = \frac{ix}{4} + (i+1)$ for all $x \in H_1$ and $B_i : \mathbb{R} \to \mathbb{R}$ define by $B_i y = \frac{iy}{2} + (i+1)$ for all $y \in H_2$. Let $M_A : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_A x = \{\frac{x}{4}\}$ for all $x \in \mathbb{R}$ and $M_B : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_B x = \{\frac{y}{2}\}$ for all $y \in \mathbb{R}$. Let $Ax = x$, for all $x \in \mathbb{R}$. Let $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ and $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$ for all $i = 1, 2, ..., N$. Then $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*) = x^*$ for all $x^* \in \mathbb{R}$.*

Proof of Solution. It is easy to observe A_i is $\frac{4}{i}$-inverse strongly monotone mapping and B_i is $\frac{2}{i}$-inverse strongly monotone mapping. By definition of A_i, B_i and a_i, b_i, we have

$$\sum_{i=1}^{N} a_i A_i x = \sum_{i=1}^{N} (\frac{3}{4^i} + \frac{1}{N4^N}) A_i x = \sum_{i=1}^{N} (\frac{3}{4^i} + \frac{1}{N4^N})(\frac{ix}{4} + (i+1)),$$

and

$$\sum_{i=1}^{N} b_i B_i y = \sum_{i=1}^{N} (\frac{2}{3^i} + \frac{1}{N3^N}) B_i y = \sum_{i=1}^{N} (\frac{2}{3^i} + \frac{1}{N3^N})(\frac{iy}{2} + (i+1)).$$

Then $\Omega = \{-4\}$. From definition of A, we have $L = 1$. Choose $\lambda_A = 0$, $\lambda_B = 0$ and $\gamma = \frac{1}{10}$, we have $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*) = x^*$ for all $x^* \in \mathbb{R}$.

So Example 4 shows that Lemma 8 is not true because $\lambda_A = 0$ and $\lambda_B = 0$. □

3. Main Result

We prove a strong convergence theorem to approximate a common solution of SCVIP and hierarchical fixed point problem of nonexpansive mapping.

Theorem 1. *Let H_1, H_2 be real Hilbert spaces. Let $M_A : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping and $M_B : H_2 \to 2^{H_2}$ be a multi-valued maximal monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $A_i : H_1 \to H_1$ be α_i-inverse strongly monotone with $\eta_A = \min_{i=1}\{\alpha_i\}$ and $B_i : H_2 \to H_2$ be β_i-inverse strongly monotone with $\eta_B = \min_{i=1}\{\beta_i\}$. Let $S, T : H_1 \to H_1$ be two nonexpansive mappings. Assume that $\mathbb{F} = \Phi \cap \Omega \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $u, x_1 \in H_1$ and*

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_n), \\ y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \tag{20}$$

where $\{\mu_n\}$, $\{\varphi_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$, $\{\sigma_n\} \subseteq [0, 1]$ with $\mu_n + \varphi_n + \theta_n = 1$ for all $n \geq 1$, $\lambda_A \in (0, 2\eta_A)$, $\lambda_B \in (0, 2\eta_B)$ and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of A^*A. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty,$

(ii) $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0,$
(iii) $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$
(iv) $\lim_{n\to\infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty,$
(v) $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0,$
(vi) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$, $a_i > 0$ *and* $b_i > 0$ *for all* $i = 1, 2, ..., N$.

Then $\{x_n\}$ *converges strongly to* $z_0 \in \mathbb{F}$*, where* $z_0 = P_{\mathbb{F}} u$.

Proof. Step1. First, we prove that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded.

We will show that $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)$ and $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)$ are nonexpansive mapping. Since A_i is α_i-inverse strongly monotone with $\eta_A = \min_{i=1}\{\alpha_i\}$, we have

$$\begin{aligned}\left\|(I - \lambda_A \sum_{i=1}^{N} a_i A_i)x - (I - \lambda_A \sum_{i=1}^{N} a_i A_i)y\right\|^2 &= \left\|(x-y) - \lambda_A(\sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y)\right\|^2 \\ &\leq \|x-y\|^2 - 2\lambda_A \sum_{i=1}^{N} a_i \langle x - y, A_i x - A_i y\rangle \\ &\quad + \lambda_A^2 \sum_{i=1}^{N} a_i \|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2 - 2\lambda_A \sum_{i=1}^{N} a_i \alpha_i \|A_i x - A_i y\|^2 \\ &\quad + \lambda_A^2 \sum_{i=1}^{N} a_i \|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2 + \lambda_A \sum_{i=1}^{N} a_i(\lambda_n - 2\eta_A)\|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2.\end{aligned}$$

Thus $I - \lambda_A \sum_{i=1}^{N} a_i A_i$ is a nonexpansive mapping, for all $i = 1, 2, ..., N$. By using the same proof, we obtain that $I - \lambda_B \sum_{i=1}^{N} b_i B_i$ is a nonexpansive mapping. Since $J_{\lambda_A}^{M_A}$ and $J_{\lambda_B}^{M_B}$ are nonexpnsive mapping, we have $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)$ and $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)$ are nonexpansive mapping.

Let $p \in \mathbb{F}$ then $p \in H_1$ and $p \in \Phi$ which $Tp = p$. Now, we estimate

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)T x_n) - p\|^2 \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(\sigma_n(Sx_n - p) + (1-\sigma_n)(Tx_n - p))\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|(\sigma_n(Sx_n - p) + (1-\sigma_n)(Tx_n - p))\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2 + \alpha_n(1-\sigma_n)\|Tx_n - p\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2 + \alpha_n(1-\sigma_n)\|x_n - p\|^2 \\ &= (1-\alpha_n\sigma_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2. \end{aligned} \tag{21}$$

Since $p \in \mathbb{F}$, then $p \in \Omega$ and $J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)p = p$ and $J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i)Ap = Ap$. By Lemma 8, we have

$$J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(p - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ap) = p.$$

By Lemma 7, we have

$$\begin{aligned}
&\|u_n - p\|^2 \\
&= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_n) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(p - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ap)\|^2 \\
&\leq \|x_n - p\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_n - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ap\|^2 \\
&\leq \|x_n - p\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_n\|^2 \\
&\leq \|x_n - p\|^2. \qquad (22)
\end{aligned}$$

By Equations (21) and (22), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - p\|^2 \\
&\leq \mu_n \|u - p\|^2 + \varphi_n \|y_n - p\|^2 + \theta_n \|u_n - p\|^2 \\
&\leq \mu_n \|u - p\|^2 + \varphi_n \Big[(1 - \alpha_n \sigma_n)\|x_n - p\|^2 + \alpha_n \sigma_n \|Sx_n - p\|^2\Big] + \theta_n \|x_n - p\|^2 \\
&= \mu_n \|u - p\|^2 + \varphi_n \|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 + \theta_n \|x_n - p\|^2 \\
&= \mu_n \|u - p\|^2 + (1 - \mu_n)\|x_n - p\|^2 - \varphi_n \alpha_n \sigma_n \|x_n - p\|^2 + \varphi_n \alpha_n \sigma_n \|Sx_n - p\|^2 \\
&\leq (1 - \mu_n)\|x_n - p\|^2 + \mu_n \|u - p\|^2 + \mu_n \alpha_n \sigma_n \|Sx_n - p\|^2. \qquad (23)
\end{aligned}$$

From Lemma 1(i), therefore $\{x_n\}$ is bounded. So are $\{u_n\}$,$\{y_n\}$.
Step2. Show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - \mu_{n-1} u - \varphi_{n-1} y_{n-1} - \theta_{n-1} u_{n-1}\| \\
&= \|(\mu_n - \mu_{n-1})u + (\varphi_n - \varphi_{n-1})y_{n-1} + \varphi_n (y_n - y_{n-1}) \\
&\quad + (\theta_n - \theta_{n-1})u_{n-1} + \theta_n (u_n - u_{n-1})\| \\
&\leq |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| + \varphi_n \|y_n - y_{n-1}\| \\
&\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| + \theta_n \|u_n - u_{n-1}\|. \qquad (24)
\end{aligned}$$

From definition of u_n, Lemma 7 and $\gamma \in (0, \frac{1}{L})$, we have

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 &= \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_n) \\
&\quad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_{n-1} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_n \\
&\quad - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2.
\end{aligned}$$

It implies that

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\|. \qquad (25)$$

From definition of y_n, we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|(1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)T x_n) \\
&\quad - [(1-\alpha_{n-1})x_{n-1} + \alpha_{n-1}(\sigma_{n-1} S x_{n-1} + (1-\sigma_{n-1})T x_{n-1})]\| \\
&= \|(x_n - x_{n-1}) - \alpha_n x_n + \alpha_{n-1}x_{n-1} + \alpha_n x_{n-1} - \alpha_n x_{n-1} \\
&\quad + \alpha_n\sigma_n S x_n - \alpha_{n-1}\sigma_{n-1} S x_{n-1} + \alpha_n\sigma_n S x_{n-1} - \alpha_n\sigma_n S x_{n-1} \\
&\quad + \alpha_n T x_n - \alpha_{n-1} T x_{n-1} + \alpha_n T x_{n-1} - \alpha_n T x_{n-1} \\
&\quad - \alpha_n\sigma_n T x_n + \alpha_{n-1}\sigma_{n-1} T x_{n-1} + \alpha_n\sigma_n T x_{n-1} - \alpha_n\sigma_n T x_{n-1}\| \\
&= \|(1-\alpha_n)(x_n - x_{n-1}) + (\alpha_{n-1} - \alpha_n)x_{n-1} \\
&\quad + (\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1})S x_{n-1} + \alpha_n\sigma_n(S x_n - S x_{n-1}) \\
&\quad + \alpha_n(1-\sigma_n)(T x_n - T x_{n-1}) + (\alpha_n - \alpha_{n-1})T x_{n-1} \\
&\quad + (\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n)T x_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\quad + |\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1}|\|S x_{n-1}\| + \alpha_n\sigma_n\|S x_n - S x_{n-1}\| \\
&\quad + \alpha_n(1-\sigma_n)\|T x_n - T x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T x_{n-1}\| \\
&\quad + |\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n|\|T x_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\quad + |\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1}|\|S x_{n-1}\| + \alpha_n\sigma_n\|x_n - x_{n-1}\| \\
&\quad + \alpha_n(1-\sigma_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T x_{n-1}\| \\
&\quad + |\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n|\|T x_{n-1}\| \\
&= \|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\quad + |\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1}|\|S x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T x_{n-1}\| \\
&\quad + |\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n|\|T x_{n-1}\|. \qquad (26)
\end{aligned}$$

From Equations (24)–(26), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| + \varphi_n\|y_n - y_{n-1}\| \\
&\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| + \theta_n\|u_n - u_{n-1}\| \\
&\leq |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| \\
&\quad + \varphi_n[\|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\quad + |\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1}|\|S x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T x_{n-1}\| \\
&\quad + |\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n|\|T x_{n-1}\|] \\
&\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| + \theta_n\|x_n - x_{n-1}\| \\
&= (\varphi_n + \theta_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| \\
&\quad + \varphi_n|\alpha_{n-1} - \alpha_n|\|x_{n-1}\| + \varphi_n|\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1}|\|S x_{n-1}\| \\
&\quad + \varphi_n|\alpha_n - \alpha_{n-1}|\|T x_{n-1}\| + \varphi_n|\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n|\|T x_{n-1}\| \\
&\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\| \\
&\leq (1-\mu_n)\|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}|\|u\| + |\varphi_n - \varphi_{n-1}|\|y_{n-1}\| \\
&\quad + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| + |\alpha_n\sigma_n - \alpha_{n-1}\sigma_{n-1}|\|S x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|T x_{n-1}\| + |\alpha_{n-1}\sigma_{n-1} - \alpha_n\sigma_n|\|T x_{n-1}\| \\
&\quad + |\theta_n - \theta_{n-1}|\|u_{n-1}\|.
\end{aligned}$$

By Lemma 1(i), conditions (i) and (iii), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{27}$$

From definition of u_n, we have

$$\begin{aligned} x_{n+1} - u_n &= \mu_n u + \varphi_n y_n + \theta_n u_n - u_n \\ &= \mu_n(u - u_n) + \varphi_n(y_n - u_n). \end{aligned} \tag{28}$$

From Equations (21) and (22), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\mu_n u + \varphi_n y_n + \theta_n u_n - p\|^2 \\ &= \mu_n\|u - p\|^2 + \varphi_n\|y_n - p\|^2 + \theta_n\|u_n - p\|^2 \\ &\quad - \mu_n\varphi_n\|u - y_n\|^2 - \mu_n\theta_n\|u - u_n\|^2 - \varphi_n\theta_n\|y_n - u_n\|^2 \\ &\le \mu_n\|u - p\|^2 + \varphi_n\|y_n - p\|^2 + \theta_n\|u_n - p\|^2 - \varphi_n\theta_n\|y_n - u_n\|^2 \\ &\le \mu_n\|u - p\|^2 + \varphi_n[(1 - \alpha_n\sigma_n)\|x_n - p\|^2 + \alpha_n\sigma_n\|Sx_n - p\|^2] \\ &\quad + \theta_n\|x_n - p\|^2 - \varphi_n\theta_n\|y_n - u_n\|^2 \\ &= \mu_n\|u - p\|^2 + \varphi_n\|x_n - p\|^2 - \varphi_n\alpha_n\sigma_n\|x_n - p\|^2 \\ &\quad + \varphi_n\alpha_n\sigma_n\|Sx_n - p\|^2 + \theta_n\|x_n - p\|^2 - \varphi_n\theta_n\|y_n - u_n\|^2 \\ &\le \mu_n\|u - p\|^2 + (1 - \mu_n)\|x_n - p\|^2 + \varphi_n\alpha_n\sigma_n\|Sx_n - p\|^2 \\ &\quad - \varphi_n\theta_n\|y_n - u_n\|^2 \\ &\le \mu_n\|u - p\|^2 + \|x_n - p\|^2 + \varphi_n\alpha_n\sigma_n K - \varphi_n\theta_n\|y_n - u_n\|^2, \end{aligned}$$

where $K = \sup_n\{\|Sx_n - p\|^2\}$. It follow that

$$\begin{aligned} \varphi_n\theta_n\|y_n - u_n\|^2 &\le \mu_n\|u - p\|^2 + \varphi_n\alpha_n\sigma_n K + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \mu_n\|u - p\|^2 + \varphi_n\alpha_n\sigma_n K + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\ &\le \mu_n\|u - p\|^2 + \varphi_n\alpha_n\sigma_n K + \|x_n - x_{n+1}\|L_1, \end{aligned}$$

where $L_1 = \sup_n\{\|x_n - p\| + \|x_{n+1} - p\|\}$. From Equation (27), conditions (i), (ii) and (v), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{29}$$

From Equations (28) and (29), we have

$$\begin{aligned} \|x_{n+1} - u_n\| &= \|\mu_n(u - u_n) + \varphi_n(y_n - u_n)\| \\ &\le \mu_n\|u - u_n\| + \varphi_n\|y_n - u_n\|. \end{aligned}$$

From Equation (29) and a condition (i), we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{30}$$

Since

$$\begin{aligned} \|x_n - u_n &= \|x_n - x_{n+1} + x_{n+1} - u_n\| \\ &\le \|x_n - x_{n-1}\| + \|x_{n+1} - u_n\|. \end{aligned}$$

From Equations (28) and (30), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{31}$$

Since

$$\begin{aligned} \|x_n - y_n &= \|x_n - u_n + u_n - y_n\| \\ &\leq \|x_n - u_n\| + \|u_n - y_n\|. \end{aligned}$$

From Equations (29) and (31), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{32}$$

Step3. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.
We have

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Tx_n\|. \tag{33}$$

Since $\{x_n\}$ is bounded and the mappings S, T are nonexpansive then there exists a $K_1 > 0$ such that $\|Sx_n - Tx_n\| \leq K_1$, for all $n \geq 0$. Now, we estimate

$$\begin{aligned} \|y_n - Tx_n\| &= \|(1-\alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n) - Tx_n\| \\ &= \|(1-\alpha_n)(x_n - Tx_n) + \alpha_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n - Tx_n)\| \\ &= \|(1-\alpha_n)(x_n - Tx_n) + \alpha_n(\sigma_n Sx_n - \sigma_n Tx_n)\| \\ &\leq (1-\alpha_n)\|x_n - Tx_n\| + \alpha_n\sigma_n\|Sx_n - Tx_n\| \\ &\leq (1-\alpha_n)\big[\|x_n - y_n\| + \|y_n - Tx_n\|\big] + \alpha_n\sigma_n\|Sx_n - Tx_n\| \\ &\leq (1-\alpha_n)\|x_n - u_n\| + (1-\alpha_n)\|y_n - Tx_n\| + \alpha_n\sigma_n\|Sx_n - Tx_n\|, \end{aligned}$$

which implies

$$\begin{aligned} \alpha_n\|y_n - Tx_n\| &\leq (1-\alpha_n)\|x_n - y_n\| + \alpha_n\sigma_n\|Sx_n - Tx_n\| \\ &\leq \|x_n - y_n\| + \alpha_n\sigma_n K_1. \end{aligned}$$

It follow that

$$\|y_n - Tx_n\| \leq \frac{\|x_n - y_n\|}{\alpha_n} + \sigma_n K_1. \tag{34}$$

Since $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n\sigma_n} = 0$, we have $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n} = \lim_{n\to\infty} \sigma_n \cdot \frac{\|x_n - y_n\|}{\alpha_n\sigma_n} = 0$.
From $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n} = 0$, Equation (34) and a condition (v), we have

$$\lim_{n\to\infty} \|y_n - Tx_n\| = 0. \tag{35}$$

Thus, it follows from Equations (32), (33) and (35), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{36}$$

Step4. $x^* \in \mathbb{F}$
Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which converges weakly to x^*. We may assume that

$$\liminf_{n\to\infty} \langle -x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle = \lim_{k\to\infty} \langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle,$$

and
$$\liminf_{n\to\infty}\langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle = \lim_{k\to\infty}\langle Sx_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle.$$

We will show that $x^* \in Fix(T)$. Assume that $x^* \notin Fix(T)$, then $x^* \neq Tx^*$ and using Opial's property of Hilbert space and Equation (35), we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k} - x^*\| &< \liminf_{k\to\infty}\|x_{n_k} - Tx^*\| \\ &\leq \liminf_{k\to\infty}\left(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tx^*\|\right) \\ &\leq \liminf_{k\to\infty}\|x_{n_k} - x^*\|,\end{aligned}$$

which is a contradiction. Therefore, $x^* \in Fix(T)$.

Next, we show that $x^* \in \Phi$. Consider

$$\begin{aligned}y_n - x_n &= (1-\alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n) - x_n \\ &= \alpha_n\sigma_n(Sx_n - x_n) + \alpha_n(1-\sigma_n)(Tx_n - x_n),\end{aligned}$$

which implies

$$\begin{aligned}Sx_n - x_n &= \frac{y_n - x_n}{\alpha_n\sigma_n} - \frac{\alpha_n(1-\sigma_n)(Tx_n - x_n)}{\alpha_n\sigma_n} \\ &= \frac{y_n - x_n}{\alpha_n\sigma_n} + \frac{(1-\sigma_n)(I-T)x_n}{\sigma_n}.\end{aligned}$$

It follows that
$$Sx_n - x_n - \frac{y_n - x_n}{\alpha_n\sigma_n} = \frac{(1-\sigma_n)(I-T)x_n}{\sigma_n}.$$

Since T is nonexpansive, we have $I - T$ is monotone. Let $x \in Fix(T)$, we have

$$\begin{aligned}&\langle Sx_n - x_n - \frac{y_n - x_n}{\alpha_n\sigma_n}, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\ &= \frac{(1-\sigma_n)}{\sigma_n}\langle (I-T)x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\ &= \frac{(1-\sigma_n)}{\sigma_n}\langle (I-T)x_n - (I-T)(x - \frac{y_n - x_n}{\alpha_n}) + (I-T)(x - \frac{y_n - x_n}{\alpha_n}), \\ &\quad x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\ &= \frac{(1-\sigma_n)}{\sigma_n}\Big[\langle (I-T)x_n - (I-T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\ &\quad + \langle (I-T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle\Big] \\ &\leq \frac{(1-\sigma_n)}{\sigma_n}\langle (I-T)(x - \frac{y_n - x_n}{\alpha_n}), x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\ &\leq \frac{(1-\sigma_n)}{\sigma_n}\Big\|(I-T)(x - \frac{y_n - x_n}{\alpha_n})\Big\|\Big\|x - \frac{y_n - x_n}{\alpha_n} - x_n\Big\| \\ &= \frac{(1-\sigma_n)}{\sigma_n}\Big\|(I-T)(x - \frac{y_n - x_n}{\alpha_n}) - (I-T)x\Big\|\Big\|x - \frac{y_n - x_n}{\alpha_n} - x_n\Big\| \\ &\leq 2(1-\sigma_n)\frac{\|y_n - x_n\|}{\alpha_n\sigma_n}\Big\|x - \frac{y_n - x_n}{\alpha_n} - x_n\Big\|,\end{aligned}$$

which implies

$$\langle Sx_n - x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \leq 2(1-\sigma_n)\frac{\|y_n - x_n\|}{\alpha_n\sigma_n}\Big\|x - \frac{y_n - x_n}{\alpha_n} - x_n\Big\| + \langle \frac{y_n - x_n}{\alpha_n\sigma_n}, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \leq 3\frac{\|y_n - x_n\|}{\alpha_n\sigma_n}\Big\|x - \frac{y_n - x_n}{\alpha_n} - x_n\Big\|. \tag{37}$$

Since $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n} = 0$, we have

$$\lim_{k\to\infty} \langle \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle = 0. \tag{38}$$

From Equation (38) and $\frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} + x_{n_k} \rightharpoonup x^*$, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\langle -x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\
&= \lim_{k\to\infty}\langle -x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle \\
&= \lim_{k\to\infty}\langle -x_{n_k} - (x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}) + (x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}), x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle \\
&= \lim_{k\to\infty}\Big[\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle \\
&\quad - \langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle\Big] \\
&= \lim_{k\to\infty}\Big[\langle x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle - \langle x, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle \\
&\quad + \langle \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle\Big] \\
&= \|x - x^*\|^2 - \langle x, x - x^*\rangle.
\end{aligned} \tag{39}$$

Since S is weakly continuous and $\frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} + x_{n_k} \rightharpoonup x^*$, we obtain

$$\liminf_{n\to\infty}\langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle = \lim_{k\to\infty}\langle Sx_{n_k}, x - \frac{y_{n_k} - x_{n_k}}{\alpha_{n_k}} - x_{n_k}\rangle = \langle Sx^*, x - x^*\rangle. \tag{40}$$

From Equations (37), (39) and (40), we have

$$\begin{aligned}
\langle Sx^* - x^*, x - x^*\rangle &= \langle Sx^*, x - x^*\rangle - \langle x^*, x - x^*\rangle \\
&= \langle Sx^*, x - x^*\rangle + \|x - x^*\|^2 - \langle x, x - x^*\rangle \\
&= \liminf_{n\to\infty}\Big[\langle Sx_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle - \langle x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle\Big] \\
&= \liminf_{n\to\infty}\langle Sx_n - x_n, x - \frac{y_n - x_n}{\alpha_n} - x_n\rangle \\
&\leq \liminf_{n\to\infty} 3\frac{\|y_n - x_n\|}{\alpha_n\sigma_n}\Big\|x - \frac{y_n - x_n}{\alpha_n} - x_n\Big\| \\
&\leq 0.
\end{aligned}$$

Hence x^* solve Hierarchical fixed point problem, i.e., $x^* \in \Phi$.

Next, we show that $x^* \in \Omega$. Assume that $x^* \neq J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*)$. Applying the Opial's property, Equation (31) and Lemma 7, we have

$$
\begin{aligned}
&\liminf_{k\to\infty} \|x_{n_k} - x^*\| \\
&\quad < \liminf_{k\to\infty} \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*\| \\
&\quad = \liminf_{k\to\infty} \|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n_k} \\
&\qquad + J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n_k} \\
&\qquad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*\| \\
&\quad \le \liminf_{k\to\infty} \Big[\|x_{n_k} - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n_k}\| \\
&\qquad + \|J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x_{n_k} - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n_k} \\
&\qquad - J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*\|\Big] \\
&\quad \le \liminf_{k\to\infty} \Big[\|x_{n_k} - u_{n_k}\| + \|x_{n_k} - x^*\| \\
&\qquad - \gamma(1-\gamma L)\|(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax_{n_k} - (I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*\|\Big] \\
&\quad \le \liminf_{k\to\infty} \Big[\|x_{n_k} - u_{n_k}\| + \|x_{n_k} - x^*\|\Big] \\
&\quad = \liminf_{k\to\infty} \|x_{n_k} - x^*\|.
\end{aligned}
$$

This is a contradiction. Then $x^* = J_{\lambda_A}^{M_A}(I - \lambda_A \sum_{i=1}^{N} a_i A_i)(x^* - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B \sum_{i=1}^{N} b_i B_i))Ax^*$. From Lemma 8, we have $x^* \in \Omega$. Therefore, $x^* \in \mathbb{F}$.

Step5. Finally, we will prove that $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$.

We show that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_{\mathbb{F}}u$. We may assume the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle. \tag{41}$$

Since $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$ and $x^* \in \mathbb{F}$. By Equations (13) and (41), we have

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle \le 0 \tag{42}$$

From Equations (21) and (22), we have

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\mu_n u + \varphi_n y_n + \theta u_n - z_0\|^2 \\
&= \|\mu_n(u - z_0) + \varphi_n(y_n - z_0) + \theta(u_n - z_0)\|^2 \\
&\le \|\varphi_n(y_n - z_0) + \theta(u_n - z_0)\|^2 + 2\langle \mu_n(u - z_0), x_{n+1} - z_0\rangle \\
&\le \varphi_n\|y_n - z_0\|^2 + \theta\|u_n - z_0\|^2 + 2\mu_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\le \varphi_n[(1 - \alpha_n\sigma_n)\|x_n - z_0\|^2 + \alpha_n\sigma_n\|Sx_n - z_0\|^2] \\
&\quad + \theta\|x_n - z_0\|^2 + 2\mu_n\langle u - z_0, x_{n+1} - z_0\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varphi_n\|x_n - z_0\|^2 + \varphi_n\alpha_n\sigma_n\|Sx_n - z_0\|^2 \\
&\quad + \theta\|x_n - z_0\|^2 + 2\mu_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\mu_n)\|x_n - z_0\|^2 + \varphi_n\alpha_n\sigma_n\|Sx_n - z_0\|^2 \\
&\quad + 2\mu_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned}
$$

Applying Lemma 1(ii), conditions (i), (iv) and Equation (42), we can conclude that the $\{x_n\}$ converges strongly to $z_0 = P_{\mathbb{F}}u$. This completes the proof. □

Next, we have the following strong convergence to approximation a common element of solution the set of SMVI and hierarchical fixed point problem of nonexpansive mapping.

Corollary 1. *Let H_1, H_2 be real Hilbert spaces. Let $M_A : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping and $M_B : H_2 \to 2^{H_2}$ be a multi-valued maximal monotone mapping. Let $F : H_1 \to H_2$ be a bounded linear operator with its adjoint operator F^*. Let $A : H_1 \to H_1$ be α-inverse strongly monotone and $B : H_2 \to H_2$ be β-inverse strongly monotone. Let $S, T : H_1 \to H_1$ be two nonexpansive mappings. Assume that $\mathbb{F} = \Phi \cap \Theta \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $u, x_1 \in H_1$ and*

$$
\begin{cases}
u_n = J_{\lambda_A}^{M_A}(I - \lambda_A A)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \lambda_B B))Fx_n), \\
y_n = (1-\alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1-\sigma_n)Tx_n), \\
x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n,
\end{cases}
\tag{43}
$$

where $\{\mu_n\}$, $\{\varphi_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$, $\{\sigma_n\} \subseteq [0,1]$ with $\mu_n + \varphi_n + \theta_n = 1$ for all $n \geq 1$, $\lambda_A \in (0, 2\alpha)$, $\lambda_B \in (0, 2\beta)$ and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of F^*F. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty$,
(ii) $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0$,
(iii) $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$
(iv) $\lim_{n\to\infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty$,
(v) $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n\sigma_n} = 0$,

Then $\{x_n\}$ converges strongly to $z_0 \in \mathbb{F}$, where $z_0 = P_{\mathbb{F}}u$.

Proof. Put $A_i \equiv A$ and $B_i \equiv B$ for all $i = 1, 2, ..., N$ in Theorem 1. From Theorem 1, we obtain the desired result. □

4. Application

4.1. Split Zero Point Problem

Let H be a real HIlbert space. Let $M : H \to 2^H$ be a maximal monotone operator. Then *the zero point problem* is to find $x^* \in H$ such that

$$0 \in Mx^*, \tag{44}$$

such an $x^* \in H$ is called a zero point of M. The set of zero point of M is denoted by $M^{-1}(0)$.

Let H_1 and H_2 be two real Hilbert spaces. Setting $A_i \equiv 0$ and $B_i \equiv 0$ for all $i = 1, 2, .., N$ in SCVIP, then SCVIP reduce to *the split zero point problem*: Find $x^* \in H_1$ such that

$$0 \in M_A x^*, \tag{45}$$

and

$$y^* \in Ax^* \text{ such that } 0 \in M_B y^*, \tag{46}$$

where $A : H_1 \to H_2$ is bounded linear operator, $M_A : H_1 \to 2^{H_1}$ and $M_B : H_2 \to 2^{H_2}$ are multi-valued mapping. The set of all solution of this problem is denoted by $\Omega_2 = \{x \in M_A^{-1}(0) : Ax \in M_B^{-1}(0)\}$.

The split zero point problem which consists of the special cases, split feasibility problem, variational inequalities, etc., which is used in practice as a model in machine learning, image processing and linear inverse problem.

Next, we give the strong convergence theorem for solving the split zero point problem and the hierarchical fixed point problem of nonexpansive mapping.

Corollary 2. *Let H_1, H_2 be real Hilbert spaces. Let $M_A : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping and $M_B : H_2 \to 2^{H_2}$ be a multi-valued maximal monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $S, T : H_1 \to H_1$ be two nonexpansive mappings. Assume that $\mathbb{F} = \Phi \cap \Omega_2 \neq \emptyset$. Let the iterative sequence generated by hybrid iterative algorithm:*

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(x_n - \gamma A^* J_{\lambda_B}^{M_B} A x_n), \\ y_n = (1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)T x_n), \\ x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n, \end{cases} \tag{47}$$

where $\{\delta_n\}$, $\{\varphi_n\}$, $\{\eta_n\}$, $\{\alpha_n\}$, $\{\sigma_n\} \subseteq [0,1]$ with $\delta_n + \varphi_n + \eta_n = 1$ for all $n \geq 1$, and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of A^*A. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \mu_n = 0, \sum_{n=1}^{\infty} \mu_n = \infty$,
(ii) $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0$,
(iii) $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^{\infty} |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$
(iv) $\lim_{n\to\infty} \sigma_n = 0, \sum_{n=1}^{\infty} \sigma_n < \infty$,
(v) $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$,

Then $\{x_n\}$ converges strongly to $z \in \mathbb{F}$, where $z = P_{\mathbb{F}} u$.

Proof. Put $A_i \equiv 0$ and $B_i \equiv 0$ for all $i = 1, 2, ..., N$ in Theorem 1. From Theorem 1, we obtain the desired conclusion. □

4.2. Split Combination of Variational Inequalities Problem

Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let h be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. The subdifferential ∂h of h is defined by

$$\partial h(x) = \{z \in H : h(x) + \langle z, u - x \rangle \leq h(u), \ \forall u \in H\},$$

for all $x \in H$. From Rockafellar [16], we get that ∂h is a maximal monotone operator. Let i_C be the indicator function of C, i.e.,

$$i_C = \begin{cases} 0\,; \ if\ x \in C, \\ +\infty\,; \ if\ x \notin C. \end{cases}$$

Then i_C is a proper, lower semicontinuous and convex function on H and so the subdifferential ∂i_C of i_C is a maximal monotone operator. The resolvent operator $J_r^{\partial i_C}$ of ∂i_C for $\lambda > 0$ defined by $J_r^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x)$, $x \in H$, then we have $J_r^{\partial i_C}(x) = P_C x$ for all $x \in H$ and $\lambda > 0$; see more detail [18]. Moreover, let $h : H \to H$ be a single valued operator, we have $x \in VI(H, h, \partial i_C) = VI(C, h)$.

Setting $M_A = \partial i_{H_1}$ and $M_B = \partial i_{H_2}$ in Equations (6) and (7), then SCVIP reduce to *the split combination of variational inequality problem*, that is find $x^* \in H_1$ such that

$$\langle \sum_{i=1}^{N} a_i A_i x^*, x - x^* \rangle \geq 0, \ \forall x \in H_1, \tag{48}$$

and

$$y^* = Ax^* \in H_2 \text{ such that } \langle \sum_{i=1}^{N} b_i B_i y^*, y - y^* \rangle \geq 0, \ \forall y \in H_2, \tag{49}$$

where $A : H_1 \to H_2$ is bounded linear operator and $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$. The set of all this is denoted by $\Omega_3 = \{x \in VI(H_1, \sum_{i=1}^N a_i A_i) : \ Ax \in VI(H_2, \sum_{i=1}^N b_i B_i)\}$.

Remark 3. *If $M_A = \partial i_{H_1}$ and $M_B = \partial i_{H_2}$, then we have Ω reduce to Ω_3.*

Proof. We will show that $VI(H_1, \sum_{i=1}^N a_i A_i, M_A) = VI(H_1, \sum_{i=1}^N a_i A_i)$. We have for $x^* \in H_1$. Consider,

$$
\begin{aligned}
x^* \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A) &\Leftrightarrow \theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + M_A x^* \\
&\Leftrightarrow \theta_{H_1} \in \sum_{i=1}^N a_i A_i x^* + \partial i_{H_1}(x^*) \\
&\Leftrightarrow -\sum_{i=1}^N a_i A_i x^* \in \partial i_{H_1}(x^*) \\
&\Leftrightarrow \langle \sum_{i=1}^N a_i A_i x^*, x - x^* \rangle \geq 0, \ \forall x \in H_1, \\
&\Leftrightarrow x^* \in VI(H_1, \sum_{i=1}^N a_i A_i).
\end{aligned}
$$

Similarly, we also have $VI(H_2, \sum_{i=1}^N b_i B_i, M_B) = VI(H_2, \sum_{i=1}^N b_i B_i)$. Then $\Omega \equiv \Omega_3$ where $M_A = \partial i_{H_1}$ and $M_B = \partial i_{H_2}$. □

The split combination of variational inequality problem has played an essential role for concrete problems in dynamic emission tomographic image reconstruction, signal recovery problems, beam-forming problems, power-control problems, bandwidth allocation problems and optimal control problems.

Next, we establish a strong convergence theorem for solving the split combination of variational inequality problem and hierarchical fixed point problem of nonexpansive mapping by using a modified Halpern iterative method as follows:

Theorem 2. *Let H_1, H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $A_i : H_1 \to H_1$ be α_i-inverse strongly monotone with $\eta_A = \min_{i=1}\{\alpha_i\}$ and $B_i : H_2 \to H_2$ be β_i-inverse strongly monotone with $\eta_B = \min_{i=1}\{\beta_i\}$. Let $S, T : H_1 \to H_1$ be two nonexpansive mappings. Assume that $\mathbb{F} = \Phi \cap \Omega_3 \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $u, x_1 \in H_1$ and*

$$
\begin{cases}
u_n = P_{H_1}(I - \lambda_A \sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - P_{H_2}(I - \lambda_B \sum_{i=1}^N b_i B_i))Ax_n), \\
y_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n), \\
x_{n+1} = \mu_n u + \varphi_n y_n + \theta_n u_n,
\end{cases} \tag{50}
$$

where $\{\mu_n\}$, $\{\varphi_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$, $\{\sigma_n\} \subseteq [0,1]$ with $\mu_n + \varphi_n + \theta_n = 1$ for all $n \geq 1$, $\lambda_A \in (0, 2\eta_A)$, $\lambda_B \in (0, 2\eta_B)$ and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of A^*A. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \mu_n = 0, \sum_{n=1}^\infty \mu_n = \infty$,
(ii) $0 < c \leq \varphi_n, \theta_n \leq d < 1, \exists c, d > 0$,
(iii) $\sum_{n=1}^\infty |\mu_{n+1} - \mu_n| < \infty, \sum_{n=1}^\infty |\varphi_{n+1} - \varphi_n| < \infty, \sum_{n=1}^\infty |\theta_{n+1} - \theta_n| < \infty$
(iv) $\lim_{n\to\infty} \sigma_n = 0, \sum_{n=1}^\infty \sigma_n < \infty$,
(v) $\lim_{n\to\infty} \frac{\|x_n - y_n\|}{\alpha_n \sigma_n} = 0$,
(vi) $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 1$, *$a_i > 0$ and $b_i > 0$ for all $i = 1, 2, ..., N$.*

Then $\{x_n\}$ converges strongly to $z_0 \in \mathbb{F}$, where $z_0 = P_{\mathbb{F}} u$.

Proof. Put $M_A = \partial i_{H_1}$ and $M_B = \partial i_{H_2}$ in Theorem 1. Using the same method in Theorem 1, we have the desired conclusion. □

5. Numerical

The purpose of this section is to give a numerical example to support some of our. The following example given for supporting Theorem 1 and example show that Theorem 1 is not true if condition (iv) fails, but conditions (i), (ii), (iii), (v) and (vi) are satisfied.

Since Theorem 1 can solve hierarchical fixed point problem for a nonexpansive mapping and SCVIP which our problems can modify for concrete problem in signal processing, image reconstruction, intensity-modulated radiationtherapy treatment planning and sensor networks in computerized tomography. So, we give a numerical example as follows:

Example 5. *Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. For every $i = 1, 2, .., N$, let the mapping $A_i : \mathbb{R} \to \mathbb{R}$ define by $A_i x = \frac{x}{4i}$ for all $x \in H_1$ and $B_i : \mathbb{R} \to \mathbb{R}$ define by $B_i y = \frac{y}{3i}$ for all $y \in H_2$, respectively, let $M_A, M_B : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_A(x) = \{2x\}$, for all $x \in \mathbb{R}$ and $M_B(y) = \{2y\}$, for all $y \in \mathbb{R}$. Let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = -2x$, for all $x \in \mathbb{R}$ and let $\gamma \in (0, \frac{1}{4})$, so we choose $\gamma = \frac{1}{10}$. Let the mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = \max\{0, -x\}$, for all $x \in \mathbb{R}$ and let the mapping $S : \mathbb{R} \to \mathbb{R}$ be defined by $Sx = \min\{0, \frac{x}{2}\}$, for all $x \in \mathbb{R}$. Setting $\{\mu_n\} = \{\frac{1}{5n}\}$, $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$, $\{\theta_n\} = \{\frac{8n-4}{15n}\}$, $\{\alpha_n\} = \{\frac{1}{n}\}$ and $\{\sigma_n\} = \{\frac{1}{4n^2}\}$, $\forall n \in \mathbb{N}$. For every $i = 1, 2, ..., N$, suppose that $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ and $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$. Then $\{x_n\}$ converges strongly to a point $x^* = 0 \in \mathbb{F}$.*

Proof of Solution. It is easy to check that a_i and b_i satisfies all the conditions of Theorem 1 and A_i is $\frac{1}{4i}$-inverse strongly monotone and B_i is $\frac{1}{3i}$-inverse strongly monotone for all $i = 1, 2, ..., N$. We choose $\lambda_A = \frac{1}{4N}$, $\lambda_B = \frac{1}{3N}$. Since $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$, we obtain

$$\sum_{i=1}^{N} a_i A_i x = \sum_{i=1}^{N} \left(\frac{3}{4^i} + \frac{1}{N4^N}\right) \frac{x}{4i}.$$

Then $0 \in VI(H_1, \sum_{i=1}^N a_i A_i, M_A)$. Since $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$, we have

$$\sum_{i=1}^{N} b_i B_i y = \sum_{i=1}^{N} \left(\frac{2}{3^i} + \frac{1}{N3^N}\right) \frac{y}{3i}.$$

Then $0 \in VI(H_2, \sum_{i=1}^N b_i B_i, M_B)$. Thus $\{0\} = \Omega$.

It is easy to observe that T, S are nonexpansive mappings with $Fix(T) = \{0\}$, $Fix(S) = \{0\}$. Hence $\Phi = \{0\}$. Therefore $\mathbb{F} = \Phi \cap \Omega = \{0\}$.

For every $n \in \mathbb{N}$, $\{\mu_n\} = \{\frac{1}{5n}\}$, $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$, $\{\theta_n\} = \{\frac{8n-4}{15n}\}$, $\{\alpha_n\} = \{\frac{1}{n}\}$ and $\{\sigma_n\} = \{\frac{1}{4n^2}\}$, then the sequence $\{\mu_n\}$, $\{\varphi_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$ and $\{\sigma_n\}$ satisfy all the conditions of Theorem 1. We rewrite (20) as follows:

$$\begin{cases} u_n = J_{\lambda_A}^{M_A}(I - \frac{1}{4N}\sum_{i=1}^N a_i A_i)(x_n - \gamma A^*(I - J_{\lambda_B}^{M_B}(I - \frac{1}{3N}\sum_{i=1}^N b_i B_i))Ax_n), \\ y_n = (1 - \frac{1}{n})x_n + \frac{1}{0}(\frac{1}{4n^2}Sx_n + (1 - \frac{1}{4n^2})Tx_n), \\ x_{n+1} = \frac{1}{5n}u + \frac{7n+1}{15n}y_n + \frac{8n-4}{15n}u_n, \end{cases} \tag{51}$$

Choose $u = -1$, $x_1 = 1$, $N = 100$ and $n = 100$. The numerical for the sequence $\{x_n\}$ are shown Table 1 and Figure 1. □

Table 1. The values of $\{x_n\}$ with $N = 100, n = 100$.

n	x_n
1	1.0000
2	−0.0767
3	−0.1178
4	−0.1125
5	−0.1027
⋮	⋮
50	−0.0139
⋮	⋮
96	−0.0073
97	−0.0072
98	−0.0071
99	−0.0070
100	−0.0070

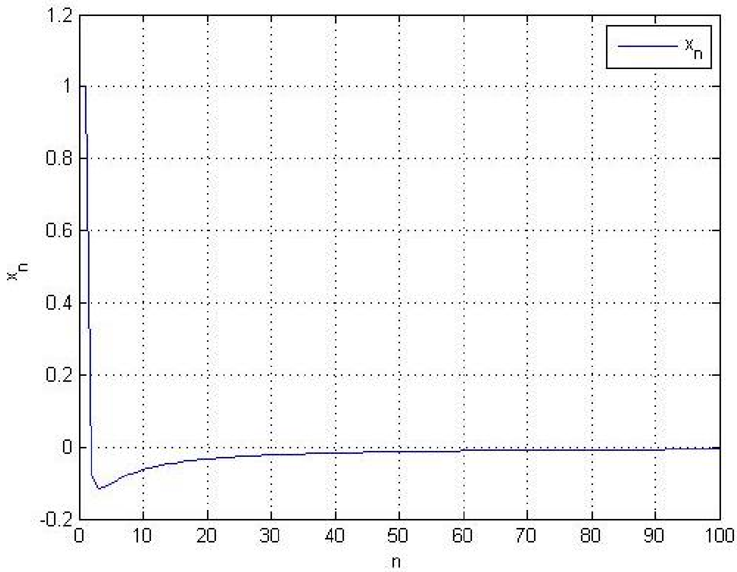

Figure 1. The sequence $\{x_n\}$ converges strongly to 0 with initial values $x_1 = 1$, $N = 100$ and $n = 100$.

Example 6. *Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. For every $i = 1, 2, .., N$, let the mapping $A_i : \mathbb{R} \to \mathbb{R}$ define by $A_i x = \frac{x}{4i}$ for all $x \in H_1$ and $B_i : \mathbb{R} \to \mathbb{R}$ define by $B_i y = \frac{y}{3i}$ for all $y \in H_2$, respectively, let $M_A, M_B : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_A(x) = \{2x\}$, for all $x \in \mathbb{R}$ and $M_B(y) = \{2y\}$, for all $y \in \mathbb{R}$. Let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = -2x$, for all $x \in \mathbb{R}$ and let $\gamma \in (0, \frac{1}{4})$, so we choose $\gamma = \frac{1}{10}$. Let the mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = \max\{0, -x\}$, for all $x \in \mathbb{R}$ and let the mapping $S : \mathbb{R} \to \mathbb{R}$ be defined by $Sx = \min\{0, \frac{x}{2}\}$, for all $x \in \mathbb{R}$. Setting $\{\mu_n\} = \{\frac{1}{5n}\}$, $\{\varphi_n\} = \{\frac{7n+1}{15n}\}$, $\{\theta_n\} = \{\frac{8n-4}{15n}\}$, $\{\alpha_n\} = \{\frac{1}{n}\}$ and $\{\sigma_n\} = \{n\}$, $\forall n \in \mathbb{N}$. For every $i = 1, 2, ..., N$, suppose that $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$ and $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$. Then $\{x_n\}$ is divegence.*

Proof of Solution. Note that the sequence $\{\mu_n\}$, $\{\varphi_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$, a_i and b_i satisfies the conditions (i), (ii), (iii), (v) and (vi) from Theorem 1, while assumption (iv) does not converge to 0 since

$$\lim_{n\to\infty} n = \infty.$$

Choose $u = -1$, $x_1 = 1$, $N = 100$ and $n = 25$. The numerical for the sequence $\{x_n\}$ are shown in Table 2 and Figure 2. Therefore, $\{x_n\}$ does not converge to 0. □

Table 2. The values of $\{x_n\}$ with $N = 100, n = 25$.

n	x_n
1	1.0000
2	−0.0767
3	−0.1717
4	−0.3525
5	−0.4736
⋮	⋮
17	−12.0108
⋮	⋮
21	−41.7292
22	−57.2056
23	−78.5297
24	−107.9388
25	−148.5342

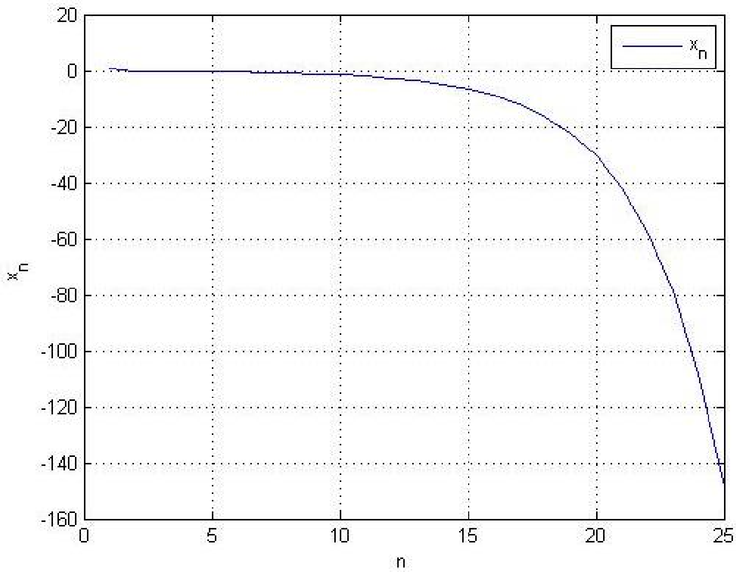

Figure 2. The sequence $\{x_n\}$ is divergence with initial values $x_1 = 1$, $N = 100$ and $n = 25$.

Next, we give example to support out some result in a two dimensional space of real numbers.

Example 7. *Let $H_1 = H_2 = \mathbb{R}^2$, with the inner product defined by $\langle x, y\rangle = xy = x_1 \cdot y_1 + x_2 \cdot y_2$, for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and induced usual norm $\| \cdot \|$ defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. For every $i = 1, 2, .., N$, let the mapping $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ define by $A_i x = \frac{x}{3i}$ for all $x = (x_1, x_2) \in H_1$ and $B_i : \mathbb{R}^2 \to \mathbb{R}^2$ define by $B_i y = \frac{y}{4i}$ for all $y = (y_1, y_2) \in H_2$, respectively, let $M_A, M_B : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $M_A(x) = \{x\}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $M_B(y) = \{3y\}$, for all $y = (y_1, y_2) \in \mathbb{R}^2$. Let the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $A(x) = 3x$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and let $\gamma \in (0, \frac{1}{9})$, so we choose $\gamma = \frac{1}{15}$. Let the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Tx = \frac{x}{3}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and let the mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Sx = \min\{0, \frac{x}{5}\}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Setting $\{\mu_n\} = \{\frac{1}{4n}\}$, $\{\varphi_n\} = \{\frac{7n+1}{12n}\}$, $\{\theta_n\} = \{\frac{5n-4}{12n}\}$, $\{\alpha_n\} = \{\frac{1}{n}\}$ and $\{\sigma_n\} = \{\frac{1}{4n^2}\}$, $\forall n \in \mathbb{N}$. For every $i = 1, 2, ..., N$, suppose that $a_i = \frac{9}{10^i} + \frac{1}{N10^N}$ and $b_i = \frac{2}{3^i} + \frac{1}{N3^N}$. Then $\{x_n\}$ converges strongly to a point $x^* = (0, 0) \in \mathbb{F}$.*

Proof of Solution. It is easy to check that a_i and b_i satisfies all the conditions of Theorem 1 and A_i is $\frac{1}{3i}$-inverse strongly monotone and B_i is $\frac{1}{4i}$-inverse strongly monotone for all $i = 1, 2, ..., N$. We choose $\lambda_A = \frac{1}{5N}$, $\lambda_B = \frac{1}{7N}$. Thus $\{(0, 0)\} = \Omega$.

For definition of T and S, then T and S are nonexpansive mapping with $Fix(T) = \{(0,0)\}$. Hence $\Phi = \{(0,0)\}$. Therefore $\mathbb{F} = \Phi \cap \Omega = \{(0,0)\}$.

For every $n \in \mathbb{N}$, $\{\mu_n\} = \{\frac{1}{4n}\}$, $\{\varphi_n\} = \{\frac{7n+1}{12n}\}$, $\{\theta_n\} = \{\frac{5n-4}{12n}\}$, $\{\alpha_n\} = \{\frac{1}{n}\}$ and $\{\sigma_n\} = \{\frac{1}{4n^2}\}$, then the sequence $\{\mu_n\}$, $\{\varphi_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$ and $\{\sigma_n\}$ satisfy all the conditions of Theorem 1.

From Theorem 1, we can conclude that the sequence $\{x_n\}$ converges to $(0,0)$. □

6. Conclusions

(i) Table 1 and Figure 1 show that the sequence $\{x_n\}$ converges to 0, where $\{0\} = \Phi \cap \Omega$.
(ii) Table 2 and Figure 2 show that the sequence $\{x_n\}$ diverge, where condition (iv) is violated since $\lim_{n\to\infty} \sigma_n \neq 0$.

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