*mathematics*

*Article*

# Generalized $F$-Contractions on Product of Metric Spaces

**Nicolae Adrian Secelean [1,*] and Mi Zhou [2]**

1 Department of Mathematics and Computer Science, Faculty of Sciences, Lucian Blaga University of Sibiu, 550012 Sibiu, Romania
2 School of Science and Technology, University of Sanya, Hainan 572000, China; mizhou330@126.com
* Correspondence: nicolae.secelean@ulbsibiu.ro

Received: 16 October 2019; Accepted: 30 October 2019; Published: 3 November 2019

**Abstract:** Our purpose in this paper is to extend the fixed point results of a $\psi F$-contraction introduced by Secelean N.A. and Wardowski D. ($\psi F$-Contractions: Not Necessarily Nonexpansive Picard Operators, *Results. Math.* **70**(3), 415–431 (2016)) defined on a metric space $X$ into itself to the case of mapping defined on the product space $X^I$, where $I$ is a set of positive integers (natural numbers). Some improvements to the conditions imposed on function $F$ and space $X$ are provided. An illustrative example is given.

**Keywords:** $F$-contraction; $\psi F$-contraction, generalized $\psi F$-contraction; fixed point

**MSC:** Primary 47H09; Secondary 47H10; 45D05

## 1. Introduction

As it is well known, Banach contraction principle is of crucial importance in fixed point theory with many applications in various fields of mathematics. This initiated several authors to extend and improve the purpose of that theory either by generalizing the domain of the mapping or by extending the contractive conditions.

In [1] D. Wardowski introduced a new type of contractive self-map $T$ on a metric space $(X, \mathrm{d})$, so called $F$-contraction. This is defined by the inequality $F\big(\mathrm{d}(Tx, Ty)\big) + \tau \leq F\big(\mathrm{d}(x, y)\big)$ for all $x, y \in X$ with $Tx \neq Ty$, where $\tau > 0$ and $F : (0, \infty) \to \mathbb{R}$ satisfies the conditions (F1)-(F3) (see Definition 1 below). Wardowski proved that, whenever $(X, \mathrm{d})$ is complete, every $F$-contraction has a unique fixed point which is the limit of the Picard iterations. He also showed that $F$-contractions are generalizations of Banach contractions.

In last decades, there is a sustained endeavor of many researchers obtain new classes of Picard mappings by extend and improve the survey of $F$-contractions by generalizing the function $F$ and the spaces with metric type structures. In this respect, in [2] N.A. Secelean considers $F$-contractions defined on the product space $X^I$ with values in $X$, where $I$ is a set of positive integers, and proved two fixed point theorems for such mappings. N.A. Secelean and D. Wardowski [3] introduced a new concept $\psi F$-contraction which strictly generalized $F$-contraction and proved that it is also a Picard operator (i.e., it has a unique fixed point which is the limit of Picard iteration). They also given an example of $\psi F$-contraction which is neither contractive nor nonexpansive map.

In this paper we generalize the fixed point result given in [3] for $\psi F$-contractions defined on complete metric space $X$ by extending this mappings on product metric space $X^I$ endowed with the sup metric, where $I$ is a set of positive integers. We also extend the results obtained in [2,4]. We highlight that the improvement provided in the present paper consists also in imposing only the condition (F1) for function $F$ and by replacing the completeness of the space $X$ with a less restrictive one: orbitally completeness. An illustrative example are given.

## 2. Preliminaries: $F$-Contractions; $\psi F$-Contractions

We give here a brief exposition of $F$-contractions defined by D. Wardowski [1] and some of its extensions, namely $\psi F$-contractions, introduced by N.A. Secelean and D. Wardowski in [3].

In this paper we denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ the set of all real numbers, all positive real numbers and all positive integers, respectively. We will also write $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

If $\nu, \lambda \in \overline{\mathbb{R}}_+$, by "$\nu > \lambda$" we understand $\nu > \lambda$ if $\lambda \in \mathbb{R}_+$ and $\nu = \infty$ otherwise.

Throughout this section $(X, \mathrm{d})$ denotes a metric space.

**Definition 1** ([1] (Def.2.1))**.** *Let denote by $\mathfrak{F}$ the set of all functions $F : (0, \nu) \to \mathbb{R}$, $\nu >$ diam $X$, satisfying:*

*(F1) $F$ is strictly increasing, i.e., for every $t, s \in (0, \nu)$, $t < s$, one has $F(t) < F(s)$,*

*(F2) for every sequence $(t_n) \subset (0, \nu)$, $\lim_n t_n = 0$ if and only if $\lim_n F(t_n) = -\infty$,*

*(F3) there exists $\lambda \in (0, 1)$ such that $\lim\limits_{t \searrow 0} t^\lambda F(t) = 0$,*

*where diam means the diameter.*

*A mapping $T : X \to X$ is called $F$-*contraction *if there are $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F\big(\mathrm{d}(Tx, Ty)\big) \le F\big(\mathrm{d}(x, y)\big) \text{ for all } x, y \in X \text{ with } Tx \ne Ty. \tag{1}$$

For each $\mu \in \overline{\mathbb{R}}_+$, we denote by $\Psi^*_\mu$ the class of all increasing functions $\psi : (-\infty, \mu) \to (-\infty, \mu)$ such that $\psi^n(t) \longrightarrow -\infty$, for every $t \in (-\infty, \mu)$, where $\psi^n$ denotes the $n$-th composition of $\psi$.

If $F$ satisfies (F1) and (F2), $\psi \in \Psi^*_\mu$, $\mu = \sup_{0<t<\nu}$, $\nu >$ diam $X$, a mapping $T : X \to X$ is said to be *$\psi F$-contraction* whenever

$$Tx \ne Ty \Rightarrow F\big(\mathrm{d}(Tx, Ty)\big) \le \psi\Big(F\big(\mathrm{d}(x, y)\big)\Big).$$

The following results are easy to be verified:

**Remark 1.** *(1) All $F$-contractions are contractive maps and every Banach contraction with ratio $r \in (0, 1)$ is an $F$-contraction with $F(t) = \ln t$ and $\tau = -\ln r$.*

*(2) Every $\psi F$-contraction is an $F$-contraction if we take $\psi(t) = t - \tau$.*

In the next theorems similar results as the Banach contraction principle for $F$-contractions and $\psi F$-contractions are established.

**Theorem 1** ([1](Th. 2.1))**.** *Assume that $(X, \mathrm{d})$ is a complete metric space, $F \in \mathfrak{F}$ and $T : X \to X$ is an $F$-contraction. Then $T$ has a unique fixed point $\xi$ and, for each $x \in X$, the sequence $(T^n x)_n$ converges to $\xi$.*

**Theorem 2** ([3] (Th. 3.3))**.** *Let $T : X \to X$ be a $\psi F$-contraction, where $F : (0, \nu) \to \mathbb{R}$ satisfies (F1), (F2), $\nu >$ diam $X$, and $\psi \in \Psi^*_\mu$ be continuous, $\mu > \sup F$. If $(X, \mathrm{d})$ is complete, then $T$ has a unique fixed point $\xi$ and, for every $x \in X$, the sequence $(T^n x)_n$ converges to $\xi$.*

## 3. The Results

Inspired by [2,3], we will generalize the concept of $F$-contraction.

Let $(X, \mathrm{d})$ be a metric space.

Let $I$ be a nonempty set of positive integers $I$ and denote

$$X^I := \{x = (x_i)_{i \in I};\ x_i \in X,\ \sup_{i,j \in I} \mathrm{d}(x_i, x_j) < \infty\}.$$

There is no loss of generality in assuming that $I = \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, if $I$ is finite and $I = \mathbb{N}$ otherwise. In this circumstance, $X^I = X^m = \{x = (x_1, \ldots, x_m);\ x_i \in X,\ i = 1, \ldots, m\}$, where $m = \operatorname{card} I$ and, respectively, $X^I$ is the $l^\infty(X)$ space of all bounded sequences in $X$.

We say that $\xi \in X$ is a *fixed point* of an operator $T : X^I \to X$ if $T(\xi, \xi, \ldots) = \xi$.

Set $\nu > \sup_{x,y \in X} \mathrm{d}(x, y)$. Throughout the paper, we denote by $\mathcal{F}$ the class of nondecreasing functions $F : (0, \nu) \to \mathbb{R}$ and, for $\mu = \sup_{0<t<\nu} F(t) \in \overline{\mathbb{R}}_+$, by $\Psi_\mu$ the family of all increasing and upper semi-continuous mappings $\psi : (-\infty, \mu) \to (-\infty, \mu)$ such that $\psi(t) < t$ for all $t \in (-\infty, \mu)$ (several examples of such mappings $\psi$ can be found in [3]).

**Definition 2.** *Let us consider $F \in \mathcal{F}$ and $\psi \in \Psi_\mu$.*

*A mapping $T : X^I \to X$ is said to be a* generalized $\psi F$-contraction *if, for every $x = (x_i)$, $y = (y_i) \in X^I$,*

$$Tx \neq Ty \Rightarrow F\big(\mathrm{d}(Tx, Ty)\big) \leq \psi\big(F\big(\sup_{i \in I} \mathrm{d}(x_i, y_i)\big)\big). \tag{2}$$

We provide first the fixed point result for the case when $I$ is a finite set and, next, we prove a theorem for the general case.

### *3.1. Generalized $\psi F$-Contractions on $X^m$*

In this subsection, the product metric space $X^I$, where $I$ is finite, is considered.

Let $T : X^m \to X$ be a function and, for a given $\alpha = (x_0, x_1, \ldots, x_{m-1}) \in X^m$, we define the *orbit* of $T$ at $\alpha$ by $\mathcal{O}(\alpha) = \{x_n\} = \{x_n,\ n = 0, 1, \ldots\}$, where $x_{k+1} = T(x_k, x_{k-1}, \ldots, x_{k-m+1})$ for all $k \geq m - 1$.

We say that the map $T$ is *orbitally continuous* at a point $\xi \in X$ if, for every $\alpha \in X^m$ such that $x_n \underset{n}{\longrightarrow} \xi$ and every subsequences $(x^1_{n_k}), \ldots, (x^m_{n_k})$, one has $\lim_k T(x^1_{n_k}, \ldots, x^m_{n_k}) = T(\xi, \ldots, \xi)$, where $\{x_n\} = \mathcal{O}(\alpha)$. If $T$ is orbitally continuous at every $\xi \in X$, we say that it is *orbitally continuous*.

The space $X$ is *T-orbitally complete* if, for each $\alpha \in X^m$ and every Cauchy subsequence $(x_{n_k})$ of $(x_n)$, where $\{x_n\} = \mathcal{O}(\alpha)$, is convergent.

**Definition 3.** *Let us consider $F \in \mathcal{F}$ and $\psi \in \Psi_\mu$.*

*A mapping $T : X^m \to X$ is called* strong orbitally generalized $\psi F$-contraction *(SOG $\psi F$-contraction for short) if, for every $\alpha \in X^m$, one has*

$$F\big(\mathrm{d}\big(T(x_k, \ldots, x_{k-m+1}), T(y_k, \ldots, y_{k-m+1})\big)\big) \leq \psi\big(F\big(\max_{0 \leq i \leq m-1} \mathrm{d}(x_{k-i}, y_{k-i})\big)\big), \tag{3}$$

*for all $x = (x_i)$, $y = (y_i) \subset \overline{\mathcal{O}(\alpha)}$ and all $k \geq m - 1$ with $T(x_k, \ldots, x_{k-m+1}) \neq T(y_k, \ldots, y_{k-m+1})$, where the bar means the closure.*

*$T$ is an* weak orbitally generalized $\psi F$-contraction *(WOG $\psi F$-contraction for short) if, for every $\alpha \in X^m$, one has*

$$F\big(\mathrm{d}\big(T(x_k, \ldots, x_{k-m+1}), T(x_l, \ldots, x_{l-m+1})\big)\big) \leq \psi\big(F\big(\max_{0 \leq i \leq m-1} \mathrm{d}(x_{k-i}, x_{l-i})\big)\big), \tag{4}$$

*for all $k, l \geq 1$ such that $T(x_{k+1}, \ldots, x_{k-m+2}) \neq T(x_k, \ldots, x_{k-m+1})$, where $\{x_n\} = \mathcal{O}(\alpha)$.*

It is obvious that every generalized $\psi F$-contraction is a SOG $\psi F$-contraction and every SOG $\psi F$-contraction is an WOG $\psi F$-contraction.

In the following, we provide some generalizations of Theorems 1 and 2. We need first the following three results.

**Lemma 1** ([5] (L. 2.1))**.** *If $\psi \in \Psi_\mu$, $\mu \in \overline{\mathbb{R}}_+$, then $\lim_n \psi^n(t) = -\infty$, for all $t < \mu$.*

**Lemma 2** ([6] (L.3.2(a))). *Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a nondecreasing map and $(t_k)_k$ a sequence of positive real numbers. Then*

$$F(t_k) \underset{k}{\longrightarrow} -\infty \ \Rightarrow \ t_k \underset{k}{\longrightarrow} 0.$$

**Proposition 1** ([7] (Prop.3)). *Let $(x_n)$ be a sequence of elements from $X$ and $\Delta$ be a subset of $(0, \nu)$, $\nu \in \overline{\mathbb{R}}_+$, such that $(0, \nu) \setminus \Delta$ is dense in $(0, \nu)$. If $\mathrm{d}(x_n, x_{n+1}) \underset{n}{\longrightarrow} 0$ and $(x_n)$ is not Cauchy, then there exist $\eta \in (0, \nu) \setminus \Delta$, $n_0 \in \mathbb{N}$ and the sequences of natural numbers $(m_k)$, $(n_k)$ such that*

1. $\mathrm{d}(x_{m_k}, x_{n_k}) \searrow \eta$, $k \longrightarrow \infty$,
2. $\mathrm{d}(x_{m_k+p}, x_{n_k+q}) \longrightarrow \eta$, $k \longrightarrow \infty$, $p, q \in \{0, 1\}$.

Our first new result is the next.

**Theorem 3.** *Let $m \in \mathbb{N}$, $F \in \mathcal{F}$, $\psi \in \Psi_\mu$ and $T : X^m \to X$ be an WOG $\psi F$-contraction which is orbitally continuous. If the space $(X, \mathrm{d})$ is $T$-orbitally complete, then there is $\xi \in X$ such that $T(\xi, \ldots, \xi) = \xi$. If, further,*

$$\mathrm{d}\big(T(u, \ldots, u), T(v, \ldots, v)\big) \neq \mathrm{d}(u, v), \ \ \forall\, u, v \in X,\ u \neq v, \tag{5}$$

*then $\xi$ is unique and, for every $\alpha \in X^m$, $x_n \underset{n}{\longrightarrow} \xi$, where $\{x_n\} = \mathcal{O}(\alpha)$.*

**Proof.** There is no loss of generality in assuming $m = 2$, for the cases $m = 1$ and $m \geq 3$ one can proceed analogously.

Assume that $T$ is an WOG $\psi F$-contraction.

Set $x_0, x_1 \in X$ and, for each $k \geq 1$, $x_{k+1} := T(x_k, x_{k-1})$. We also define $\gamma_k := \max\{\mathrm{d}(x_{k+1}, x_k), \mathrm{d}(x_k, x_{k-1})\}$, $k = 1, 2, \ldots$

If there is $k_0 \in \mathbb{N}$ such that $\gamma_{k_0} = 0$, then $x_{k-1} = x_k = x_{k+1}$ and, taking $\xi = x_{k-1}$, one obtains $T(\xi, \xi) = \xi$.

Now suppose that $\gamma_k > 0$ for all $k \geq 1$ and fix $k \geq 3$.

If $\gamma_k = \mathrm{d}(x_{k+1}, x_k)$, then

$$F\big(\mathrm{d}(x_{k+1}, x_k)\big) = F\big(\mathrm{d}(T(x_k, x_{k-1}), T(x_{k-1}, x_{k-2}))\big)$$
$$\leq \psi\big(F\big(\max\{\mathrm{d}(x_k, x_{k-1}), \mathrm{d}(x_{k-1}, x_{k-2})\}\big)\big) = \psi\big(F(\gamma_{k-1})\big).$$

In the other case, we have

$$F\big(\mathrm{d}(x_k, x_{k-1})\big) = F\big(\mathrm{d}(T(x_{k-1}, x_{k-2}), T(x_{k-2}, x_{k-3}))\big) \leq \psi\big(F(\gamma_{k-2})\big).$$

Consequently,

$$F(\gamma_k) = F\big(\max\{\mathrm{d}(x_{k+1}, x_k), \mathrm{d}(x_k, x_{k-1})\}\big) \leq \max\big\{\psi\big(F(\gamma_{k-1})\big), \psi\big(F(\gamma_{k-2})\big)\big\}$$
$$= \psi\big(\max\{F(\gamma_{k-1}), F(\gamma_{k-2})\}\big), \tag{6}$$

where, in the last equality, we used the monotonicity of $\psi$.

By (6), we have for $k \geq 4$,

$$F(\gamma_{k-1}) \leq \psi\big(\max\{F(\gamma_{k-2}), F(\gamma_{k-3})\}\big) < \max\{F(\gamma_{k-2}), F(\gamma_{k-3})\} \tag{7}$$

so

$$\psi^2\big(F(\gamma_{k-1})\big) < \psi^2\big(\max\{F(\gamma_{k-2}), F(\gamma_{k-3})\}\big) = \max\big\{\psi^2\big(F(\gamma_{k-2})\big), \psi^2\big(F(\gamma_{k-3})\big)\big\}. \tag{8}$$

From (6) and (8), we obtain

$$\psi(F(\gamma_k)) \le \psi^2\big(\max\{F(\gamma_{k-1}), F(\gamma_{k-2})\}\big) = \max\big\{\psi^2(F(\gamma_{k-1})), \psi^2(F(\gamma_{k-2}))\big\}$$
$$< \max\big\{\psi^2(F(\gamma_{k-2})), \psi^2(F(\gamma_{k-3}))\big\}. \tag{9}$$

From (7), one has

$$\psi(F(\gamma_{k-1})) \le \psi^2\big(\max\{F(\gamma_{k-2}), F(\gamma_{k-3})\}\big) = \max\big\{\psi^2(F(\gamma_{k-2})), \psi^2(F(\gamma_{k-3}))\big\}. \tag{10}$$

Now, by (6), (9) and (10), it follows

$$F(\gamma_{k+1}) \le \psi^2\big(\max\{F(\gamma_{k-2}), F(\gamma_{k-3})\}\big).$$

Inductively, we obtain

$$F(\gamma_{k+1}) \le \psi^{p+1}\big(\max\{F(\gamma_{k-2p}), F(\gamma_{k-2p-1})\}\big), \text{ whether } 0 \le p \le \frac{k}{2} - 1.$$

Thus, if $k$ is an even number,

$$F(\gamma_{k+1}) \le \psi^{\frac{k}{2}-1}\big(\max\{F(\gamma_2), F(\gamma_1)\}\big),$$

and, if $k$ is odd,

$$F(\gamma_{k+1}) \le \psi^{\frac{k-3}{2}}\big(\max\{F(\gamma_3), F(\gamma_2)\}\big).$$

From Lemma 1, we deduce that $F(\gamma_k) \underset{k}{\longrightarrow} -\infty$ and, from Lemma 2, that

$$\mathrm{d}(x_{k+1}, x_k) \underset{k}{\longrightarrow} 0. \tag{11}$$

Now, assume that the sequence $(x_n)$ is not Cauchy and let $\Delta$ be the set of discontinuities of $F$. Since $F$ is monotonic, it follows that $\Delta$ is at most countable, and so $(0, \nu) \setminus \Delta$ is dense in $(0, \nu)$. According to Proposition 1, one can find $\eta \in (0, \nu) \setminus \Delta$ and sequences $(m_k)$, $(n_k)$ such that

$$\mathrm{d}(x_{m_k}, x_{n_k}) \searrow \eta,\ \mathrm{d}(x_{m_k+1}, x_{n_k+1}) \longrightarrow \eta, k \longrightarrow \infty.$$

Since $\eta > 0$, there exists $K \in \mathbb{N}$ such that $\mathrm{d}(x_{m_k+1}, x_{n_k+1}) > 0$ for all $k \ge K$. Therefore, we get

$$F\big(\mathrm{d}(x_{m_k+1}, x_{n_k+1})\big) = F\big(\mathrm{d}(T(x_{m_k}, x_{m_k-1}), T(x_{n_k}, x_{n_k-1}))\big)$$
$$\le \psi\big(F\big(\max\{\mathrm{d}(x_{m_k}, x_{n_k}), \mathrm{d}(x_{m_{k-1}}, x_{n_{k-1}})\}\big)\big)$$
$$\le \psi\big(F\big(\mathrm{d}(x_{m_{k-1}}, x_{m_k}) + \mathrm{d}(x_{m_k}, x_{n_k}) + \mathrm{d}(x_{n_k}, x_{n_{k-1}})\big)\big),\ \forall k \ge K.$$

Letting $k \longrightarrow \infty$, using (11), the continuity of $F$ at $\eta$, and the fact that $\psi$ is upper semi-continuous, one obtains

$$F(\eta) \le \limsup_{t \longrightarrow F(\eta)} \psi(t) \le \psi(F(\eta)) < F(\eta),$$

a contradiction. Therefore, $(x_n)$ is Cauchy, hence, $X$ being $T$-orbitally complete, is convergent. Let $\xi \in X$ be its limit.

Now, using the orbitally continuity of $T$ at $\xi$, one has

$$\xi = \lim_k x_k = \lim_k T(x_{k-1}, \ldots, x_{k-m}) = T(\xi, \ldots, \xi).$$

Finally, the uniqueness of $\xi$ follows obviously from (5). $\square$

**Corollary 1.** *Let $m \in \mathbb{N}$, $F \in \mathcal{F}$, $\psi \in \Psi_\mu$ and $T : X^m \to X$ be an SOG $\psi F$-contraction. If the space $(X, \mathrm{d})$ is $T$-orbitally complete and (5) holds, then there is a unique $\xi \in X$ such that $T(\xi, \dots, \xi) = \xi$ and, for every $\alpha \in X^m$, $x_n \underset{n}{\longrightarrow} \xi$, where $\{x_n\} = \mathcal{O}(\alpha)$.*

**Proof.** Choose $\xi \in X$ and $\alpha \in X^m$ such that $x_n \underset{n}{\longrightarrow} \xi$, where $\{x_n\} = \mathcal{O}(\alpha)$. If $(x^1_{n_k})$,...,$(x^m_{n_k})$ are subsequences of $(x_n)$, then, by (3),

$$F\big(\mathrm{d}\big(T(x^1_{n_k}, \dots, x^m_{n_k}), T(\xi, \dots, \xi)\big)\big) \le \psi\big(F\big(\max_{1\le j\le m}\{\mathrm{d}(x^j_{n_k}, \xi)\}\big)\big) < F\big(\max_{1\le j\le m}\{\mathrm{d}(x^j_{n_k}, \xi)\}\big)$$

hence

$$\mathrm{d}\big(T(x^1_{n_k}, \dots, x^m_{n_k}), T(\xi, \dots, \xi)\big) \le \max_{1\le j\le m}\{\mathrm{d}(x^j_{n_k}, \xi)\} \underset{k}{\longrightarrow} 0,$$

so $T$ is orbitally continuous at $\xi$.

The conclusion now follows from Theorem 3. $\square$

**Corollary 2.** *Let $m \in \mathbb{N}$, $F \in \mathcal{F}$, $\psi \in \Psi_\mu$ and $T : X^m \to X$ be a generalized $\psi F$-contraction. If the space $(X, \mathrm{d})$ is $T$-orbitally complete, then there is a unique $\xi \in X$ such that $T(\xi, \dots, \xi) = \xi$ and, for every $\alpha \in X^m$, $x_n \underset{n}{\longrightarrow} \xi$, where $\{x_n\} = \mathcal{O}(\alpha)$.*

**Proof.** Firstly, we prove that $T$ satisfies the condition (5). Let $u, v \in X$, $u \ne v$. One has

$$F\big(\mathrm{d}\big(T(u, \dots, u), T(v \dots, v)\big)\big) \le \psi\big(F\big(\mathrm{d}(u, v)\big)\big) < F\big(\mathrm{d}(u, v)\big),$$

hence $\mathrm{d}\big(T(u, \dots, u), T(v \dots, v)\big) < \mathrm{d}(u, v)$.

Next, we apply Corollary 1. $\square$

**Remark 2.** *In the particular case $m = 1$, several results concerning $F$-contractions in the literature can be obtained. The improvement of these results also consists in the fact that the requirement for $F$ is just to satisfy (F1) and that we consider an arbitrary mapping $t \mapsto \psi(t)$ satisfying some minimal properties instead of $t - \tau$.*

*Corollary 2 generalized ([2],Th. 2.2).*

**Example 1.** *Let us consider $X = (1, 2]$ endowed with the standard metric and $T : X \times X \to X$ given by $T(x, y) = \sqrt{x + y}$. Then*

*1. $X$ is uncomplete while it is orbitally complete because, for every $x_0, x_1 \in X$, the sequence $x_n = T(x_{n-1}, x_{n-2})$, $n \ge 2$, converges to 2.*

*2. $T$ is a generalized $\psi F$-contraction, where $F : (0, \nu) \to \mathbb{R}$, $F(t) = -1/t$, $,\nu > 1$, and $\psi : (-\infty, -1/\nu) \to (-\infty, -1/\nu)$, $\psi(t) = \sqrt[3]{t^3 - 1}$.*

*3. $T$ has a unique fixed point $\xi = 2$ and for every $\alpha \in X^m$, $x_n \underset{n}{\longrightarrow} \xi$, where $\{x_n\} = \mathcal{O}(\alpha)$.*

**Proof.** 1. It is obvious that $x_n \le 2$ for every $n \ge 0$.

Let $\lambda = \min\{x_0, x_1\}$ and define $y_n = \sqrt{y_{n-1} + y_{n-2}}$ for all $n \ge 2$, where $y_0 = y_1 = \lambda$. Then $y_n \le x_n$ for every $n \ge 0$. One can easily prove, by induction, that $(y_n)_n$ is nondecreasing so, being upper bounded, is convergent and its limit is 2. Consequently, from the sandwich rule, $\lim_n x_n = 2$.

2. We fix $x, y, z, t \in X$ with $x + y \ne z + t$ and denote $a = x + y$, $b = z + t$, $M = \max\big\{|z - x|, |t - y|\big\}$. Hence $a, b \in (2, 4]$, so $2\sqrt{2} < \sqrt{a} + \sqrt{b}$. Thus

$$2\sqrt{2}\,\big|\sqrt{a} - \sqrt{b}\big| < \big|\sqrt{a} - \sqrt{b}\big| \cdot \big|\sqrt{a} + \sqrt{b}\big| = |a - b|.$$

Consequently

$$2\sqrt{2}\left|\sqrt{x+y}-\sqrt{z+t}\right| < \left|(x+y)-(z+t)\right| = \left|(x-z)+(y-t)\right|$$
$$\leq |z-x|+|t-y| \leq 2\max\left\{|z-x|,|t-y|\right\}. \tag{12}$$

In order to obtain (2), we need prove the inequality

$$-\frac{1}{\left|\sqrt{x+y}-\sqrt{z+t}\right|} \leq -\frac{\sqrt[3]{M^3+1}}{M},$$

that is

$$\left|\sqrt{x+y}-\sqrt{z+t}\right| \leq \frac{M}{\sqrt[3]{M^3+1}}.$$

This follows from (12) and $\sqrt[3]{M^3+1} < \sqrt{2}$.

3. The assertion follows from Corollary 2. □

### 3.2. Generalized $\psi F$-Contractions on $X^I$

In this subsection $I$ is an arbitrary subset of $\mathbb{N}$.

For a function $f : X^I \to X$ we define $\widetilde{f} : X \to X$ to be $\widetilde{f}(t) := f(\tilde{t})$, where $\tilde{t} = (x_i)_{i\in I}$, $x_i = t$, for all $i \in I$ (the constant sequence).

We will follows the construction from [2].

Let us consider a mapping $T : X^I \to X$. For a given $x = (x_i)_{i\in I} \in X^I$, the *iterative sequence* $(y_k)_{k\geq 0}$ associated with $T$ at $x$ is defined by $y_0 = Tx$, $y_k = T\big(\widetilde{T}^k(x_1), \widetilde{T}^k(x_2), \dots\big)$, for every $k \geq 1$.

In order to prove the next theorem, we need the following elementary result.

**Lemma 3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $A \subset \mathbb{R}$ a bounded set. Then*
*$(a)$ if $f$ is nondecreasing, then $\sup f(A) \leq f(\sup A)$;*
*$(b)$ if $f$ is continuous, then $\sup f(A) \geq f(\sup A)$;*
*$(c)$ if $A$ is finite and $f$ is nodecreasing, then $\sup f(A) = f(\sup A)$.*

In the following, we provide a version of Corollary 2 by using another successive approximation of the fixed point and, also, a fixed point theorem, in the case when $I$ is infinite.

**Theorem 4.** *Let $F \in \mathcal{F}$, $\psi \in \Psi_\mu$ and $T : X^I \to X$ be a generalized $\psi F$-contraction and assume that the space $(X, \mathrm{d})$ is $\widetilde{T}$-orbitally complete. If $I$ is finite or $F$ is continuous, then there exists a unique $\xi \in X$ such that $T(\xi, \xi, \dots) = \xi$ and $\widetilde{T}^p(t) \underset{p}{\longrightarrow} \xi$ for every $t \in X$. Furthermore, $\xi$ is the limit of the iterative sequence $(y_k)_{k\geq 0}$ associated with $T$ at any $x = (x_i)_{i\in I} \in X$.*

**Proof.** Taking $m = 1$ in Corollary 2, it follows that $\widetilde{T}$ is a (generalized) $\psi F$-contraction and there is a unique $\xi \in X$ such that $\widetilde{T}(\xi) = \xi$ and $\widetilde{T}^p(t) \underset{p}{\longrightarrow} \xi$ for all $t \in X$.

Choose $x = (x_i) \in X^I$. We will prove that the iterative sequence $(y_k)_{k\geq 0}$ associated with $T$ at $x$ converges to $\xi$.

First, we observe that, for each $k \geq 1$,

$$F\big(\mathrm{d}(\xi, \widetilde{T}^k(t))\big) = F\big(\mathrm{d}\big(\widetilde{T}(\xi), \widetilde{T}(\widetilde{T}^{k-1}(t))\big)\big) \leq \psi\big(F\big(d(\xi, \widetilde{T}^{k-1}(t))\big)\big)$$
$$\leq \dots \leq \psi^k\big(F\big(\mathrm{d}(\xi, t)\big)\big), \tag{13}$$

for all $t \in X$ satisfying $\xi \neq \widetilde{T}^k(t)$.

Set $K = \{k \in \mathbb{N};\ y_k \neq \xi\}$. If $K$ is finite, then, clearly, $y_k \longrightarrow \xi$. Assume that $K$ is infinite. For each $k \in K$, set $I_k = \{i \in I;\ \widetilde{T}^k(x_i) \neq \xi\}$. Then $I_k \neq \emptyset$ and, by (13), $\widetilde{T}^j(x_i) \neq \xi$ for all $j \leq k$ and $i \in I_k$. Then, from hypothesis (continuity of $F$ or boundedness of $I$), Lemma 3 and (13), one has

$$F\big(\mathrm{d}(\xi, y_k)\big) \leq \psi\big(F\big(\sup_{i\in I} \mathrm{d}(\xi, \widetilde{T}^k(x_i))\big)\big) = \psi\big(F\big(\sup_{i\in I_k} \mathrm{d}(\xi, \widetilde{T}^k(x_i))\big)\big)$$

$$\leq \psi\big(\sup_{i\in I_k} F\big(\mathrm{d}(\xi, \widetilde{T}^k(x_i))\big)\big) \leq \psi\big(\sup_{i\in I_k} \psi^k\big(F\big(\mathrm{d}(\xi, x_i)\big)\big) < \sup_{i\in I_k} \psi^k\big(F\big(\mathrm{d}(\xi, x_i)\big)$$

$$\leq \psi^k\big(F\big(\sup_{i\in I_k} \mathrm{d}(\xi, x_i)\big)\big) \leq \psi^k(M)$$

for every $k \in K$, $k \geq 1$, where $M = F\big(\sup_{i\in I} \mathrm{d}(\xi, x_i)\big) < \infty$ because the sequence $(x_i)$ is bounded.

From the previous inequalities and Lemma 1, we deduce that $F\big(\mathrm{d}(\xi, y_k)\big) \underset{k\in K}{\longrightarrow} -\infty$, hence, according to Lemma 2, we get $\mathrm{d}(\xi, y_k) \underset{k\in K}{\longrightarrow} 0$. Thus, since $y_k = \xi$ for all $k \notin K$, $y_k \longrightarrow \xi$. $\square$

In ([2], Ex. 2.2), one can find a particular example of $\psi F$-contraction on a product metric space $X^I$, $I \subset \mathbb{N}$, which is not generalized Banach contraction, where $\psi(t) = t - \tau$, $\tau > 0$, $(X, \mathrm{d})$ is complete, $F$ continuous.

**Author Contributions:** conceptualization, N.A.S. and M.Z.; methodology, N.A.S. and M.Z.; validation, N.A.S. and M.Z.; writing–original draft preparation, N.A.S. and M.Z.; writing–review and editing, N.A.S.; supervision, M.Z.

**Funding:** The first author is financed by Lucian Blaga University of Sibiu & Hasso Plattner Foundation research grants LBUS-IRG-2019-05.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *87*, 94. [CrossRef]
2. Secelean N.A. Generalized $F$-iterated function systems on product of metric spaces. *J. Fixed Point Theory Appl.* **2015**, *17*, 575–595. [CrossRef]
3. Secelean, N.A.; Wardowski, D. $\psi F$-Contractions: Not Necessarily Nonexpansive Picard Operators. *Results Math.* **2016**, *70*, 415–431. [CrossRef]
4. Secelean, N.A. Generalized Iterated Function Systems on the space $l^\infty(X)$. *J. Math. Anal. Appl.* **2014**, *410*, 847–858. [CrossRef]
5. Secelean, N.A.; Wardowski, D. New Fixed Point Tools in Non-metrizable Spaces. *Results Math.* **2017**, *72*, 919–935. [CrossRef]
6. Secelean, N.A. Iterated function systems consisting of $F$-contractions. *Fixed Point Theory Appl.* **2013**, *2013*, 277. [CrossRef]
7. Turinici, M. Wardowski implicit contractions in metric spaces. *arXiv* **2012**, arXiv:1211.3164.