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A New Scheme Using Cubic B-Spline to Solve Non-Linear Differential Equations Arising in Visco-Elastic Flows and Hydrodynamic Stability Problems

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Abstract: This study deals with the numerical solution of the non-linear differential equations (DEs) arising in the study of hydrodynamics and hydro-magnetic stability problems using a new cubic B-spline scheme (CBS). The main idea is that we have modified the boundary value problems (BVPs) to produce a new system of linear equations. The algorithm developed here is not only for the approximation solutions of the 10^{th} order BVPs but also estimate from 1st derivative to 10^{th} derivative of the exact solution as well. Some examples are illustrated to show the feasibility and competence of the proposed scheme.

Keywords: non-linear differential equation; cubic B-spline; central finite difference approximations; absolute errors

MSC: 34K10; 34K28; 42A10; 65D05; 65D07

1. Introduction

Recent research in the field of hydrodynamic and hydromagnetics stability have found the presence of a family of problems in differential equations (DEs) of a high order, and which have real mathematical interest. There are various approximate (numerical) methods in the literature that have been used for the solution of boundary value problems (BVPs). The existence and uniqueness to finding the solution of higher order BVPs are systematically examined in [1]. The BVPs of higher order DEs have been examined due to their significance and the potential for applications in applied sciences. To find the analytical solutions of such BVPs analytically is very tough and are available in very few cases. Very few researchers have tried the numerical solution of 10^{th} order BVPs. Some of the approximate techniques have been established over the years to the numerical solution for these kinds of BVPs. In [2,3], the authors has solved 10^{th} and 12^{th} order BVPs using the Adomian decomposition method (ADM) involving Green's function. The homotopy perturbation approach was utilized in [4] to solve BVPs of 10^{th} order. When a uniform magnetic field is applied across the fluid in the direction of gravity, the instability sets now as ordinary convection and it is modeled by 10^{th} order BVPs as discussed in [5]. In [6], established approximate techniques for solving the 10^{th} order non-linear BVPs occurring in thermal instability.

Numerical methods for the solution of non-linear BVPs of order 2 m were found in [7]. An effective numerical procedure DTM for solving some linear and non-linear BVPs of 10^{th} order is discussed in [8]. In [9,10], the BVPs of 9^{th} and 10^{th} order are considered by adopting homotopy perturbation technique and the modified-variational iteration technique. Also the variational iterative technique was adopted in [11] for solving the 10^{th} order BVPs. Wazwaz [12–15] proposed modified form of ADM for solving 6^{th}, 8^{th}, 10^{th} and 12^{th} order.

The study of non-polynomial spline [16] of 11^{th} degree is a key element to solve 10^{th} order BVPs. In [17], it is depicted that the DEs that describe the 10^{th} order model to incorporate a 3rd order model of enlistment machine, two equations for dynamic power control, two equations for receptive power control, and three equations for edge pitch control. A 10^{th} order nonlinear dynamic model was developed in [18] to turn mobile robots that incorporate slip between the driven wheels and the ground. Based on binary six-point and eight-point approximating subdivision scheme, two collocation algorithms are constructed by [19,20] to find the solution of BVPs. The 4^{th} order linear BVPs using a new cubic B-spline were solved in [21]. Authors explained the 10^{th} and 12^{th} order BVPs by using the Galerkin weighted residual technique in [22]. The 5^{th}, 6^{th} and 8^{th} order linear and non-linear BVPs by using the cubic B-spline scheme (CBS) method were solved in [23–25]. The higher (10^{th} and 11^{th}) degree splines were tested in [26,27] for solving 10^{th} order BVPs. In [28] they practiced 2nd order finite difference schemes for the mathematical solutions of the 8^{th}, 10^{th} and 12^{th} order Eigen-value problems. Galerkin method with septic B-spline and quintic B-spline was adopted in [29,30] for solving 10^{th} order BVPs. Quintic B-spline and septic-B spline collocation methods was discussed in [31,32] to find solution of a 10^{th} order BVPs.

For discrete methods, e.g., Adomian decomposition, shooting, homotopy perturbation, finite differences and variational-iterative technique, only give discrete approximate values of the unknown $y(x)$. For fitting curve to data we require further data processing methods. To overcome these disadvantages, we introduced a new CBS scheme for the solution of 10^{th} order BVPs. The algorithm developed here is not only for the approximation solutions of the 10^{th} order boundary value problems(BVPs) employing CBS but also estimate derivatives of 1st order to 10^{th} order (where boundary conditions (BCs) are defined) of the exact solution as well.

The rest of the paper is organized as follows. The construction of CBS is presented in Section 2. In Section 3, the CBS scheme is utilized as an interpolating function in the solution of 10^{th} order nonlinear BVPs. The results and discussion are presented in Section 4. Also some problems are considered in this section to show the efficiency of the CBS scheme. Finally, the concluding remarks are given in the final section.

2. The Construction of CBS

In this section, we construct the CBS basis functions for solving numerically the non-linear equations arising in the study of hydrodynamics and hydro-magnetic stability problems. To find the approximate solution at nodal points defined in the region $[a, b]$. For an interval $\Omega = [a, b]$, we divide it into n sub-intervals $\Omega_i = [\kappa_i, \kappa_{i+1}];\ i = 0, 1, 2, ..., n-1$, by the equidistant knots. For this range, we select equidistant points such that

$$\Omega_\imath = \kappa_\imath = a + \imath h, \tag{1}$$

such that

$$\Omega = \{a = \kappa_0, ..., \kappa_n = b\}, \tag{2}$$

i.e., $\kappa_\imath = a + \imath h, (\imath = 0, ..., n)$ and $h = \frac{b-a}{n}$.

Assume $S_3(\Omega) = \{p(t) \in C^2[a,b]\}$ such that $p(t)$ converted to to cubic-polynomial on separately sub interval$(\kappa_\imath, \kappa_{\imath+1})$. The basis function is defined as

$$M_\imath(\kappa) = \frac{1}{6h^3} \begin{cases} (\kappa - \kappa_{\imath-2})^3, & \text{if } \kappa \in [\kappa_{\imath-2}, \kappa_{\imath-1}], \\ h^3 + 3h^2(\kappa - \kappa_{\imath-1}) + 3h(\kappa - \kappa_{\imath-1})^2 - 3 \\ (\kappa - \kappa_{\imath-1})^3, & \text{if } \kappa \in [\kappa_{\imath-1}, \kappa_\imath], \\ h^3 + 3h^2(\kappa_{\imath+1} - \kappa) + 3h(\kappa_{\imath+1} - \kappa)^2 - 3 \\ (\kappa_{\imath+1} - \kappa)^3, & \text{if } \kappa \in [\kappa_\imath, \kappa_{\imath+1}], \\ (\kappa_{\imath+2} - \kappa)^3, & \text{if } \kappa \in [\kappa_{\imath+1}, \kappa_{\imath+2}], \\ 0, & \text{otherwise}, \end{cases}$$

for $(\imath = 2,\ 3,\ 4, ..., n-2)$. Considering one and all $M_\imath(\kappa)$ is also a piece-wise cubic with knots at Ω, simultaneously $M_\imath(\kappa) \in S_3(\Omega)$.

Assume $\Psi = \{M_\imath\};\ (\imath = -1, 0, 1, 2 \ldots n, n+1)$ be linearly independent and let $M_3(\Omega) = span\Psi$. Thus $M_3(\Omega)$ is $(n+3)$ dimensional and $M_3(\Omega) = S_3(\Omega)$. Let $s(\kappa)$ be the cubic-B spline function interpolating at the nodal points and $s(\kappa) \in S_3(\Omega)$. Then $s(\kappa)$ can be written as

$$s(\kappa) = \sum_{\imath=-1}^{n+1} \jmath_\imath M_\imath(\kappa).$$

Consequently now for a function $w(\kappa)$, there happened to be a distinctive cubic-B spline $s(\kappa) = \sum_{\imath=-1}^{n+1} \jmath_\imath M_\imath(\kappa)$, satisfying the interpolating conditions:

$$w(\kappa_\imath) = s(\kappa_\imath) = \frac{\jmath_{\imath-1} + 4\jmath_\imath + \jmath_{\imath+1}}{6}, \tag{3}$$

for $\imath = 0, \ldots, n$.

The values of $M_\imath(\kappa)$, and its derivatives $M_\imath{}^{(1)}(\kappa)$, $M_\imath{}^{(2)}(\kappa)$ at nodal points are required and these derivatives are tabulated in Table 1.

Table 1. Values of $M_\imath(\kappa)$ and its derivatives.

	$M_\imath\ (\kappa)$	$M_\imath{}^{(1)}\ (\kappa)$	$M_\imath{}^{(2)}\ (\kappa)$
$\kappa_{\imath-2}, \kappa_{\imath+2}$	0	0	0
$\kappa_{\imath-1}$	1/6	$1/2h$	$1/h^2$
$\kappa_\imath$	4/6	0	$-2/h^2$
$\kappa_{\imath+1}$	1/6	$-1/2h$	$1/h^2$
otherwise	0	0	0

Assume $m_\imath = s^{(1)}(\kappa_\imath)$ and $\aleph_\imath = s^{(2)}(\kappa_\imath)$ then from

$$m_\imath = s^{(1)}(\kappa_\imath) = w^{(1)}(\kappa_\imath) - \frac{1}{180}h^4 w^{(5)}(\kappa_\imath) + O(h^6) \tag{4}$$

$$w^{(1)}(\kappa) = s^{(1)}(\kappa_\imath) = \frac{\jmath_{\imath+1} - \jmath_{\imath-1}}{2h} \tag{5}$$

$$\aleph_\imath = s^{(2)}(\kappa_\imath) = w^{(2)}(\kappa_\imath) - \frac{1}{12}h^2 w^{(4)}(\kappa_\imath) + \frac{1}{360}h^4 w^{(6)}(\kappa_\imath) + O(h^6) \tag{6}$$

$$w^{(2)}(\kappa) = s^{(2)}(\kappa_\imath) = \frac{\jmath_{\imath+1} - 2\jmath_\imath + \jmath_{\imath-1}}{h^2}, \quad (7)$$

$\aleph_\imath$ may be used to determine numerical-difference formulas for $w^{(3)}(\kappa_\imath), w^{(4)}(\kappa_\imath)$ such that ($\imath = 1$ *to* $n-1$), for $w^{(5)}(\kappa_\imath), w^{(6)}(\kappa_\imath)$ such that ($\imath = 2$ *to* $n-2$), for $w^{(7)}(\kappa_\imath)$, $w^{(8)}(\kappa_\imath)$ such that ($\imath = 3$ *to* $n-3$) and $w^{(9)}(\kappa_\imath)$, $w^{(10)}(\kappa_\imath)$ such that ($\imath = 4$ *to* $n-4$) like so the errors can be obtained by using Taylor-series

$$\begin{cases} \frac{\aleph_{\imath+1} - \aleph_{\imath-1}}{2h} = \frac{s^{(3)}(\kappa_{\imath-}) + s^{(3)}(\kappa_{\imath+})}{2} = w^{(3)}(\kappa_\imath) + \frac{1}{12}h^2 w^{(5)}(\kappa_\imath) + O(h^4); \\ w^{(3)}(\kappa) = s^{(3)}(\kappa_\imath) = \frac{\jmath_{\imath+2} - 2\jmath_{\imath+1} + 2\jmath_{\imath-1} - \jmath_{\imath-2}}{2h^3}, \\ \frac{\aleph_{\imath+1} - 2\aleph_\imath + \aleph_{\imath-1}}{h^2} = \frac{s^{(3)}(\kappa_{\imath-}) - s^{(3)}(\kappa_{\imath+})}{h} = w^{(4)}(\kappa_\imath) - \frac{1}{720}h^4 w^{(8)}(\kappa_\imath) + O(h^6); \\ w^{(4)}(\kappa) = s^{(4)}(\kappa_\imath) = \frac{\jmath_{\imath+2} - 4\jmath_{\imath+1} + 6\jmath_\imath - 4\jmath_{\imath-1} + \jmath_{\imath-2}}{h^4}, \\ \frac{\aleph_{\imath+2} - 2\aleph_{\imath+1} + 2\aleph_{\imath-1} - \aleph_{\imath-2}}{2h^3} = w^{(5)}(\kappa_\imath) + O(h^2); \\ w^{(5)}(\kappa) = s^{(5)}(\kappa_\imath) = \frac{\jmath_{\imath+3} - 4\jmath_{\imath+2} + 5\jmath_{\imath+1} + 5\jmath_{\imath-1} + 4\jmath_{\imath-2} - \jmath_{\imath-3}}{2h^5}. \end{cases} \quad (8)$$

Similarly (see [31]),

$$\begin{cases} w^{(6)}(\kappa_\imath) = s^{(6)}(\kappa_\imath) = \frac{\jmath_{\imath+3} - 6\jmath_{\imath+2} + 15\jmath_{\imath+1} - 20\jmath_\imath + 15\jmath_{\imath-1} - 6\jmath_{\imath-2} + \jmath_{\imath-3}}{h^6}, \\ w^{(7)}(\kappa_\imath) = s^{(7)}(\kappa_\imath) = \frac{\jmath_{\imath+4} - 6\jmath_{\imath+3} + 14\jmath_{\imath+2} - 14\jmath_{\imath+1} + 14\jmath_{\imath-1} - 14\jmath_{\imath-2} + 6\jmath_{\imath-3} - \jmath_{\imath-4}}{2h^7}, \\ w^{(8)}(\kappa_\imath) = s^{(8)}(\kappa_\imath) = \frac{1}{h^8}(\jmath_{\imath+4} - 8\jmath_{\imath+3} + 28\jmath_{\imath+2} - 56\jmath_{\imath+1} + 70\jmath_\imath - 56\jmath_{\imath-1} + 28\jmath_{\imath-2} - 8\jmath_{\imath-3} + \jmath_{\imath-4}), \\ w^{(9)}(\kappa_\imath) = s^{(9)}(\kappa_\imath) = \frac{1}{2h^9}(\jmath_{\imath+5} - 8\jmath_{\imath+4} + 27\jmath_{\imath+3} - 48\jmath_{\imath+2} + 42\jmath_{\imath+1} - 42\jmath_{\imath-1} + 48\jmath_{\imath-2} - 27\jmath_{\imath-3} + 8\jmath_{\imath-4} - \jmath_{\imath-5}). \end{cases} \quad (9)$$

3. The 10^{th} Order Nonlinear BVPs

In this section, we consider the 10^{th} order nonlinear BVPs arising in the study of hydrodynamics stability and visco-elastic flows.

$$w^{(10)}(\kappa) = f(\kappa, w(\kappa), w^{(1)}(\kappa), w^{(2)}(\kappa), w^{(3)}(\kappa), w^{(4)}(\kappa), w^{(5)}(\kappa), w^{(6)}(\kappa), w^{(7)}(\kappa), w^{(8)}(\kappa), w^{(9)}(\kappa)), \kappa \in [a, b], \quad (10)$$

with BCs

$$\begin{aligned} &w(a) = \lambda_0, && w^{(1)}(a) = \lambda_1, && w^{(2)}(a) = \lambda_2, \\ &w^{(3)}(a) = \lambda_3, && w^{(4)}(a) = \lambda_4, && w(b) = \chi_0, \\ &w^{(1)}(b) = \chi_1, && w^{(2)}(b) = \chi_2, && w^{(3)}(b) = \chi_3, \\ &w^{(4)}(b) = \chi_4, \end{aligned} \quad (11)$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4$ are given real constants, $(a_\imath(\kappa);\ \imath = 1, 2, ..., 10)$ and f is continuous in interval $[a, b]$.
The Taylor, series for $w^{(10)}(\kappa_\imath)$ at the preferred collocation points alongside central difference (see [31]), we have

$$w^{(10)}(\kappa_\imath) = \frac{1}{h^6}\Big(w_{\imath+3}{}^{(4)}(\kappa_\imath) - 6w_{\imath+2}{}^{(4)}(\kappa_\imath) + 15w_{\imath+1}{}^{(4)}(\kappa_\imath) - 20w_\imath{}^{(4)}(\kappa_\imath) + 15 w_{\imath-1}{}^{(4)}(\kappa_\imath) - 6w_{\imath-2}{}^{(4)}(\kappa_\imath) + w_{\imath-3}{}^{(4)}(\kappa_\imath)\Big). \quad (12)$$

Equation (9) can be written as

$$\begin{aligned}
\frac{\aleph_{\imath-2}-2\aleph_{\imath-3}+\aleph_{\imath-4}}{h^2}&=w^{(4)}(\kappa_{\imath-3})-\frac{1}{720}h^4w^{(8)}(\kappa_{\imath-3})+O(h^6),\\
\frac{\aleph_{\imath-1}-2\aleph_{\imath-2}+\aleph_{\imath-3}}{h^2}&=w^{(4)}(\kappa_{\imath-2})-\frac{1}{720}h^4w^{(8)}(\kappa_{\imath-2})+O(h^6),\\
\frac{\aleph_{\imath}-2\aleph_{\imath-1}+\aleph_{\imath-2}}{h^2}&=w^{(4)}(\kappa_{\imath-1})-\frac{1}{720}h^4w^{(8)}(\kappa_{\imath-1})+O(h^6),\\
\frac{\aleph_{\imath+2}-2\aleph_{\imath+1}+\aleph_{\imath}}{h^2}&=w^{(4)}(\kappa_{\imath+1})-\frac{1}{720}h^4w^{(8)}(\kappa_{\imath+1})+O(h^6),\\
\frac{\aleph_{\imath+3}-2\aleph_{\imath+2}+\aleph_{\imath+1}}{h^2}&=w^{(4)}(\kappa_{\imath+2})-\frac{1}{720}h^4w^{(8)}(\kappa_{\imath+2})+O(h^6),\\
\frac{\aleph_{\imath+4}-2\aleph_{\imath+3}+\aleph_{\imath+2}}{h^2}&=w^{(4)}(\kappa_{\imath+3})-\frac{1}{720}h^4w^{(8)}(\kappa_{\imath+3})+O(h^6).
\end{aligned} \tag{13}$$

Substituting Equation (13) into Equation (12), we obtain

$$\begin{aligned}
&\frac{1}{h^8}(\aleph_{\imath+4}-8\aleph_{\imath+3}+28\aleph_{\imath+2}-56\aleph_{\imath+1}+70\aleph_{\imath}-56\aleph_{\imath-1}+28\aleph_{\imath-2}-8\aleph_{\imath-3}+\aleph_{\imath-4})\\
&=w^{(10)}(\kappa_\imath)+O(h^2).
\end{aligned} \tag{14}$$

Since $\aleph_\imath=\frac{\jmath_{\imath+1}-2\jmath_\imath+\jmath_{\imath-1}}{h^2}$ so, Equation (14) becomes

$$\begin{aligned}
w^{(10)}(\kappa_\imath)=&\frac{1}{h^8}\Big(\frac{\jmath_{\imath+5}-2\jmath_{\imath+4}+\jmath_{\imath+3}}{h^2}-8(\frac{\jmath_{\imath+4}-2\jmath_{\imath+3}+\jmath_{\imath+2}}{h^2})+28(\frac{\jmath_{\imath+3}-2\jmath_{\imath+2}+\jmath_{\imath+1}}{h^2})-56(\frac{\jmath_{\imath+2}-2\jmath_{\imath+1}+\jmath_{\imath}}{h^2})\\
&+70(\frac{\jmath_{\imath+1}-2\jmath_{\imath}+\jmath_{\imath-1}}{h^2})-56(\frac{\jmath_{\imath}-2\jmath_{\imath-1}+\jmath_{\imath-2}}{h^2})+28(\frac{\jmath_{\imath-1}-2\jmath_{\imath-2}+\jmath_{\imath-3}}{h^2})-8(\frac{\jmath_{\imath-2}-2\jmath_{\imath-3}+\jmath_{\imath-4}}{h^2})+\\
&\frac{\jmath_{\imath-3}-2\jmath_{\imath-4}+\jmath_{\imath-5}}{h^2}\Big).
\end{aligned} \tag{15}$$

After some simplifications the above equation becomes

$$\begin{aligned}
&w^{(10)}(\kappa_\imath)=s^{(10)}(\kappa_\imath)=\frac{1}{h^{10}}\Big(\jmath_{\imath+5}-10\jmath_{\imath+4}+45\jmath_{\imath+3}-120\jmath_{\imath+2}+210\jmath_{\imath+1}-252\jmath_\imath\\
&+210\jmath_{\imath-1}-120\jmath_{\imath-2}+45\jmath_{\imath-3}-10\,\jmath_{\imath-4}+\jmath_{\imath-5}\Big).
\end{aligned} \tag{16}$$

Let $w(\kappa_\imath)=s(\kappa_\imath)=\sum_{\imath=-1}^{n+1}\jmath_\imath M_\imath(\kappa_\imath)$ be the accurate solution of non-linear 10^{th} order BVPs

$$\begin{aligned}
&w^{(10)}(\kappa_\imath)=f(\kappa_\imath,w(\kappa_\imath),w^{(1)}(\kappa_\imath),w^{(2)}(\kappa_\imath),w^{(3)}(\kappa_\imath),w^{(4)}(\kappa_\imath),\\
&w^{(5)}(\kappa_\imath),w^{(6)}(\kappa_\imath),w^{(7)}(\kappa_\imath),w^{(8)}(\kappa_\imath),w^{(9)}(\kappa_\imath)),\kappa_\imath\in[a,b].
\end{aligned} \tag{17}$$

Imposing Equations (3), (5), (7), (8) and (9) into Equation (17), we have

$$\begin{aligned}
&\frac{1}{h^{10}}\ (\jmath_{\imath+5}-10\jmath_{\imath+4}+45\jmath_{\imath+3}-120\jmath_{\imath+2}+210\jmath_{\imath+1}-252\jmath_\imath+210\jmath_{\imath-1}-120\jmath_{\imath-2}+45\jmath_{\imath-3}\\
&-10\,\jmath_{\imath-4}+\jmath_{\imath-5})=f_\imath\Big(\kappa_\imath,\frac{1}{6}(\jmath_{\imath-1}+4\jmath_\imath+\jmath_{\imath+1}),\frac{1}{2h}(\jmath_{\imath+1}-\jmath_{\imath-1}),\frac{1}{h^2}(\jmath_{\imath+1}-2\jmath_\imath+\jmath_{\imath-1}),\\
&\frac{1}{2h^3}(\jmath_{\imath+2}-2\jmath_{\imath+1}+2\jmath_{\imath-1}-\jmath_{\imath-2}),\frac{1}{h^4}(\jmath_{\imath+2}-4\jmath_{\imath+1}+6\jmath_\imath-4\jmath_{\imath-1}+\jmath_{\imath-2}),\frac{1}{2h^5}(\jmath_{\imath+3}\\
&-4\jmath_{\imath+2}+5\jmath_{\imath+1}+5\jmath_{\imath-1}+4\jmath_{\imath-2}-\jmath_{\imath-3}),\frac{1}{h^6}(\jmath_{\imath+3}-6\jmath_{\imath+2}+15\jmath_{\imath+1}-20\jmath_\imath+15\jmath_{\imath-1}\\
&-6\jmath_{\imath-2}+\jmath_{\imath-3}),\frac{1}{2h^7}(\jmath_{\imath+4}-6\jmath_{\imath+3}+14\jmath_{\imath+2}-14\jmath_{\imath+1}+14\jmath_{\imath-1}-14\jmath_{\imath-2}+6\jmath_{\imath-3}-\jmath_{\imath-4}),\\
&\frac{1}{h^8}(\jmath_{\imath+4}-8\jmath_{\imath+3}+28\jmath_{\imath+2}-56\jmath_{\imath+1}+70\jmath_\imath-56\jmath_{\imath-1}+28\jmath_{\imath-2}-8\jmath_{\imath-3}+\jmath_{\imath-4}),\frac{1}{2h^9}(\jmath_{\imath+5}-8\jmath_{\imath+4}\\
&+27\jmath_{\imath+3}-48\jmath_{\imath+2}+42\jmath_{\imath+1}-42\jmath_{\imath-1}+48\jmath_{\imath-2}-27\jmath_{\imath-3}+8\,\jmath_{\imath-4}-\jmath_{\imath-5})\Big),\kappa\in[a,b].
\end{aligned} \tag{18}$$

Equation (18) we will produce a new system consisting of $(n-7)$ linear equations $(\imath = 4, 5, ..., n-4)$ with $(n+3)$ unknowns $\jmath_\imath$ where $(\imath = -1, 0, ..., n+1)$, therefore ten further equations are required. From given BCs at $\kappa = a$, we have five equations:

$$\begin{aligned} w(a) = \lambda_0 \;&\Rightarrow\; \jmath_{-1} + 4\jmath_0 + \jmath_1 = 6\lambda_0 \\ w^{(1)}(a) = \lambda_1 \;&\Rightarrow\; -\jmath_{-1} + \jmath_1 = 2\lambda_1 h \\ w^{(2)}(a) = \lambda_2 \;&\Rightarrow\; \jmath_{-1} - 2\jmath_0 + \jmath_1 = \lambda_2 h^2 \\ w^{(3)}(a) = \lambda_3 \;&\Rightarrow\; \jmath_2 - 2\jmath_1 + 2\jmath_{-1} - \jmath_{-2} = 2\lambda_3 h^3 \\ w^{(4)}(a) = \lambda_4 \;&\Rightarrow\; \jmath_2 - 4\jmath_1 + 6\jmath_0 - 4\jmath_{-1} + \jmath_{-2} = \lambda_4 h^4, \end{aligned} \tag{19}$$

similarly from $\kappa = b$ there will be other five equations

$$\begin{aligned} w(b) = \chi_0 \;&\Rightarrow\; \jmath_{n-1} + 4\jmath_n + \jmath_{n+1} = 6\chi_0 \\ w^{(1)}(b) = \chi_1 \;&\Rightarrow\; -\jmath_{n-1} + \jmath_{n+1} = 2\chi_1 h \\ w^{(2)}(b) = \chi_2 \;&\Rightarrow\; \jmath_{n-1} - 2\jmath_n + \jmath_{n+1} = \chi_2 h^2 \\ w^{(3)}(b) = \chi_3 \;&\Rightarrow\; \jmath_{n+2} - 2\jmath_{n+1} + 2\jmath_{n-1} - \jmath_{n-2} = 2\chi_3 h^3 \\ w^{(4)}(b) = \chi_4 \;&\Rightarrow\; \jmath_{n+2} - 4\jmath_{n+1} + 6\jmath_n - 4\jmath_{n-1} + \jmath_{n-2} = \chi_4 h^4. \end{aligned} \tag{20}$$

Omitting the order of the error of terms, the exact solution $w(\kappa_\imath) = s(\kappa_\imath) = \sum_{\imath=-1}^{n+1} \jmath_\imath M_\imath(\kappa_\imath)$ is accomplished by finding solution of the discussed above linear system of $(n+3)$ equations in $(n+3)$ unknowns considering the Equations (18)–(20).

4. Convergence Analysis

Let $\widehat{w}(\kappa)$ be the exact solution of the Equations (10)–(12) and also $\widehat{s}(\kappa)$ be the CBS approximation to $\widehat{w}(\kappa)$. Therefore, we have

$$\widehat{w}(\kappa_\imath) = \widehat{s}(\kappa_\imath) = \sum_{\imath=-1}^{n+1} \widehat{\jmath}_\imath M_\imath(\kappa_\imath), \tag{21}$$

where

$$\widehat{\jmath} = \widehat{\jmath}_{imath} = \left[\widehat{\jmath}_{-1}, \; \widehat{\jmath}_0, \; \widehat{\jmath}_1, \; ..., \widehat{\jmath}_{n+1}\right]^T.$$

Also, we have assume that $s'(\kappa)$ be the computed cubic B spline approximation to $\widehat{s}(\kappa)$, namely

$$w'(\kappa_\imath) = s'(\kappa_\imath) = \sum_{i=-1}^{n+1} \jmath'_i M_\imath(\kappa_\imath),$$

$$\jmath' = \jmath'_i = \left[\jmath'_{-1}, \; \jmath'_0, \; \jmath'_1, \; ..., \jmath'_{n+1}\right]^T.$$

To approximate the error $\|\widehat{w}(\kappa_\imath)) - \widehat{s}(\kappa_\imath))\|_\infty$ we have to estimate error $\|\widehat{w}(\kappa_\imath)) - s'(\kappa_\imath))\|_\infty$ and $\|w'(\kappa_\imath)) - \widehat{s}(\kappa_\imath))\|_\infty$ seperately

The system of $(n+3) \times (n+3)$ matrix can be written as:

$$B\jmath = G.$$

Then, we have

$$B\widehat{\jmath} = \widehat{G} \tag{22}$$

and

$$B\jmath' = G'. \tag{23}$$

Now, by subtracting Equations (22) and (23), we obtain

$$B(\jmath' - \hat{\jmath}) \ = \ G' - \hat{G},$$

where B is an $(n+3) \times (n+3)$-dimensional band matrix,
and

$$G \ = \ \left[G_{-1}, \ \ G_0, \ \ G_1, ..., G_{n+1}\right]^T,$$

where T denoting transpose.

We can write

$$(\jmath' - \hat{\jmath}) \ = \ B^{-1}(G' - \hat{G}). \tag{24}$$

Taking the infinity norm from Equation (24), we obtain

$$\|(\jmath' - \hat{\jmath})\|_\infty \ = \ \|B^{-1}\|_\infty \|G' - \hat{G}\|_\infty.$$

The B-spline $M \ = \ M_\imath \ = \ \{M_{-1}, \ \ M_0, \ \ M_1, \ \ ..., M_{n+1}\}$ satisfy the following property

$$\Big| \sum_{i=-1}^{n+1} \jmath_i' M_\imath(\kappa_\imath) \Big| \le 1.$$

Using [24]

$$\|B^{-1}\|_\infty \|G' - \hat{G}\|_\infty \le \flat h^2.$$

$$\|(\jmath' - \hat{\jmath})\|_\infty \le \flat h^2. \tag{25}$$

$$s'(\kappa_\imath)) - \hat{s}(\kappa_\imath) \ = \ (\jmath' - \hat{\jmath}) \sum_{i=-1}^{n+1} M_\imath(\kappa_\imath).$$

$$\|s'(\kappa_\imath) - \hat{s}(\kappa_\imath)\|_\infty \ = \ \|(\jmath' - \hat{\jmath}) \sum_{i=-1}^{n+1} M_\imath(\kappa_\imath)\|_\infty.$$

$$\|s'(\kappa_\imath) - \hat{s}(\kappa_\imath)\|_\infty \le \|(\jmath' - \hat{\jmath})\|_\infty \Big| \sum_{i=-1}^{n+1} M_\imath(\kappa_\imath)\Big| \le \flat h^2. \tag{26}$$

$$\|\hat{w}(\kappa_\imath) - s'(\kappa_\imath)\|_\infty \le \rho h^4. \tag{27}$$

$$\|\hat{w}(\kappa_\imath) - \hat{s}(\kappa_\imath)\|_\infty \le \|\hat{w}(\kappa_\imath) - s'(\kappa_\imath)\|_\infty \ + \ \|s'(\kappa_\imath) - \hat{s}(\kappa_\imath)\|_\infty. \tag{28}$$

Using Equations (26) and (27) in Equation (28)

$$\|\hat{w}(\kappa_\imath) - \hat{s}(\kappa_\imath)\|_\infty \le \flat h^2 \ + \ \rho h^4 \ = \ \ell h^2.$$

which proves that this method is second order convergent and $\|\hat{w}(\kappa) - \hat{s}(\kappa)\|_\infty \le \ell h^2$.

5. Results and Discussions

To test the accuracy of CBS method, three problems are discussed and compared with the existing methods in this section.

5.1. Problem 1

We consider the following DEs arising in viscoelastic flows and hydrodynamic stability problems as given in [29,31]

$$w^{(10)}(\kappa) = \frac{14175}{4}(\jmath + w(\kappa) + 1)^{11}; \ 0 \le \kappa \le 1;$$

subject to BCs;

$$w(0) = w(1) = 0, \qquad w^{(1)}(0) = -\frac{1}{2} = -w^{(2)}(0), \qquad w^{(1)}(1) = 1,$$

$$w^{(2)}(1) = 4, \quad w^{(3)}(0) = \frac{3}{4}, \quad w^{(3)}(1) = 12, \quad w^{(4)}(0) = \frac{3}{2}, \quad w^{(4)}(1) = 48.$$

the exact solution of given equation is $w(\kappa) = \frac{2}{2-\kappa} - \kappa - 1$. The values of fifteen unknowns $\jmath_i$ from the Equations (18)–(20) are

$\jmath_{-2} = 0.10849167$, $\jmath_3 = -0.12456626$, $\jmath_8 = -0.13626667$,
$\jmath_{-1} = 0.05166667$, $\jmath_4 = -0.15061957$, $\jmath_9 = $ -0.08666667,
$\jmath_0 = -0.00083333$, $\jmath_5 = -0.16684713$, $\jmath_{10} = -0.0066667$,
$\jmath_1 = -0.04833333$, $\jmath_6 = -0.17169449$, $\jmath_{11} = 0.11333333$,
$\jmath_2 = -0.09000833$, $\jmath_7 = -0.16277005$, $\jmath_{12} = 0.28773333$.

Tables 2 and 3 analyzed the exact solution and cubic B-spline scheme (CBS) solution of problem 1 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$ respectively. Figures 1–3 analyze the exact solution with cubic B-spline scheme (CBS) solution of problem 1 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$ graphically. Table 4 analyze the errors at those derivatives where boundary conditions (BCs) are defined in problem 1 at $h = \frac{1}{10}$.

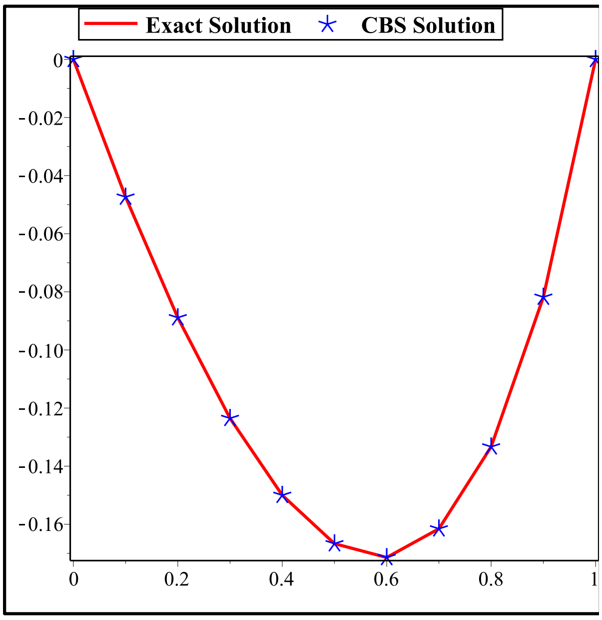

Figure 1. Problem 1 at $h = \frac{1}{10}$.

Table 2. Analyzing exact solution and cubic B-spline scheme (CBS) solution of problem 1 at $h = \frac{1}{10}$.

κ	Exact Solution	CBS Solution	Absolute Error
0	0	0	0×10^{0}
0.1	−0.0473684	−0.0473665	1.900×10^{-06}
0.2	−0.0888889	−0.0888822	6.670×10^{-05}
0.3	−0.1235294	−0.1235488	3.810×10^{-05}
0.4	−0.1500000	−0.1509819	1.020×10^{-04}
0.5	−0.1666667	−0.1669504	1.720×10^{-04}
0.6	−0.1714286	−0.1714992	2.030×10^{-05}
0.7	−0.1615385	−0.1615302	1.700×10^{-06}
0.8	−0.1333333	−0.1333172	9.160×10^{-05}
0.9	−0.0818182	−0.0818000	2.180×10^{-05}
1	0	0	0×10^{0}

Table 3. Analyzing exact solution and CBS solution of problem 1 at $h = \frac{1}{5}$.

κ	Exact Solution	CBS Solution	Absolute Error of CBS
0	0	0	0×10^{0}
0.2	−0.0888889	−0.0888000	8.890×10^{-05}
0.4	−0.1500000	−0.1500222	2.980×10^{-05}
0.6	−0.1714286	−0.1714778	1.150×10^{-05}
0.8	−0.1333333	−0.1333000	3.730×10^{-05}
1	0	0	0×10^{0}

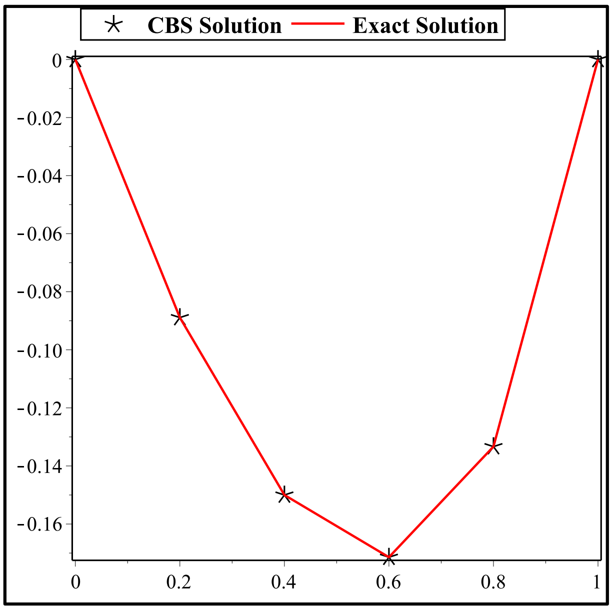

Figure 2. Problem 1 at $h = \frac{1}{5}$.

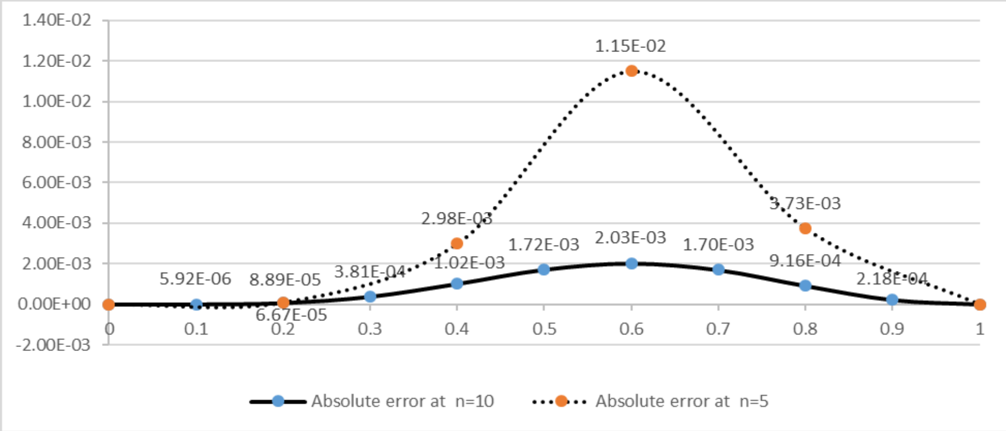

Figure 3. Problem 1 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$.

Table 4. Errors at derivatives where boundary conditions (BCs) are defined in problem 1 at $h = \frac{1}{10}$.

κ	CBS-Solution of $w^{(1)}(\kappa)$	CBS-Solution of $w^{(2)}(\kappa)$	CBS-Solution of $w^{(3)}(\kappa)$	CBS-Solution of $w^{(4)}(\kappa)$
0	−0.5	0.5	0.75	1.5
0.1	−0.4459	0.5825	1.0585	1.93853
0.2	−0.3812	0.7117	1.3398	2.54026
0.3	−0.3031	0.8505	1.3543	3.38062
0.4	−0.2114	0.9826	1.4378	4.57764
0.5	−0.1054	1.1380	1.9730	6.32099
0.6	0.0204	1.3772	3.0994	8.92485
0.7	0.1771	1.7579	4.6624	12.92780
0.8	0.3805	2.3097	6.4105	19.29012
0.9	0.6480	3.0400	8.4517	29.80422
1	1	4	12	48

5.2. Problem 2

We consider the following problem as given in [16]

$$w^{(10)}(\kappa) = 9!(e^{-10w(\kappa)} - \frac{2}{(1+\kappa)^{10}});\ 0 \le \kappa \le e^{1/2-1}$$

subject to BCs;

$$w(0) = 0, \qquad w\left(e^{1/2-1}\right) = \frac{1}{2}, \qquad w^{(1)}(0) = -w^{(2)}(0) = 1, \qquad w^{(1)}\left(e^{1/2-1}\right) = e^{\left(\frac{-1}{2}\right)},$$

$$w^{(2)}\left(e^{1/2-1}\right) = -e^{(-1)}, \qquad w^{(3)}(0) = 2, \qquad w^{(3)}\left(e^{1/2-1}\right) = 2e^{\left(-\frac{3}{2}\right)},$$

$$w^{(4)}(0) = -6, \qquad w^{(4)}\left(e^{1/2-1}\right) = -6e^{(-2)},$$

the exact solution of a given equation is $w(\kappa) = \ln(1+\kappa)$ where the domain $[0, e^{1/2-1}]$ for $h = 2^{-i}e^{1/2-1}$.

The values of fifteen unknowns $\jmath_i$ from Equations (18)–(20) are

$\jmath_{-2} = -0.13805879,$ $\jmath_3 = 0.1782805\,,$ $\jmath_8 = 0.4183388\,,$
$\jmath_{-1} = -0.0662749\,,$ $\jmath_4 = 0.2311044\,,$ $\jmath_9 = 0.4601370\,,$
$\jmath_0 = 0.0007014\,,$ $\jmath_5 = 0.2812925\,,$ $\jmath_{10} = 0.5002580\,,$
$\jmath_1 = 0.0634693\,,$ $\jmath_6 = 0.3290924\,,$ $\jmath_{11} = 0.5388309\,,$
$\jmath_2 = 0.1225218\,,$ $\jmath_7 = 0.3747130\,,$ $\jmath_{12} = 0.57597018\,.$

Tables 5 and 6 analyzed the exact solution and cubic B-spline scheme (CBS) solution of problem 2 at $h = 0.064872$ and $h = 0.12974426$ respectively. Figures 4–6 analyze the exact solution with cubic B-spline scheme (CBS) solution of problem 2 at $h = 0.064872$ and $h = 0.12974426$ graphically. Table 7 analyze the errors at those derivatives where boundary conditions (BCs) are defined in problem 2 at $h = 0.064872$.

Table 5. Analyzing exact solution and CBS-solution of problem 2 at $h = 0.064872$.

κ	**Exact Solution**	**CBS Solution**	**Absolute Error of CBS**
0	0	0	0×10^{0}
0.065	0.06285473	0.0628501	4.650×10^{-06}
0.13	0.12199129	0.1219728	1.850×10^{-05}
0.195	0.17782512	0.1777914	3.370×10^{-05}
0.259	0.23070570	0.2306651	4.060×10^{-05}
0.324	0.28092982	0.2808945	3.530×10^{-05}
0.389	0.32875164	0.3287292	2.250×10^{-05}
0.454	0.37439053	0.3743805	1.000×10^{-05}
0.519	0.41803711	0.4180342	2.920×10^{-06}
0.584	0.45985807	0.4598575	5.980×10^{-07}
0.648	0.5	0.5	0×10^{0}

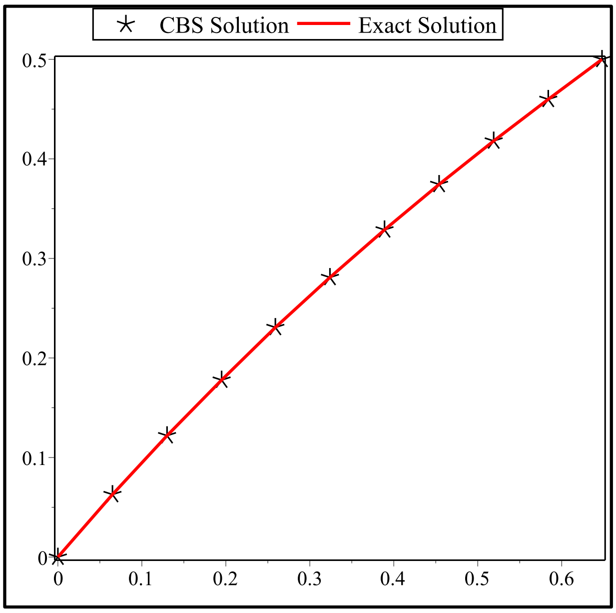

Figure 4. Problem 1 at $h = 0.064872$.

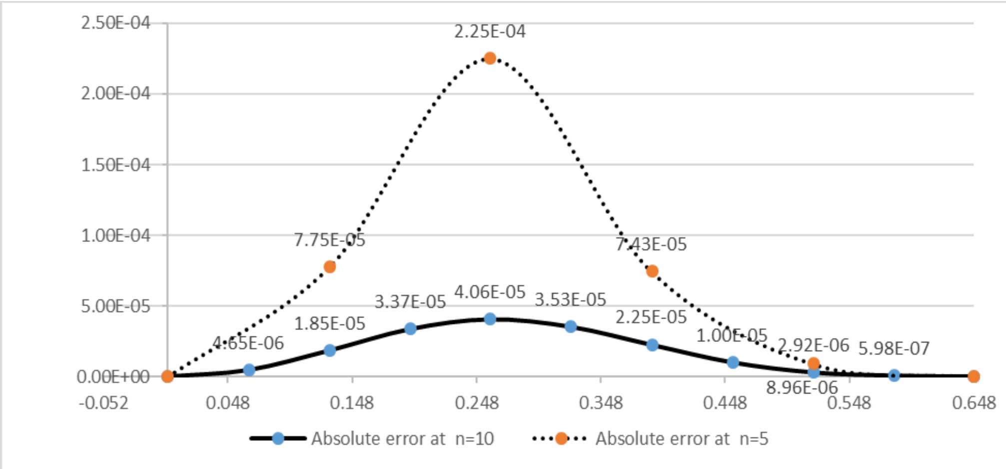

Figure 5. Problem 2 at $h = 0.064872$ and $h = 0.12974426$.

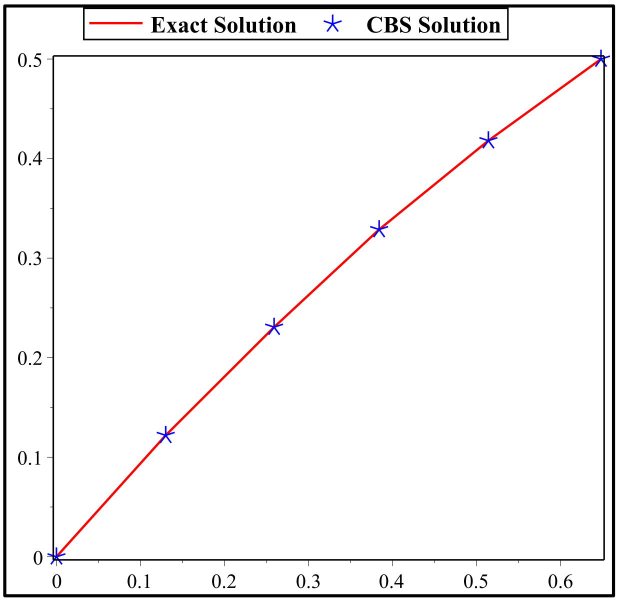

Figure 6. Problem 1 at $h = 0.12974426$.

Table 6. Analyzing the exact solution and CBS solution of problem 2 at $h = 0.12974426$.

κ	Exact Solution	CBS Solution	Absolute Error of CBS
0	0	0	0×10^{0}
0.130	0.1219912	0.1219138	7.750×10^{-05}
0.259	0.2307057	0.2304804	2.250×10^{-04}
0.389	0.3287516	0.3286773	7.430×10^{-05}
0.519	0.41803711	0.4180281	8.960×10^{-06}
0.649	0.5	0.5	0×10^{0}

Table 7. Errors at derivatives where BCs are defined in problem 2 at $h = 0.064872$.

κ	CBS Solution of $w^{(1)}(\kappa)$	CBS Solution of $w^{(2)}(\kappa)$	CBS Solution of $w^{(3)}(\kappa)$	CBS Solution of $w^{(4)}(\kappa)$
0	1	−1	2	−6
0.065	0.93893	−0.88288	1.67530	−4.66533031
0.13	0.88490	−0.78264	1.42971	−3.68168892
0.195	0.83690	−0.69738	1.20485	−2.94393959
0.259	0.79396	−0.62632	1.00104	−2.38195984
0.324	0.75524	−0.56751	0.83630	−1.94791017
0.389	0.72004	−0.51781	0.72045	−1.60848492
0.454	0.68786	−0.47403	0.64400	−1.34006674
0.519	0.65840	−0.43426	0.58187	−1.12562522
0.584	0.63139	−0.39854	0.51160	−0.95268994
0.648	0.60653	−0.36788	0.44626	−0.81200035

5.3. Problem 3

We consider the following equation as given in [29,33]

$$w^{(10)}(\kappa) + e^{-\kappa}(w(\kappa))^2 = e^{-3\kappa} + e^{-\kappa};\ 0 \leq z' \leq 1$$

subject to BCs;

$$w(0) = w^{(2)}(0) = w^{(4)}(0) = -w^{(1)}(0) = -w^{(3)}(0) = 1,$$

$$w(0) = w^{(2)}(0) = w^{(4)}(0) = -w^{(1)}(0) = -w^{(3)}(0) = e^{-1}$$

the exact solution of given equation is $w(\kappa) = e^{-\kappa}$. The values of fifteen unknowns J_i the Equations (18)–(20) are

$J_{-2} = 1.21938333$, $J_{-1} = 1.10333333$, $J_0 = 0.99833333$, $J_1 = -0.9033333$, $J_2 = -0.5333053$, $J_3 = -0.73961579$, $J_4 = -0.66924328$, $J_5 = 0.605557470$, $J_6 = 0.547923909$, $J_7 = 0.495772367$, $J_8 = -0.44858605$, $J_9 = 0.405893650$, $J_{10} = 0.36726630$, $J_{11} = 0.33231776$, $J_{12} = 0.30069852$.

Tables 8 and 9 analyzed the exact solution and cubic B-spline scheme (CBS) solution of problem 3 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$ respectively. Figures 7–9 analyze the exact solution with cubic B-spline scheme (CBS) solution of problem 3 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$ graphically. Table 10 analyze the errors at those derivatives where boundary conditions (BCs) are defined in problem 3 at $h = \frac{1}{10}$.

Table 8. Analyzing exact solution and CBS solution of problem 3 at $h = \frac{1}{10}$.

κ	Exact Solution	CBS Solution	Absolute Error of CBS
0	1	1	0
0.1	0.9048374	0.9048417	4.250×10^{-06}
0.2	0.8187308	0.8187471	1.630×10^{-05}
0.3	0.7408182	0.7408483	3.010×10^{-05}
0.4	0.6703200	0.6703577	3.770×10^{-05}
0.5	0.6065307	0.6065662	3.550×10^{-05}
0.6	0.5488116	0.5488376	2.590×10^{-05}
0.7	0.4965853	0.4965999	1.460×10^{-05}
0.8	0.4493290	0.4493350	6.080×10^{-06}
0.9	0.4065697	0.4065712	1.500×10^{-06}
1	0.3678794	0.3678794	0

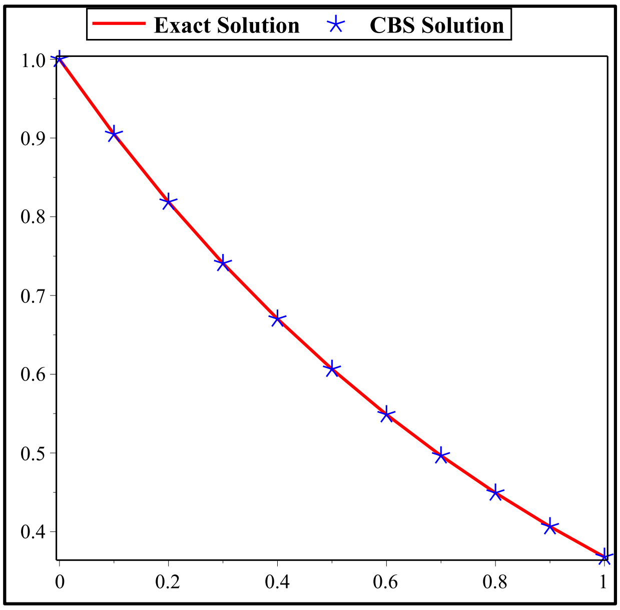

Figure 7. Problem 1 at $h = \frac{1}{10}$.

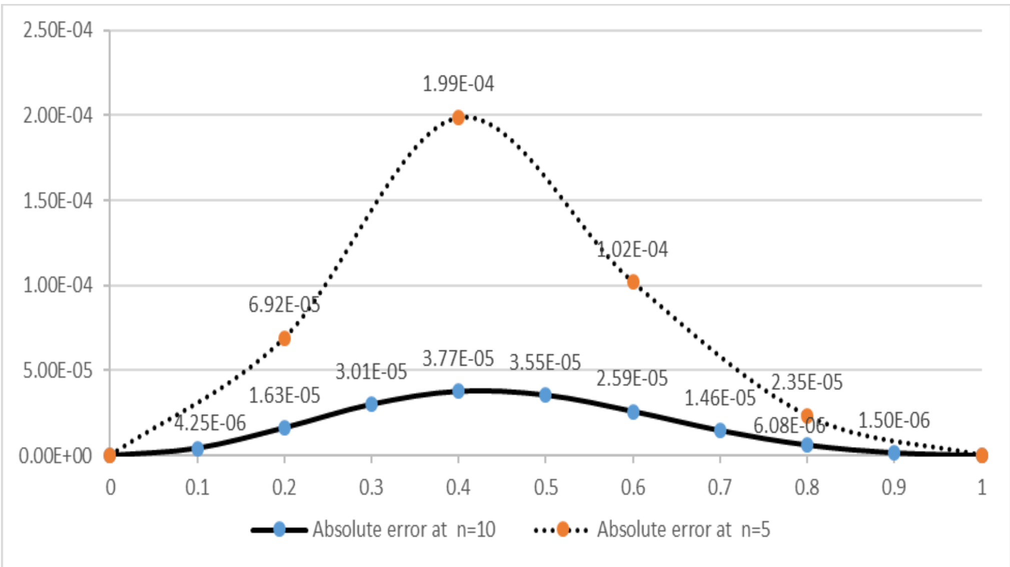

Figure 8. Problem 3 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$.

Table 9. Analyzing exact solution and CBS solution of problem 3 at $h = \frac{1}{5}$.

κ	**Exact Solution**	**CBS**	**Absolute Error of CBS**
0	1	1	0×10^{0}
0.2	0.8187308	0.8188000	6.920×10^{-05}
0.4	0.6703200	0.6705188	1.990×10^{-04}
0.6	0.5488116	0.5489132	1.020×10^{-04}
0.8	0.4493290	0.4493525	2.350×10^{-05}
1	0.3678794	0.3678794	0×10^{0}

Table 10. Errors at derivatives where BCs are defined in problem 3 at $h = \frac{1}{10}$.

κ	CBS Solution of $w^{(2)}(\kappa)$, $w^{(4)}(\kappa)$	CBS Solution of $w^{(1)}(\kappa)$, $w^{(3)}(\kappa)$
0	1	−1
0.1	0.90482409	−0.90484731
0.2	0.81870008	−0.81872330
0.3	0.74077662	−0.74079984
0.4	0.67027318	−0.67029640
0.5	0.60648354	−0.60650676
0.6	0.54876876	−0.54879198
0.7	0.49655070	−0.49657392
0.8	0.44930633	−0.44932955
0.9	0.40656241	−0.40658563
1	0.36787944	−0.36787944

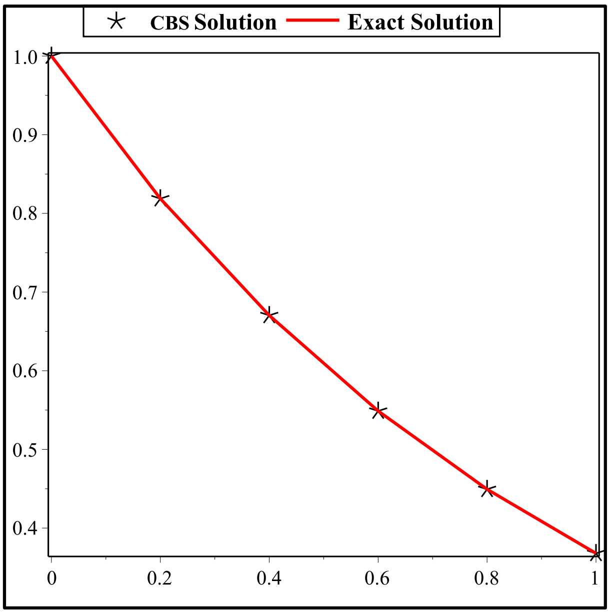

Figure 9. Problem 1 at $h = \frac{1}{10}$ and $h = \frac{1}{5}$.

6. Conclusions

In this study, we present new scheme using CBS of some non-linear differential equations arising in visco-elastic flows and hydrodynamic stability problems. The proper selection for the choice of the scheme and an appropriate of adjustment BCs may cause elasticity for the betterment of the results. The new CBS scheme proposed in this study is very simple to apply in solving the non-linear DEs compared with some existing schemes. An advantage of using the CBS scheme is that it gives a spline function on each new time line which can be applied to achieve the numerical solutions at any stage in the space direction.

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