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Approximations of Fixed Points in the Hadamard Metric Space $CAT_p(0)$

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Abstract: In this paper, we consider the recently introduced $CAT_p(0)$, where the comparison triangles belong to ℓ_p, for $p \geq 2$. We first establish an inequality in these nonlinear metric spaces. Then, we use it to prove the existence of fixed points of asymptotically nonexpansive mappings defined in $CAT_p(0)$. Moreover, we discuss the behavior of the successive iteration introduced by Schu for these mappings in Banach spaces. In particular, we prove that the successive iteration generates an approximate fixed point sequence.

Keywords: fixed point; generalized $CAT(0)$ spaces; Hadamard metric spaces; hyperbolic metric spaces; Lipschitzian mapping; modified Mann iteration

MSC: primary 47H09; 47H10

1. Introduction

Hadamard metric spaces, also known as complete $CAT(0)$ metric spaces, play an important role when dealing with the geometry of Bruhat–Tits building, metric trees, Hadamard manifolds, or simply connected nonpositively curved symmetric spaces. In fact, the power of Hadamard spaces goes beyond geometry. For example, $CAT(0)$ geometry was used to solve an interesting problem in Dynamical billiards [1]. In a very simplistic way, Hadamard metric spaces are the nonlinear version of Hilbert vector spaces.

A metric Hadamard space (M, d) is characterized by the inequality (1) [2,3], known as the inequality of Bruhat and Tits, i.e., for any $a, b, x \in M$, there exists $c \in M$ such that

$$4\, d(x,c)^2 + d(a,b)^2 \leq 2\, d(x,a)^2 + 2\, d(x,b)^2. \tag{1}$$

It is easy to check that c is a metric midpoint of a and b, i.e., $d(c,a) = d(c,b) = d(a,b)/2$. Note that in the linear Hilbert spaces, the inequality (1) becomes an equality. A Hadamard space for which the inequality (1) is an equality are known as flat Hadamard spaces. They are isomorphic to closed convex subsets of Hilbert spaces.

To understand this inequality, one has to look at the formal definition of a $CAT(0)$ space with the comparison triangles taken in the Euclidean plane $\mathbb{R}^2$. In this case, the triangles in M are kind of slimmer than the comparison triangles in $\mathbb{R}^2$. This fundamental property motivated the authors of [4] to consider metric spaces for which the corresponding triangles are taken in general Banach spaces. The most natural example is to take these comparison triangles in ℓ_p, for $p > 1$. The authors of [4] called these more general metric spaces "generalized $CAT_p(0)$". In this work, we continue investigating the properties of these metric spaces and establish some existence fixed point results and their approximations.

For readers interested in metric fixed point theory, we recommend the book [5]. For more on geodesic metric spaces, we recommend the excellent book [2].

2. Basic Definitions and Preliminaries

Most of the terminology of geodesic metric spaces is taken from the work in [2]. Consider a metric space (M,d). A geodesic function $\zeta : [0,1] \to M$ is any function that satisfies $d(\zeta(\alpha),\zeta(\beta)) = |\alpha-\beta| d(\zeta(0),\zeta(1))$, for every $\alpha,\beta \in [0,1]$. (M,d) is said to be a geodesic space if, for every two points $a,b \in M$, there exists a function $\zeta : [0,1] \to M$ such that $\zeta(0) = a$ and $\zeta(1) = b$ and is geodesic. Throughout, we will use the notation $\zeta(\alpha) = (1-\alpha)\, a \oplus \alpha\, b$, for $\alpha \in (0,1)$. (M,d) is said to be uniquely geodesic if any two points in M are connected by a unique geodesic. In this case, the range of the unique geodesic function connecting a and b will be denoted by $[a,b]$, i.e., $[a,b] = \{(1-\alpha)\, a \oplus \alpha\, b;\ \alpha \in [0,1]\}$.

Normed vector spaces are natural examples of geodesic metric spaces. Complete Riemannian manifolds, and polyhedral complexes of piecewise constant curvature are examples of nonlinear geodesic metric spaces. In these two examples, it is not obvious to show the existence of geodesics and show that they are unique. To determine when such spaces are uniquely geodesic is also a very hard task.

Geodesic triangles are naturally introduced in geodesic metric spaces. Indeed, let (M,d) be a geodesic metric space. Any three points—$x,y,z \in M$—will define a geodesic triangle $\Delta(x,y,z)$, which consists of the three given points called its vertices and the geodesic segments between each pair of vertices also known as the edges of $\Delta(x,y,z)$. Comparison triangles are crucial to the definition of $CAT(0)$ spaces [6]. Given a geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (M,d), a triangle $\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ is said to be a comparison triangle to $\Delta(x_1,x_2,x_3)$ whenever

$$d_{\mathbb{R}^2}\left(\bar{x}_i,\bar{x}_j\right) = d\left(x_i,x_j\right),$$

holds for any $i,j \in \{1,2,3\}$. The point $\bar{x} = \alpha\, \bar{x}_i + (1-\alpha)\, \bar{x}_j$ is called a comparison point for $x = \alpha\, x_i \oplus (1-\alpha)\, x_j$, for any $\alpha \in [0,1]$ and $i \neq j$.

Throughout (M,d) stands for a uniquely geodesic metric space.

Definition 1. *[2] Let $\Delta = \Delta(x_1,x_2,x_3)$ be a geodesic triangle in M and $\overline{\Delta} = \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ be a comparison triangle for Δ in $\mathbb{R}^2$. We say that Δ satisfies the $CAT(0)$ inequality if for any $x,y \in \Delta$ and their comparison points $\bar{x},\bar{y} \in \overline{\Delta}$, the following holds,*

$$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y}).$$

(M,d) is said to be a $CAT(0)$ space if all geodesic triangles satisfy the $CAT(0)$ inequality. A Hadamard metric space is any complete $CAT(0)$ space [7].

Let (M,d) be a $CAT\,(0)$ space. Let x,y_1,y_2 be in M. If $m = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$ is the midpoint of y_1 and y_2, then the $CAT\,(0)$ inequality implies:

$$d\left(x,m\right)^2 + \frac{1}{4}d\left(y_1,y_2\right)^2 \leq \frac{1}{2}d\left(x,y_1\right)^2 + \frac{1}{2}d\left(x,y_2\right)^2.$$

Strictly convex Banach spaces are obviously uniquely geodesic. It is well known that a normed vector space is a $CAT(0)$ space if and only if it is a pre-Hilbert space [2].

A recent extension to $CAT(0)$ spaces was initiated in [4]. It is based on the idea that comparison triangles belong to a general Banach spaces instead of the Euclidean plane.

Definition 2. *[4] $(\mathbb{E}, \|.\|)$ be a Banach space. The geodesic metric space (M, d) is said to be a $CAT_{\mathbb{E}}(0)$ space if for any geodesic triangle Δ in M, there exists a comparison triangle $\overline{\Delta}$ in $\mathbb{E}$ such that*

$$d(x,y) \leq \|\overline{x} - \overline{y}\|,$$

for any $x, y \in \Delta$ and their comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$. If $\mathbb{E} = \ell_p$, for $p \geq 1$, we say M is a $CAT_p(0)$ space.

It is obvious that $(\mathbb{E}, \|.\|)$ is a $CAT_{\mathbb{E}}(0)$ space. If $(\mathbb{E}, \|.\|)$ is not a pre-Hilbert space, then $(\mathbb{E}, \|.\|)$ is not a $CAT(0)$ space. In other words, Definition 2 gives a larger class of hyperbolic metric spaces. Throughout our work, we mainly focus on $CAT_p(0)$ metric spaces for $p > 2$. It is obvious that $CAT_2(0)$ space is exactly the classical $CAT(0)$ space, which has been extensively studied.

The classical inequality (1) gives information about the middle point of two points. Except that many successive iterations, which present some interesting behavior, do not involve the middle point, but a convex combination of the given two points. Therefore, it is of utmost importance to prove or discover some metric properties of this kind of convex combinations.

Next, we will discuss a property of the convex combinations that holds in $CAT_p(0)$ spaces for $p \geq 2$.

Lemma 1. *Let (M, d) be a $CAT_p(0)$ metric space, with $p \geq 2$. Then, for any x, y_1, y_2 in M and $\beta \in [0,1]$, we have*

$$d(x, \beta\, y_1 \oplus (1-\beta)\, y_2)^p + \frac{1}{2^{p-1}} \beta\, (1-\beta)\, d(y_1, y_2)^p \leq \beta\, d(x, y_1)^p + (1-\beta)\, d(x, y_2)^p.$$

Proof. Note that this inequality is valid in ℓ_p, for $p \geq 2$. Indeed, Lim [8] proved the following inequality,

$$\|\beta\, x + (1-\beta)\, y\|^p + g(\beta)\, \|x-y\|^p \leq \beta\, \|x\|^p + (1-\beta)\, \|y\|^p,$$

for any $\beta \in [0,1]$ and $x, y \in \ell_p$, where

$$g(\beta) = \beta\, (1-\beta)\, \frac{1 + [x(\beta \wedge (1-\beta))]^{p-1}}{[1 + x(\beta \wedge (1-\beta))]^{p-1}},$$

and $x(\gamma)$, for $\gamma \in [0, 1/2]$ is the unique solution to

$$(1-\gamma)x^{p-1} - \gamma - ((1-\gamma)x - \gamma)^{p-1} = 0, \quad x \in \left[\frac{\gamma}{1-\gamma}, 1\right].$$

In particular, we have $g(\beta) \geq \beta\, (1-\beta) \dfrac{1}{2^{p-1}}$, for any $\beta \in [0,1]$. Therefore,

$$\|\beta\, x + (1-\beta)\, y\|^p + \frac{\beta\, (1-\beta)}{2^{p-1}}\, \|x-y\|^p \leq \beta\, \|x\|^p + (1-\beta)\, \|y\|^p, \tag{2}$$

for any $\beta \in [0,1]$ and $x, y \in \ell_p$. Next, we turn our attention to the proof of Lemma 1. Consider the geodesic triangle $\Delta(x, y_1, y_2)$, where $x, y_1, y_2 \in M$, and $\beta \in [0,1]$. Ase M is a $CAT_p(0)$ space, there exists a comparison geodesic triangle $\overline{\Delta} = \overline{\Delta}(\bar{x}, \bar{y}_1, \bar{y}_2)$ in ℓ_p. The comparison axiom implies that

$$d(x, \beta\, y_1 \oplus (1-\beta)\, y_2) \leq \left\|\overline{x} - \left(\beta\, \bar{y}_1 + (1-\beta)\, \bar{y}_2\right)\right\| = \|\beta\, (\bar{x} - \bar{y}_1) + (1-\beta)\, (\bar{x} - \bar{y}_2)\|.$$

The inequality (2) implies that

$$\begin{array}{rcl} d(x, \beta\, y_1 \oplus (1-\beta)\, y_2)^p & \leq & \|\beta\, (\bar{x}-\bar{y}_1) + (1-\beta)\, (\bar{x}-\bar{y}_2)\|^p \\ & \leq & \beta\, \|\bar{x}-\bar{y}_1\|^p + (1-\beta)\, \|\bar{x}-\bar{y}_2\|^p \\ & & \quad - \frac{\beta\, (1-\beta)}{2^{p-1}}\, \|\bar{y}_2 - \bar{y}_1\|^p \\ & = & \beta\, d(x,y_1)^p + (1-\beta)\, d(x,y_2)^p - \frac{\beta\, (1-\beta)}{2^{p-1}}\, d(y_2,y_1)^p, \end{array}$$

which implies the conclusion of Lemma 1. □

Throughout, we will use the notation $C_p = 1/2^{p-1}$, for $p \geq 2$.

In the next section, we extend some known fixed point results in Banach spaces and $CAT(0)$ spaces to the case of $CAT_p(0)$, for $p > 2$.

3. Fixed Point Results in $CAT_p(0)$

Next, we investigate the fixed point problem for the class of asymptotically nonexpansive mappings. Note that this family of mappings was introduced by Goebel and Kirk [9] as a family of mappings that sits between the family of nonexpansive mappings [10] and the family of uniformly Lipschitzian mappings [11].

As we said before, throughout we consider (M,d) to be a geodesic metric space.

Definition 3. *[9–11] Let $J : M \to M$ be a map.*

(1) J is asymptotically nonexpansive if there exists $\{\rho_n\}$ such that $\lim_{n\to\infty} \rho_n = 1$ and

$$d(J^n(x), J^n(y)) \leq \rho_n\, d(x,y),$$

for any $x, y \in M$ and $n \in \mathbb{N}$. We can always assume that $\rho_n \geq 1$, for any $n \in \mathbb{N}$.

(2) J is uniformly Lipschitzian if there exists $\rho \geq 0$ such that

$$d(J^n(x), J^n(y)) \leq \rho\, d(x,y),$$

for any $x, y \in M$ and $n \in \mathbb{N}$.

(3) A point $x \in M$ is a fixed point of J if $J(x) = x$ holds. $Fix(J)$ will denote the set of fixed points of J.

The fixed point problem for this class of mappings was extensively investigated [12–15]. It followed two directions: The first deals with the existence of a fixed point. The second deals with the approximation of the fixed points based on algorithms initiated by Schu [16]. In this work, we will follow the same directions as well.

A powerful tool used in investigating the existence of fixed points is the concept of type functions, which plays a major role in the study of metric fixed point theory in Banach spaces. Historically, it is also known as the asymptotic center.

Definition 4. *Let (M,d) be a metric space. A function $\theta : M \to [0,+\infty)$ is a type function if there exists a bounded sequence $\{x_n\}$ in M such that*

$$\theta(a) = \limsup_{n\to\infty}\, d(x_n, a),$$

for any $a \in M$. A sequence $\{z_n\}$ in M is said to be a minimizing sequence of θ whenever

$$\lim_{n\to\infty}\, \theta(z_n) = \inf\{\theta(a);\ a \in M\}.$$

The following technical lemma shows why type functions are a powerful tool.

Lemma 2. *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be a nonempty closed convex subset of M and $\theta : C \to [0, +\infty)$ be a type function generated by a bounded sequence $\{x_n\} \subset M$. Then, the following hold.*

(1) Any minimizing sequence of θ is convergent.
(2) All minimizing sequences of θ converge to the same limit $z \in C$.
(3) z is a minimum point of θ, i.e., $\theta(z) = \inf\{\theta(x);\ x \in C\}$.

Proof. Set $\theta_0 = \inf\{\theta(x);\ x \in C\}$. Without loss of generality, we assume $\theta_0 > 0$. Let $\{z_n\}$ be a minimizing sequence of θ. Assume that $\{z_n\}$ is not Cauchy. As any subsequence of $\{z_n\}$ is also a minimizing sequence of θ, we may assume there exists $\varepsilon_0 > 0$ such that $d(z_n, z_m) \geq \varepsilon_0$, for any $n, m \in \mathbb{N}$. As C is convex, then $\frac{1}{2} z_m \oplus \frac{1}{2}\, z_{m+1} \in C$, for any $m \in \mathbb{N}$. Lemma 1 implies

$$\begin{aligned} d\left(x_n, \frac{1}{2}\, z_m \oplus \frac{1}{2}\, z_{m+1}\right)^p + \frac{1}{2^{p+1}}\, \varepsilon_0^p &\leq d\left(x_n, \frac{1}{2}\, z_m \oplus \frac{1}{2}\, z_{m+1}\right)^p \\ &\quad + \frac{1}{2^{p+1}}\, d(z_m, z_{m+1})^p \\ &\leq \frac{1}{2}\, d(x_n, z_m)^p + \frac{1}{2}\, d(x_n, z_{m+1})^p, \end{aligned}$$

for any $n, m \in \mathbb{N}$. If we let $n \to \infty$, we get

$$\theta\left(\frac{1}{2}\, z_m \oplus \frac{1}{2}\, z_{m+1}\right)^p + \frac{1}{2^{p+1}}\, \varepsilon_0^p \leq \frac{1}{2}\, \theta(z_m)^p + \frac{1}{2}\, \theta(z_{m+1})^p,$$

which implies

$$\theta_0^p + \frac{1}{2^{p+1}}\, \varepsilon_0^p \leq \frac{1}{2}\, \theta(z_m)^p + \frac{1}{2}\, \theta(z_{m+1})^p,$$

for any $m \in \mathbb{N}$. If we let $m \to \infty$, we get

$$\theta_0^p + \frac{1}{2^{p+1}}\, \varepsilon_0^p \leq \frac{1}{2}\, \theta_0^p + \frac{1}{2}\, \theta_0^p = \theta_0^p.$$

This contradiction shows that $\{z_n\}$ is Cauchy, which shows that (1) holds. To prove (2), let $\{z_m\}$ and $\{w_m\}$ be two minimizing sequences of θ. Consider the sequence $\{y_m\}$ defined by $y_{2m} = z_m$ and $y_{2m+1} = w_m$, for any $m \in \mathbb{N}$. Then, $\{y_m\}$ is also a minimizing sequence of θ. From (1) we conclude that $\{y_m\}$ is convergent. As both $\{z_m\}$ and $\{w_m\}$ are subsequence of $\{y_m\}$ we conclude that $\{z_m\}$ and $\{w_m\}$ have the same limit. The conclusion of (3) follows from the simple fact that type functions are continuous. □

In the first result, we discuss the existence of a fixed point for asymptotically nonexpansive mappings in $CAT_p(0)$ spaces.

Theorem 1. *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be a nonempty closed bounded convex subset of M. Let $J : C \to C$ be an asymptotically nonexpansive mapping. Then, J has a fixed point. Moreover, $Fix(J)$ is closed and convex.*

Proof. Let $\{\rho_n\}$ be the Lipschitz sequence associated to J. Fix $x \in C$. Consider the type function θ generated by $\{J^n(x)\}$. Let z be the minimum point of θ which exists by using Lemma 2. Therefore,

$$d(J^{n+m}(x), J^m(z)) \leq \rho_m\, d(J^n(x), z),$$

for any $n, m \in \mathbb{N}$. If we let $n \to \infty$, we will get $\theta(J^m(z)) \leq \rho_m\, \theta(z) = \rho_m\, \theta_0$, for any $m \in \mathbb{N}$. If we let $m \to \infty$, and using $\lim\limits_{n \to \infty} \rho_n = 1$, we get $\lim\limits_{m \to \infty} \theta(J^m(z)) = \theta_0$, i.e., $\{J^m(z)\}$ is a minimizing sequence of θ. Using Lemma 2, we conclude that $\{J^m(z)\}$ converges to z. As J is continuous, we conclude that $J(z) = z$. The fact that $Fix(J)$ is closed is obvious from the continuity of J. Let us prove that $Fix(J)$ is convex. Let $z_1, z_2 \in Fix(J)$ be different. As $Fix(J)$ is closed, we only need to prove that $w = \frac{1}{2}z_1 \oplus \frac{1}{2}z_2 \in Fix(J)$. Note that

$$d(z_i, J^n(w)) = d(J^n(z_i), J^n(w)) \leq \rho_n\, d(z_i, w) = \frac{\rho_n}{2}\, d(z_1, z_2),$$

for $n \in \mathbb{N}$ and $i = 1, 2$. Therefore,

$$d(z_1, z_2) \leq d(z_1, J^n(w)) + d(J^n(w), z_2) \leq \rho_n\, d(z_1, z_2),$$

for any $n \in \mathbb{N}$. As $\lim\limits_{n\to\infty} \rho_n = 1$, we conclude that

$$\lim_{n\to\infty} d(z_i, J^n(w)) = \frac{d(z_1, z_2)}{2}, \quad for\ i = 1, 2.$$

Similarly, we can show that

$$\lim_{n\to\infty} d\left(z_i, \frac{1}{2}w \oplus \frac{1}{2}J^n(w)\right) = \frac{d(z_1, z_2)}{2}, \quad for\ i = 1, 2.$$

Using Lemma 1, we get

$$d\left(z_1, \frac{1}{2}\, w \oplus \frac{1}{2}\, J^n(w)\right)^p + \frac{1}{2^{p+1}} d(w, J^n(w))^p \leq \frac{1}{2}\, d(z_1, w)^p + \frac{1}{2}\, d(z_1, J^n(w))^p,$$

for any $n \in \mathbb{N}$. If we let $n \to \infty$, we get $\lim\limits_{n\to\infty} d(w, J^n(w)) = 0$. As J is continuous, then $w \in Fix(J)$ as claimed, which completes the proof of Theorem 1. □

Note that we may refine the boundedness assumption of C by assuming that an orbit of J is bounded. In this case, the above proof still holds. The convexity of the set of fixed points is a useful information, because it will allow us to prove the existence of a common fixed point for this class of mappings for example.

Next, we discuss the behavior of the successive iterations introduced by Schu [16] for asymptotically nonexpansive. In this case, Lemma 1 will prove to be crucial.

Recall that for a bounded nonempty subset C of a metric space (M, d), $\delta(C)$ denotes the diameter of C and is defined by

$$\delta(C) = \sup\{d(c_1, c_2); c_1, c_2 \in C\}.$$

Lemma 3. *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be a nonempty closed bounded convex subset of M. Let $J : C \to C$ be an asymptotically nonexpansive mapping with $\{\rho_n\}$ as its associated Lipschitz constants. Let $\{\gamma_n\} \subset [0, 1]$, such that $\sum\limits_{n=1}^{\infty} \gamma_n(\rho_n - 1) < \infty$. The modified Mann iteration process [16] is defined by*

$$x_{n+1} = \gamma_n J^n(x_n) \oplus (1 - \gamma_n)x_n, \tag{3}$$

for any $n \in \mathbb{N}$, where $x_0 \in C$ is a fixed arbitrary point. If $z \in Fix(J)$, then $\lim\limits_{n\to\infty} d(x_n, z)$ exists.

Proof. First, note that

$$\begin{aligned} d(x_{n+1}, z) &\leq \gamma_n d(J^n(x_n), z) + (1 - \gamma_n)d(x_n, z) \\ &= \gamma_n d(J^n(x_n), J^n(z)) + (1 - \gamma_n)d(x_n, z) \\ &\leq \gamma_n(\rho_n - 1)d(x_n, z) + d(x_n, z), \end{aligned}$$

which implies $d(x_{n+1}, z) - d(x_n, z) \leq \gamma_n(\rho_n - 1)d(x_n, z)$, for any $n \in \mathbb{N}$. In particular, we have $d(x_{n+1}, z) - d(x_n, z) \leq \gamma_n(\rho_n - 1)\delta(C)$, for any $n \in \mathbb{N}$. Therefore,

$$d(x_{n+m}, z) - d(x_n, z) \leq \delta(C) \sum_{i=0}^{m-1} \gamma_{n+i}(\rho_{n+i} - 1),$$

for any $n, m \in \mathbb{N}$. If we let $m \to \infty$, we get

$$\limsup_{m\to\infty} d(x_m, z) \leq d(x_n, z) + \delta(C) \sum_{i=n}^{\infty} \gamma_i(\rho_i - 1),$$

for any $n \in \mathbb{N}$. Using the assumption $\sum\limits_{n=1}^{\infty} \gamma_n(\rho_n - 1) < \infty$, we obtain

$$\begin{aligned} \limsup_{m\to\infty} d(x_m, z) &\leq \liminf_{n\to\infty} d(x_n, z) + \delta(C) \liminf_{n\to\infty} \sum_{i=n}^{\infty} \gamma_i(\rho_i - 1) \\ &= \liminf_{n\to\infty} d(x_n, z) \\ &\leq \limsup_{m\to\infty} d(x_m, z) \leq \delta(C). \end{aligned}$$

Therefore, $\limsup\limits_{m\to\infty} d(x_m, z) = \liminf\limits_{n\to\infty} d(x_n, z)$, i.e., $\{d(x_n, z)\}$ is convergent as claimed. □

Remark 1. *In the original work of Schu [16], the conclusion of Lemma 3 is obtained under the stronger assumption* $\sum\limits_{n=1}^{\infty} (\rho_n - 1) < \infty$. *One may be confused to how we can assume a weaker assumption that involves the sequence* $\{\gamma_n\}$. *In fact, the construction of the sequence* $\{\gamma_n\}$ *is done during the computation of the sequence* $\{x_n\}$ *at the same time making sure we have the convergence of the series* $\sum\limits_{n=1}^{\infty} \gamma_n(\rho_n - 1)$. *Moreover, if we assume that* $\gamma_n \geq a > 0$, *for some* $a \in (0, 1)$, *then* $\sum\limits_{n=1}^{\infty} \gamma_n(\rho_n - 1)$ *is convergent if and only if* $\sum\limits_{n=1}^{\infty} (\rho_n - 1)$ *is convergent.*

In the next result, we show that the sequence generated by (3) almost gives a fixed point.

Theorem 2. *Let* (M, d) *be a complete* $CAT_p(0)$ *metric space, with* $p \geq 2$. *Let* C *be a nonempty closed bounded convex subset of* M. *Let* $J : C \to C$ *be an asymptotically nonexpansive mapping with* $\{\rho_n\}$ *as its associated Lipschitz constants. Assume* $\sum\limits_{n=1}^{\infty} (\rho_n - 1) < \infty$. *Let* $\{\gamma_n\} \subset [a, b]$, *where* $0 < a \leq b < 1$. *Fix* $x_0 \in C$ *and consider the sequence* $\{x_n\}$ *generated by the iteration (3). Then,*

$$\lim_{n\to\infty} d(x_n, J^n(x_n)) = 0 \text{ and } \lim_{n\to\infty} d(x_n, J^m(x_n)) = 0,$$

for any $m \geq 1$, *i.e., the sequence* $\{x_n\}$ *is said to be an approximate fixed point sequence of* T.

Proof. First, let us prove that $\lim_{n\to\infty} d(x_n, J^n(x_n)) = 0$. Using Theorem 1, J has a fixed point $z \in C$. Lemma 3 implies that $\lim_{n\to\infty} d(x_n, z) = r$ exists. Using Lemma 1, we get

$$\begin{array}{rcl} d(z, x_{n+1})^p + \frac{a(1-b)}{2^{p-1}}\, d(x_n, J^n(x_n))^p & \leq & d(z, x_{n+1})^p + \frac{\gamma_n(1-\gamma_n)}{2^{p-1}}\, d(x_n, J^n(x_n))^p \\ & \leq & \gamma_n\, d(x_n, z)^p + (1-\gamma_n)\, d(J^n(x_n), z)^p \\ & = & \gamma_n\, d(x_n, z)^p + (1-\gamma_n)\, d(J^n(x_n), J^n(z))^p \\ & \leq & (\gamma_n + \rho_n^p(1-\gamma_n))\, d(x_n, z)^p \\ & \leq & (\rho_n^p\gamma_n + \rho_n^p(1-\gamma_n))\, d(x_n, z)^p = \rho_n^p\, d(x_n, z)^p, \end{array}$$

since $\rho_n \geq 1$, which implies

$$d(x_n, J^n(x_n))^p \leq \frac{2^{p-1}}{a(1-b)}\left(\rho_n^p\, d(x_n, z)^p - d(z, x_{n+1})^p\right),$$

for any $n \in \mathbb{N}$. As

$$\lim_{n\to\infty} \rho_n^p\, d(x_n, z)^p - d(z, x_{n+1})^p = r^p - r^p = 0,$$

we conclude that $\lim_{n\to\infty} d(x_n, J^n(x_n)) = 0$. Next, we prove that for any $m \geq 1$, we have $\lim_{n\to\infty} d(x_n, J^m(x_n)) = 0$. As $\{\rho_n\}$ is convergent, it is bounded. Set $\rho = \sup_{n\in\mathbb{N}} \rho_n$. Therefore,

$$\begin{array}{rcl} d(x_n, J(x_n)) & \leq & d(x_n, J^n(x_n)) + \rho\, d(J^{n-1}(x_n), x_n) \\ & \leq & d(x_n, J^n(x_n)) + \rho^2\, d(x_n, x_{n-1}) + \rho\, d(x_n, J^{n-1}(x_{n-1})) \\ & = & d(x_n, J^n(x_n)) + \rho^2\, \gamma_{n-1}\, d(x_{n-1}, J^{n-1}(x_{n-1})) \\ & & \qquad + \rho\, (1-\gamma_{n-1})\, d(x_{n-1}, J^{n-1}(x_{n-1})) \\ & \leq & d(x_n, J^n(x_n)) + \rho(\rho+1)\, d(x_{n-1}, J^{n-1}(x_{n-1})), \end{array}$$

for any $n \geq 1$. As $\lim_{n\to\infty} d(x_n, J^n(x_n)) = 0$, we conclude that

$$\lim_{n\to\infty} d(x_n, J(x_n)) = 0.$$

Finally, fix $m \geq 1$. As we have

$$\begin{array}{rcl} d(x_n, J^m(x_n)) & \leq & \sum_{k=0}^{m-1} d(J^k(x_n), J^{k+1}(x_n)) \\ & \leq & \sum_{k=0}^{m-1} \rho\, d(x_n, J(x_n)) \\ & \leq & m\, \rho\, d(x_n, J(x_n)), \end{array}$$

for any $n \in \mathbb{N}$. Therefore, we have $\lim_{n\to\infty} d(x_n, J^m(x_n)) = 0$, for any $m \geq 1$. □

From a computational point of view, the algorithm (3) almost generated a fixed point of J. However, from a mathematical point of view, we still need to look at the convergence of $\{x_n\}$. First, we have a strong convergence as did Schu [16]. To obtain this, we need some kind of compactness assumption. Recall that $J : C \to C$ is said to be compact if the closure of $J(C)$ is compact.

Theorem 3. *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be a nonempty closed bounded convex subset of M. Let $J : C \to C$ be an asymptotically nonexpansive mapping such that J^m is compact for some $m \geq 1$. Assume $\sum_{n=1}^{\infty} (\rho_n - 1) < \infty$, where $\{\rho_n\}$ is the Lipschitz sequence associated to J. Let $\{\gamma_n\} \subset [a, b]$,* *where $0 < a \leq b < 1$. Fix $x_0 \in C$ and consider the sequence $\{x_n\}$ generated by the iteration (3). Then, $\{x_n\}$ converges strongly to a fixed point z of J, i.e., $\lim_{n\to\infty} d(x_n, z) = 0$.*

Proof. Let $m \geq 1$ such that J^m is compact. Then, there exists a subsequence $\{x_{\phi(n)}\}$ such that $\{J^m(x_{\phi(n)})\}$ converges to some point $z \in C$. Using Theorem 2, we know that $\lim_{n\to\infty} d(x_n, J^m(x_n)) = 0$, which implies $\{x_{\phi(n)}\}$ also converges to z. Again, using Theorem 2, we know that $\lim_{n\to\infty} d(x_n, J(x_n)) = 0$, which implies that $\{J(x_{\phi(n)})\}$ also converges to z. As J is continuous, we conclude that $J(z) = z$, i.e, $z \in Fix(J)$. Moreover, Lemma 3 implies $\lim_{n\to\infty} d(x_n, z) = r$ exists. As $\lim_{n\to\infty} d(x_{\phi(n)}, z) = 0$, we conclude that $r = 0$, i.e., $\lim_{n\to\infty} d(x_n, z) = 0$. In other words, the sequence $\{x_n\}$ converges to z as claimed. □

Therefore, we wonder whether a weaker convergence is happening if we relax the compactness assumption in Theorem 3. In the original work of Schu [16], the setting is a Banach space. Therefore, we may consider naturally the weak topology. In the nonlinear setting, it is still unknown what the weak topology looks like. Lim [17] introduced a convergence concept he called Δ-convergence based on the asymptotic center of a sequence. Except that this convergence does not capture the weak topology once we restrict ourselves to Banach spaces. It only happens if the Banach space enjoys the Opial property [18]. In the next result, we discard the compactness assumption.

Theorem 4. *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be a nonempty closed bounded convex subset of M. Let $J : C \to C$ be an asymptotically nonexpansive mapping. Assume $\sum_{n=1}^{\infty} (\rho_n - 1) < \infty$, where $\{\rho_n\}$ is the Lipschitz sequence associated to J. Let $\{\gamma_n\} \subset [a, b]$, where $0 < a \leq b < 1$. Fix $x_0 \in C$ and consider the sequence $\{x_n\}$ generated by the iteration (3). For any subsequence $\{x_{\phi(n)}\}$, consider the type $\theta_\phi(x) = \limsup_{n\to\infty} d(x_{\phi(n)}, x)$ on C. Then, the minimum point z of θ_ϕ is independent of the subsequence and is a fixed point of J. We say that $\{x_n\}$ Δ-converges to z.*

Proof. Consider the type function $\theta : C \to [0, \infty)$ defined by $\{x_n\}$, i.e., $\theta(x) = \limsup_{n\to\infty} d(x_n, x)$, for any $x \in C$. According to Lemma 2, the type function θ has a unique minimum point z, which is a fixed point of J. Let us prove that z is also the minimum point of any type function θ_ϕ generated by a subsequence $\{x_{\phi(n)}\}$ of $\{x_n\}$. Again according to Lemma 2, there exists a unique minimum point z_ϕ of θ_ϕ, which is also a fixed point of J. Lemma 3 implies that $\lim_{n\to\infty} d(x_n, z)$ and $\lim_{n\to\infty} d(x_n, z_\phi)$ exist. As z is the minimum point of θ, we get $\theta(z) \leq \theta(z_\phi)$, which implies $\lim_{n\to\infty} d(x_n, z) \leq \lim_{n\to\infty} d(x_n, z_\phi)$ or

$$\lim_{n\to\infty} d(x_{\phi(n)}, z) \leq \lim_{n\to\infty} d(x_{\phi(n)}, z_\phi),$$

i.e., $\theta_\phi(z) \leq \theta_\phi(z_\phi)$. The uniqueness of the minimum point of θ_ϕ implies that $z = z_\phi$, which completes the proof of Theorem 4. □

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