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Singularities of Non-Developable Surfaces in Three-Dimensional Euclidean Space

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Abstract: We study the singularity on principal normal and binormal surfaces generated by smooth curves with singular points in the Euclidean 3-space. We discover the existence of singular points on such binormal surfaces and study these singularities by the method of singularity theory. By using structure functions, we can characterize the ruled surface generated by special curves.

Keywords: principal normal surface; binormal surface; structure functions; singular points

1. Introduction

As the easiest parameterized surfaces, ruled surfaces are widely used in project practices, architecture, and computer-aided design [1,2]. However, on the ruled surfaces there may exist singular points. For that reason, many people study the classification of different types of singularities of a ruled surface. Taking advantage of Gauss curvature, ruled surfaces can be classified either as developable surfaces or as non-developable surfaces. From [3–7], we know that the generic singularities of a developable surface are the cuspidal edge, the swallowtail, and the cuspidal cross-cap. We also know that the cross-cap is the only singular point of existence on the principal normal surface of regular space curves (see [8,9]). Meanwhile, on the binormal surface of regular space curves, there are no singular points. However, for singular curves, the situation is different, and we will study the character of the singular points on such a binormal surface.

In [10], Müller gave the definition of two integral invariants, which are the pitch and the angle of pitch of a closed ruled surface in $\mathbb{R}^3$ (three-dimensional Euclidean space). For a general ruled surface, the base curve is not unique. In order to solve the uncertainty of a base curve on the ruled surface, Liu and Yuan used the uniqueness of the striction line on the general ruled surface. Since the derivative of the base curve does not identically vanish, this surface is a non-developable ruled surface. In [11], they extended the definition of pitch to non-developable ruled surfaces. In [12], Liu et al. defined structure functions, which are invariants of non-developable surfaces. They used these functions to characterize the properties of surfaces. Meanwhile, they gave the relationship between these invariants and the pitch function, the angle function of pitch of the ruled surface (see [12–14]).

In this paper, we regard the singular curve as the Frenet-type framed base curve. In Sections 3 and 4, we give the notations of the principal normal and binormal surfaces of a Frenet-type framed base curve in Euclidean 3-space and investigate the character of singular points on these surfaces. In Section 5, we give a standard equation of a non-developable ruled surface and then study its structure functions. Moreover, we give the kinematic meanings at singular points. In Section 6, we use an example to state the singular points on these non-developable surfaces.

Throughout this article, all manifolds and maps are smooth.

2. Preliminaries

In this section, we study the ruled surface generated by Frenet-type framed base curves (see [15–17]). Since there exist singular points on these surfaces, in general, we cannot construct the normal vector of these surfaces. Therefore, we regard them as framed base surfaces that are smooth surfaces with a moving frame (see [15]).

Definition 1. *We call $(\boldsymbol{f}, \boldsymbol{f}_1, \boldsymbol{f}_2) : W \to \mathbb{R}^3 \times \triangle$ a framed surface if $\boldsymbol{f}_y(y,u) \cdot \boldsymbol{f}_1(y,u) = 0$ and $\boldsymbol{f}_u(y,u) \cdot \boldsymbol{f}_1(y,u) = 0$ for all $(y,u) \in W$, where $\triangle = \{(\boldsymbol{f}_1, \boldsymbol{f}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 | \boldsymbol{f}_1 \cdot \boldsymbol{f}_2 = 0\}$, $\boldsymbol{f}_y(y,u) = (\partial \boldsymbol{f}/\partial y)(y,u)$ and $\boldsymbol{f}_u(y,u) = (\partial \boldsymbol{f}/\partial u)(y,u)$.*

If there exists $(\boldsymbol{f}_1, \boldsymbol{f}_2) : W \to \triangle$ such that $(\boldsymbol{f}, \boldsymbol{f}_1, \boldsymbol{f}_2)$ is a framed surface, then we call $\boldsymbol{f} : W \to \mathbb{R}^3$ a framed base surface. We define $\boldsymbol{f}_3(y,u) = \boldsymbol{f}_1(y,u) \times \boldsymbol{f}_2(y,u)$, then $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a moving frame along $\boldsymbol{f}(y,u)$. Thus, we have

$$\begin{pmatrix} \boldsymbol{f}_y \\ \boldsymbol{f}_u \end{pmatrix} = \begin{pmatrix} i^y_{02} & i^y_{03} \\ i^u_{02} & i^u_{03} \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{f}_{1y} \\ \boldsymbol{f}_{2y} \\ \boldsymbol{f}_{3y} \end{pmatrix} = \begin{pmatrix} 0 & i^y_{12} & i^y_{13} \\ -i^y_{12} & 0 & i^y_{23} \\ -i^y_{13} & -i^y_{23} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{f}_{1u} \\ \boldsymbol{f}_{2u} \\ \boldsymbol{f}_{3u} \end{pmatrix} = \begin{pmatrix} 0 & i^u_{12} & i^u_{13} \\ -i^u_{12} & 0 & i^u_{23} \\ -i^u_{13} & -i^u_{23} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{pmatrix},$$

where $i^y_{ij}, i^u_{ij} : W \to \mathbb{R}$, $i = 0,1,2, j = 2,3$ are smooth functions. These functions are called the basic invariants of $(\boldsymbol{f}, \boldsymbol{f}_1, \boldsymbol{f}_2)$. By the integrability conditions of the framed surface [15], we have $i^y_{02} i^u_{12} + i^y_{03} i^u_{13} = i^u_{02} i^y_{12} + i^u_{03} i^y_{13}$. We call $C_f = (C_{1f}, C_{2f}, C_{3f}) : W \to \mathbb{R}^3$ a curvature of the framed surface if

$$C_{1f} = det \begin{pmatrix} i^y_{02} & i^y_{03} \\ i^u_{02} & i^u_{03} \end{pmatrix}, C_{2f} = det \begin{pmatrix} i^y_{12} & i^y_{13} \\ i^u_{12} & i^u_{13} \end{pmatrix}, C_{3f} = -\frac{1}{2} \left\{ det \begin{pmatrix} i^y_{02} & i^y_{13} \\ i^u_{02} & i^u_{13} \end{pmatrix} - det \begin{pmatrix} i^y_{03} & i^y_{12} \\ i^u_{03} & i^u_{12} \end{pmatrix} \right\}.$$

We suppose that $(\boldsymbol{f}, \boldsymbol{f}_1, \boldsymbol{f}_2) : W \to \mathbb{R}^3 \times \triangle$ is the framed surface and $p \in W$. The surface $\boldsymbol{f}$ is a front around p if and only if $C_f(p) \neq 0$. More details are available from [15].

Next, we study the first special non-developable ruled surface, that is, the principal normal surface.

3. Principal Normal Surface along Frenet-Type Framed Base Curves

If there exists a regular unit speed curve $\boldsymbol{T} : Y \to \mathbb{S}^2$ and a C^∞ function $\iota : Y \to \mathbb{R}$ satisfies $\dot{\boldsymbol{\beta}}(y) = \iota(y)\boldsymbol{T}(y)$ for all $y \in Y$, then we call $\boldsymbol{\beta} = \boldsymbol{\beta}(y) : Y \to \mathbb{R}^3$ a Frenet -type framed base curve (FTFB curve). We call $\{\boldsymbol{T}(y), \boldsymbol{N}(y), \boldsymbol{B}(y)\}$ an orthonormal frame along $\boldsymbol{\beta}(y)$ in $\mathbb{R}^3$, where $\boldsymbol{N} = \dot{\boldsymbol{T}}/\|\dot{\boldsymbol{T}}\|$ and $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$. More details are available from [16]. The FTFB curve is one special kind of framed base curve [15,16]. As we want to intuitively observe the properties at singular points on the ruled surface, we choose this kind curve that is similar to the Frenet curve. A principal normal surface M is a map $\boldsymbol{f} : Y \times U \to \mathbb{R}^3$ given by $\boldsymbol{f}(y,u) = \boldsymbol{\beta}(y) + u\boldsymbol{N}(y)$. By direct calculations, singular points of surface M construct the set $S = \{(y,u) \in Y \times U \mid \iota(y) - u\kappa(y) = 0,\ u\tau(y) = 0\}$. We can divide them into two classes S_1 and S_2, where

$$S_1 = \{(y,0) \in Y \times U \mid \iota(y) = 0\},\ S_2 = \{(y,u) \in Y \times U \mid u = \iota(y)/\kappa(y) \neq 0,\ \tau(y) = 0\}.$$

From above, we know the points of S_1 are located on the $\boldsymbol{\beta}$.

Next, we consider the characters of singular points of surfaces.

Theorem 1. *We suppose that $\boldsymbol{\beta}(y)$ is an FTFB curve and $\boldsymbol{f}(y,u)$ is the principal normal surface of $\boldsymbol{\beta}(y)$.*

(1) *If $(y_1, 0) \in S_1$ and $\dot{\iota}(y_1)\tau(y_1) \neq 0$, then $\boldsymbol{f}(y,u)$ is a cross-cap at the point $(y_1, 0)$.*

(2) *If $(y_2, u_2) \in S_2$ and $\dot{\tau}(y_2) \neq 0$, then $\boldsymbol{f}(y,u)$ is a cross-cap at the point (y_2, u_2).*

Proof. Taking the derivative of $\boldsymbol{f}$, we can compute that

$$\frac{\partial \boldsymbol{f}}{\partial y}(y,u) = (\iota(y) - u\kappa(y))\boldsymbol{T}(y) + u\tau(y)\boldsymbol{B}(y), \;\; \frac{\partial \boldsymbol{f}}{\partial u}(y,u) = \boldsymbol{N}(y).$$

We have the second-order derivation of $\boldsymbol{f}$ as follows:

$$\frac{\partial^2 \boldsymbol{f}}{\partial u \partial y}(y,u) = -\kappa(y)\boldsymbol{T}(y) + \tau(y)\boldsymbol{B}(y),$$

$$\frac{\partial^2 \boldsymbol{f}}{\partial y^2}(y,u) = (\dot{\iota}(y) - u\dot{\kappa}(y))\boldsymbol{T}(y) + u\dot{\tau}(y)\boldsymbol{B}(y) + (\iota(y)\kappa(y) - u\kappa^2(y) - u\tau^2(y))\boldsymbol{N}(y).$$

Then we obtain

$$det\left(\frac{\partial \boldsymbol{f}}{\partial u}(y,u), \frac{\partial^2 \boldsymbol{f}}{\partial u \partial y}(y,u), \frac{\partial^2 \boldsymbol{f}}{\partial y^2}(y,u)\right) = u\dot{\tau}(y)\kappa(y) + (\dot{\iota}(y) - u\dot{\kappa}(y))\tau(y).$$

We have known that the union of S_1 and S_2 are the set of singular points of surface M. Thus, if $(y_1, 0) \in S_1$, then $det\left(\frac{\partial \boldsymbol{f}}{\partial u}(y_1,0), \frac{\partial^2 \boldsymbol{f}}{\partial y \partial y}(y_1,0), \frac{\partial^2 \boldsymbol{f}}{\partial y^2}(y_1,0)\right) = \dot{\iota}(y_1)\tau(y_1)$. And if $(y_2, u_2) \in S_2$, then $det\left(\frac{\partial \boldsymbol{f}}{\partial u}(y_2,u_2), \frac{\partial^2 \boldsymbol{f}}{\partial u \partial y}(y_2,u_2), \frac{\partial^2 \boldsymbol{f}}{\partial y^2}(y_2,u_2)\right) = u_2\dot{\tau}(y_2)\kappa(y_2)$. From [9], this completes the proof. □

Since the Bertrand curve and Bertrand mate can be regard as curves on the principal normal surface, then we consider the singular point located on such curves.

Corollary 1. *Let $\boldsymbol{\beta}(y)$ be the space Bertrand curve of an FTFB curve and $\boldsymbol{f}(y,u)$ be the principal normal surface of $\boldsymbol{\beta}(y)$.*

(1) *If y_1 is the ordinary cusp singularity of $\boldsymbol{\beta}$, then $\boldsymbol{f}(y,u)$ is a cross-cap at the point $(y_1, 0)$.*

(2) *If y_2 is the ordinary cusp singularity of the Bertrand mate of $\boldsymbol{\beta}$, then $\boldsymbol{f}(y,u)$ is a cross-cap at the point (y_2, A).*

Proof. (1) By the definition of ordinary cusp singularity [18], we know the ordinary cusp singularity y_1 of $\boldsymbol{\beta}$ satisfying $\iota(y_1) = 0$ and $\dot{\iota}(y_1) \neq 0$. From [17,19], $\boldsymbol{\beta}$ is a space Bertrand curve of an FTFB curve if and only if there exist two constants $\zeta_1 (\neq 0)$ and ζ_2 such that $\zeta_1\kappa + \zeta_2\tau = \iota$ and $\zeta_2\kappa - \zeta_1\tau \neq 0$. Reasoning $\kappa(y) \neq 0$ for all $y \in Y$, then we obtain $\tau(y_1) \neq 0$.

(2) Suppose $\boldsymbol{\beta}_m(y)$ is the Bertrand mate of $\boldsymbol{\beta}(y)$,

$$\boldsymbol{\beta}_m(y) = \boldsymbol{\beta}(y) + A\boldsymbol{N}(y),$$

where A is a non-zero constant. By differentiating $\boldsymbol{\beta}_m(y)$ and using the Frenet equation, we obtain

$$\iota_{\beta_m}(y)\boldsymbol{T}_{\beta_m}(y) = (\iota(y) - A\kappa(y))\boldsymbol{T}(y) + A\tau(y)\boldsymbol{B}(y).$$

If y_2 is the ordinary cusp singularity of $\boldsymbol{\beta}_m(y)$, then $\tau(y_2) = 0$ and $\dot{\tau}(y_2) \neq 0$. □

Because the principal normal surface has singular points, then we regard it as the framed base surface. Next, we analyze singular points of this surface by using the criterion about a framed surface (see [15]).

For a principal normal surface $\boldsymbol{f}(y,u) = \boldsymbol{\beta}(y) + u\boldsymbol{N}(y)$, if there exist two smooth functions $\theta, \phi : Y \times U \to \mathbb{R}^3$ satisfying $\boldsymbol{f}_y(y,u) \cdot \boldsymbol{f}_1(y,u) = 0$, where $\boldsymbol{f}_1(y,u) = \cos\theta(y,u)\boldsymbol{T}(y) + \sin\theta(y,u)\boldsymbol{B}(y)$ and $\boldsymbol{f}_2(y,u) = \cos\theta(y,u)\boldsymbol{N}(y) + \sin\theta(y,u)(-\sin\theta(y,u)\boldsymbol{T}(y) + \cos\theta(y,u)\boldsymbol{B}(y))$, then we have the framed surface $(\boldsymbol{f}, \ \boldsymbol{f}_1, \ \boldsymbol{f}_2) : Y \times U \to \mathbb{R}^3 \times \triangle$. We denote $\boldsymbol{f}_3(y,u) = \boldsymbol{f}_1(y,u) \times \boldsymbol{f}_2(y,u)$. Then $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a moving frame along $\boldsymbol{f}(y,u)$.

Because of the integrability condition, we have

$$F(y,u) = -\theta_u((\iota - u\kappa)sin\theta - u\tau\cos\theta) + \tau\sin\theta - \kappa\cos\theta = 0.$$

By direct calculations, we know that $C_{2f} = \kappa(y)\cos\theta(y,u) - \tau(y)\sin\theta(y,u)$ and $C_{3f} = \theta_y(y,u)/2$. Let $(y,u) \in S$ be the singular point of the principal normal surface $\boldsymbol{f}$. Because of

$$d\boldsymbol{f} = ((\iota - u\kappa)\boldsymbol{T} + u\tau\boldsymbol{B})dy + \boldsymbol{N}du,$$

the null vector field $\boldsymbol{\eta}$ can be written as $\partial/\partial y$. If $\boldsymbol{\eta}\lambda(y,u) = \lambda_y(y,u) \neq 0$, then we obtain

$$\Phi(y) = -\theta_y + (\kappa\cos\theta - \tau\sin\theta)(u\tau\cos\theta - (\iota - u\kappa)\sin\theta)y'(u).$$

By the criterion of the singular point on the framed surface [15], we can get that the surface $\boldsymbol{f}(y,u)$ is a cuspidal edge at a singular point if

(i) $(y_1, 0) \in S_1$, $\dot{\iota}(y_1) \neq 0$, $\theta_y(y_1, 0) \neq 0$ or

(ii) $(y_2, u_2) \in S_2$, $\dot{\iota}(y_2)\kappa(y_2) - \iota(y_2)\dot{\kappa}(y_2) \neq 0$, $\theta_y(y_2, u_2) \neq 0$.

The surface $\boldsymbol{f}(y,u)$ is a swallowtail at a singular point if

(iii) $(y_1, 0) \in S_1$, $\dot{\iota}(y_1) = 0$, $\ddot{\iota}(y_1) \neq 0$, $\theta_y(y_1, 0) \neq 0$ or

(iv) $(y_2, u_2) \in S_2$, $\dot{\iota}(y_2)\kappa(y_2) - \iota(y_2)\dot{\kappa}(y_2) = 0$, $\ddot{\iota}(y_2) - u_2\ddot{\kappa}(y_2) + 2u_2\dot{\tau}(y_2)\theta_y(y_2, u_2) \neq 0$, $\theta_y(y_2, u_2) \neq 0$.

The surface $\boldsymbol{f}(y,u)$ is a cuspidal cross-cap at a singular point if

(v) $(y_1, 0) \in S_1$, $\dot{\iota}(y_1) \neq 0$, $\theta_y(y_1, 0) = 0$, $\theta_{yy}(y_1, 0) \neq 0$ or

(vi) $(y_2, u_2) \in S_2$, $\dot{\iota}(y_2)\kappa(y_2) - \iota(y_2)\dot{\kappa}(y_2) \neq 0$, $\theta_y(y_2, u_2) = 0$, $\theta_{yy}(y_2, u_2) \neq 0$.

By the derivative of $F(y,u)$, any above case no exists. Therefore, we get the conclusion.

Theorem 2. *Let M be the principal normal surface of the FTFB curve $\boldsymbol{\beta}$. If M is a framed base surface, then the singular points of M are non-degenerate. But the surface M at a singular point cannot be the cuspidal edge, swallowtail, or cuspidal cross-cap.*

4. Binormal Surface along Frenet-Type Framed Base Curves

Let $\boldsymbol{\beta} = \boldsymbol{\beta}(y) : Y \to \mathbb{R}^3$ be an FTFB curve and $\{\boldsymbol{T}(y), \boldsymbol{N}(y), \boldsymbol{B}(y)\}$ be an orthonormal frame along $\boldsymbol{\beta}(y)$ in $\mathbb{R}^3$. A binormal surface M is a map $\boldsymbol{f} : Y \times U \to \mathbb{R}^3$ given by

$$\boldsymbol{f}(y,u) = \boldsymbol{\beta}(y) + u\boldsymbol{B}(y).$$

By straightforward calculations, singular points of surface M construct the set

$$S = \{(y,u) \in Y \times U \mid \iota(y) = 0, \ u\tau(y) = 0\}.$$

We can divide it into two classes S_1 and S_2, where

$$S_1 = \{(y, 0) \in Y \times U \mid \iota(y) = 0\} \text{ and } S_2 = \{(y, u) \in Y \times U \mid \iota(y) = \tau(y) = 0\}.$$

From above, we know the points of S_1 are located on the $\boldsymbol{\beta}$ and the points of S_2 construct a ruling of $\boldsymbol{f}$. By direct calculations, we have

$$det\left(\frac{\partial \boldsymbol{f}}{\partial u}(y,u), \frac{\partial^2 \boldsymbol{f}}{\partial u \partial y}(y,u), \frac{\partial^2 \boldsymbol{f}}{\partial y^2}(y,u)\right) = \tau(y)(i(y) + u\kappa(y)\tau(y)).$$

Therefore, we get the following conclusions.

Theorem 3. *Let $\boldsymbol{f}(y,u)$ be the binormal surface of the FTFB curve $\boldsymbol{\beta}(y)$. If $(y_1, 0) \in S_1$ and $\tau(y_1)i(y_1) \neq 0$, then $\boldsymbol{f}(y,u)$ is a cross-cap at the singular point $(y_1, 0)$.*

When $\boldsymbol{\beta}$ is a Mannheim mate of an FTFB curve, we know $\tau(y) \neq 0$ for all $y \in Y$ (see [19]). Therefore, the singular points of the binormal surface of $\boldsymbol{\beta}$ are only located on $\boldsymbol{\beta}$. Next, we consider the characters of these singular points.

Corollary 2. *Let $\boldsymbol{\beta}(y)$ be the Mannheim mate of an FTFB curve and $\boldsymbol{f}(y,u)$ be the binormal surface of $\boldsymbol{\beta}(y)$. If y_1 is the ordinary cusp singularity of $\boldsymbol{\beta}(y)$, then $\boldsymbol{f}(y,u)$ is a cross-cap at a singular point $(y_1, 0)$ of the surface.*

If there exist singular points on the binormal surface of the FTFB curve, then we assume that the binormal surface is a framed base surface.

For a binormal surface, $\boldsymbol{f}(y,u) = \boldsymbol{\beta}(y) + u\boldsymbol{B}(y)$ if there exist two smooth functions $\theta, \phi : Y \times U \to \mathbb{R}^3$ such that $\boldsymbol{f}_y(y,u)\boldsymbol{f}_1(y,u) = 0$, where $\boldsymbol{f}_1(y,u) = \cos\theta(y,u)\boldsymbol{T}(y) + \sin\theta(y,u)\boldsymbol{N}(y)$ and $\boldsymbol{f}_2(y,u) = \cos\phi(y,u)\boldsymbol{B}(y) + \sin\phi(y,u)(\cos\theta(y,u)\boldsymbol{N}(y) - \sin\theta(y,u)\boldsymbol{T}(y))$. Then we get the framed surface $(\boldsymbol{f}, \boldsymbol{f}_1, \boldsymbol{f}_2) : Y \times U \to \mathbb{R}^3 \times \triangle$, where $\triangle = \{(\boldsymbol{f}_1, \boldsymbol{f}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \boldsymbol{f}_1 \cdot \boldsymbol{f}_2 = 0\}$. The integrability condition is

$$F(y,u) = \tau(y)\sin\theta(y,u) + \theta(y,u)(\iota(y)\sin\theta(y,u) + y\tau(y)cos\theta(y,u)) = 0.$$

By calculations, the curvature of the surface $\boldsymbol{f}(y,u)$ is $C_f = (C_{1f}, C_{2f}, C_{3f})$, where

$$C_{1f} = -u\tau\cos\theta - \iota\sin\theta, \;\; C_{2f} = -\tau\theta_u\sin\theta, \;\; C_{3f} = -\frac{1}{2}(\kappa + \theta_y).$$

By using the criterion of the type of singular points of a framed surface in [15], we obtain the following conclusions.

Theorem 4. *Let $\boldsymbol{f}$ be the binormal surface of an FTFB curve $\boldsymbol{\beta}$. We assume that $(\boldsymbol{f}, \boldsymbol{f}_1, \boldsymbol{f}_2) : Y \times U \to \mathbb{R}^3 \times \triangle$ is a framed surface.*

(A) *Suppose that (y_2, u_2) is a singular point of $\boldsymbol{f}(y,u)$ with $\theta_y(y_2,u_2) + \kappa(y_2) \neq 0$, then $\boldsymbol{f}(y,u)$ is a cuspidal edge at (y_2, u_2) if and only if*

(1) $(y_2, u_2) \in S_2$, $\sin(y_2, u_2)i(y_2) \neq 0$ *or*

(2) $(y_2, u_2) \in S_2$, $\sin(y_2, u_2) = 0$, $i(y_2) = 0$, $u_2\dot{\tau}(y_2) \neq 0$.

(B) *Suppose that (y_2, u_2) is a singular point of $\boldsymbol{f}(y,u)$ with $\theta_y(y_2,u_2) + \kappa(y_2) = 0$, $\theta_{yu}(y_2,u_2) \neq 0$, then $\boldsymbol{f}(y,u)$ is a cuspidal cross-cap at (y_2, u_2) if and only if*

(3) $(y_2, u_2) \in S_2$, $\sin(y_2, u_2)i(y_2) \neq 0$ *or*

(4) $(y_2, u_2) \in S_2$, $\sin(y_2, u_2) = 0$, $i(y_2) = 0$, $u_2\dot{\tau}(y_2) \neq 0$.

Proof. Since $d\boldsymbol{f} = \boldsymbol{f}_y dy + \boldsymbol{f}_u du = (\iota\boldsymbol{T} - u\tau\boldsymbol{N})dy + \boldsymbol{B}du$, the null vector field $\boldsymbol{\eta}$ is $\partial/\partial y$. Suppose $(y_2, u_2) \in S_2$ is a non-degenerate singular point of $\boldsymbol{f}(y,u)$. Since $F_y(y_2, u_2) = 0$, then (y_2, u_2) should satisfy one of the following conditions:

(a) $\imath(y_2) \neq 0$ and $\cos\theta(y_2, u_2) = \dfrac{u_2\dot{\tau}(y_2)\sin\theta(y_2, u_2)}{\imath(y_2)}$ or

(b) $\sin\theta(y_2, u_2) = 0$, $\imath(y_2) = 0$ and $\dot{\tau}(y_2) \neq 0$.

At first, we consider case (a). Because of $\sin^2\theta + \cos^2\theta = 1$, then $\sin\theta(y_2, u_2) \neq 0$, that is, $-u_2\dot{\tau}(y_2)\cos\theta(y_2, u_2) - \imath(y_2)\sin\theta(y_2, u_2) \neq 0$. Hence, the singular curve δ is given by the form $\delta = (y(u), u)$, where y is a C^∞ function with $y(u_2) = u_2$. By a straightforward calculation,

$$\Phi = det\left((\boldsymbol{f}\circ\delta)', \boldsymbol{n}\circ\delta, d\boldsymbol{n}(\boldsymbol{\eta})\right) = \sin\theta\tau(\imath\sin\theta + u\tau\cos\theta)\frac{dy}{du} + \theta_y + \kappa,$$

and $\Phi'(u_2) = \theta_{yu}(y_2, u_2)$. Thus, we have the assertion (1), (3). Next, we consider the case (b). By the above conditions, we know that $-u_2\dot{\tau}(y_2)\cos\theta(y_2, u_2) \neq 0$. Hence, the singular curve δ can also be given by $\delta = (y(u), u)$. Therefore, we get $\Phi(u_2) = \theta_y(y_2, u_2) + \kappa(y_2)$ and $\Phi'(u_2) = \theta_{yu}(y_2, u_2)$. Thus, we have the assertion (2), (4). $\square$

Suppose $(y_1, 0) \in S_1$ is the non-degenerate singular point of $\boldsymbol{f}(y, u)$. Since the integrability condition $\sin\theta(y_1, 0)\tau(y_1) = 0$, then (y_1, u_1) satisfies one of the following conditions:

(c) $\sin\theta(y_1, u_1)\imath(y_1) \neq 0$ and $\tau(y_1) = 0$,

(d) $\sin\theta(y_1, 0) = 0$ and $\tau(y_1) \neq 0$.

From the case (c), we know that the singular point $(y_1, 0)$ also belongs to S_2. Thus, we omit it. In the case (d), because of $F_y(y_1, 0) = F_{yy}(y_1, 0) = 0$, then $\theta_y(y_1, 0) = 0$. Hence, the surface cannot be the cuspidal edge, swallowtail, or cuspidal cross-cap at such a singular point $(y_1, 0)$.

5. Ruled Invariant of Ruled Surface

In [10], Müller introduced two integral invariants that are the pitch and angle of pitch of a closed ruled surface in $\mathbb{R}^3$. In [11], Liu and Yuan wanted to generalize these conceptions to the general ruled surface. They wanted to use the directrix line and orthogonal trajectory of the ruling to define the pitch of a general ruled surface. But the directrix line is not unique. To solve this uncertainty, they assumed that the directrix line is the striction line of the surface. In [12–14], Liu et al. defined structure functions of a non-developable ruled surface in $\mathbb{R}^3$. Then they verified any non-developable ruled surface for which the directrix line is the striction line of the surface and the direction of ruling can be determined by the orthonormal transformations. They gave the geometric description of the structure functions.

In this paper, we focus on the principal normal and binormal surfaces generated by FTFB curves in $\mathbb{R}^3$. They are non-developable ruled surfaces. We want to investigate the structure functions of these surfaces and observe the geometric characterization of structure functions at singular points.

Firstly, let us introduce structure functions of the non-developable ruled surface. Let

$$\boldsymbol{\beta} = \boldsymbol{\beta}(s) : Y \to \mathbb{R}^3$$

be an FTFB curve and $\boldsymbol{\delta} = \boldsymbol{\delta}(s) : Y \to \mathbb{S}^2$ be a regular unit speed curve. We call $\boldsymbol{t}(s), \boldsymbol{\delta}(s), \boldsymbol{b}(s)$ the spherical Frenet frame of the spherical curve $\boldsymbol{\delta}(s)$ in $\mathbb{R}^3$, where the tangent vector and normal vector are $\boldsymbol{t}(s) = \dot{\boldsymbol{\delta}}(s)$ and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{\delta}(s)$, respectively. Then we get the following equation:

$$\begin{cases} \dot{\boldsymbol{t}}(s) = -\boldsymbol{\delta}(s) + \kappa_\beta(s)\boldsymbol{b}(s), \\ \dot{\boldsymbol{\delta}}(s) = \boldsymbol{t}(s), \\ \dot{\boldsymbol{b}}(s) = -\kappa_\beta(s)\boldsymbol{t}(s), \end{cases}$$

where $\kappa_\beta(s)$ is the spherical curvature function of $\boldsymbol{\delta}(s)$ in $\mathbb{R}^3$.

Under the above notations, if $\boldsymbol{\beta}$ is a striction line of $\boldsymbol{f}(s,u)$, we call

$$\boldsymbol{f}(s,u) = \boldsymbol{\beta}(s) + u\boldsymbol{\delta}(s)$$

the standard equation of the (non-developable) ruled surface in $\mathbb{R}^3$. Because $\boldsymbol{\beta}$ is the striction line, we have $\dot{\boldsymbol{\beta}}(s) = m(s)\boldsymbol{\delta}(s) + n(s)\boldsymbol{b}(s)$ with two smooth functions $m(s)$ and $n(s)$. We call $\kappa_\beta(s)$, $m(s)$, and $n(s)$ structure functions of the (non-developable) ruled surface $\boldsymbol{f}(s,u)$ in $\mathbb{R}^3$. These functions $\{\kappa_\beta(s), m(s), n(s)\}$ can determine the ruled surface $\boldsymbol{f}(s,u)$ under a transformation in $\mathbb{R}^3$.

Let $\mathbf{A}(s)$ be the orthogonal trajectory of the ruling on the surface $\boldsymbol{f}(s,u)$ passing through $(s_0, 0)$, then $\mathbf{A}(s)$ can be expressed as

$$\mathbf{A}(s) = \boldsymbol{\beta}(s) - \left[\int_{s_0}^{s} (\dot{\boldsymbol{\beta}}(s) \cdot \boldsymbol{\delta}(s))ds\right] \boldsymbol{\delta}(s).$$

We call

$$\sigma(s_0) = \lim_{\triangle s \to 0} \frac{[\mathbf{A}(s_0 + \triangle s) - \boldsymbol{\beta}(s_0 + \triangle s)]\, \boldsymbol{\delta}(s_0 + \triangle s)}{\triangle s} = -\dot{\boldsymbol{\beta}}(s_0) \cdot \boldsymbol{\delta}(s_0)$$

the pitch of the (non-developable) ruled surface $\boldsymbol{f}(s,u)$ at $\boldsymbol{\beta}(s_0)$, and $\sigma(s)$ the pitch function of the (non-developable) ruled surface $\boldsymbol{f}(s,u)$.

From the definition, we have $m(s) = -\sigma(s)$ and $n(s) = -det(\dot{\boldsymbol{\beta}}(s), \boldsymbol{\delta}(s), \dot{\boldsymbol{\delta}}(s))$. If $\sigma(s) = 0$ for any $s \in Y$, then we call $\boldsymbol{f}(s,u)$ the non-pitched ruled surface.

Next, we will use $\{\kappa_\beta(s), m(s), n(s)\}$ to characterize the surface generated by special framed base curves and describe the singular points.

Theorem 5. *Let $\boldsymbol{f}(s,u)$ be a non-pitched ruled surface with a structure equation. If structure functions satisfy equation $\kappa_\beta(n^2 + A^2) = -A\dot{n}$ with constant $A \neq 0$, then $\boldsymbol{f}(s,u)$ is the binormal surface of a Mannheim mate of an FTFB curve.*

Proof. Because $\sigma(s) = 0$, then $\boldsymbol{f}(s,u)$ is the binormal surface of $\boldsymbol{\beta}(s)$. By direct calculations, we obtain $\kappa(s) = -\kappa_\beta(s)$, $\tau(s) = -1$, $\iota(s) = n(s)$. From [19], the necessary and sufficient condition about which an FTFB curve is a Mannheim mate is

$$\kappa(\iota^2 + A^2\tau^2) = A(\iota\dot{\tau} - \dot{\iota}\tau), \ \ \tau \neq 0,$$

then we know $\kappa_\beta(n^2 + A^2) = -A\dot{n}$. $\square$

By the above assumptions, if s_0 is the singular point of Mannheim mate $\boldsymbol{\beta}$ of an FTFB curve, then $\dot{\iota}(s_0) = 0$. This means that s_0 is the A_2-singularity of $\boldsymbol{\beta}$.

Theorem 6. *Let $\boldsymbol{f}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a non-developable ruled surface and $\boldsymbol{\gamma}(s)$ be a striction line of $\boldsymbol{f}(s,u)$ such that $\|\boldsymbol{\delta}(s)\| = \|\dot{\boldsymbol{\delta}}(s)\| = 1$. If the structure functions of $\boldsymbol{f}(s,u)$ satisfy the situations*

1. $n(s) = \left[\int_c^s m(s)ds\right] \left\{tan\left[\int_c^s \kappa_\beta(s)ds\right]\right\}$,
2. $\boldsymbol{\beta}(s) = \boldsymbol{\gamma}(s) - \left[\int_c^s m(s)ds\right] \boldsymbol{\delta}(s)$,
3. $\int_c^s m(s)ds = \zeta_1 \cos^2\left[\int_c^s \kappa_\beta(s)ds\right] - \frac{\zeta_2}{2}\sin 2\left[\int_c^s \kappa_\beta(s)ds\right]$, *and*
4. $\zeta_1 \cos\left[\int_c^s \kappa_\beta(s)ds\right] - \zeta_2 \sin\left[\int_c^s \kappa_\beta(s)ds\right] \neq 0$,

where c, ζ_1, ζ_2 are constants, then $\boldsymbol{f}(s,u)$ is the principal normal surface of a Bertrand curve $\boldsymbol{\beta}(s)$ of an FTFB curve.

Proof. The derivative of $\boldsymbol{\beta}(s)$ has the form

$$\begin{aligned}\frac{d\boldsymbol{\beta}}{ds} &= \iota(s)\boldsymbol{T}(s)\\ &= \frac{d\boldsymbol{\gamma}}{ds} - m(s)\boldsymbol{\delta}(s) - \left[\int_c^s m(s)ds\right]\dot{\boldsymbol{\delta}}(s)\\ &= \frac{\displaystyle\int_c^s m(s)ds}{\cos\left(\displaystyle\int_c^s \kappa_\beta(s)ds\right)}\left\{\boldsymbol{b}(s)\sin\left(\int_c^s \kappa_\beta(s)ds\right) - \boldsymbol{t}(s)\cos\left(\int_c^s \kappa_\beta(s)ds\right)\right\}.\end{aligned}$$

Put $\boldsymbol{T}(s) = \boldsymbol{b}(s)\sin\left(\int_c^s \kappa_\beta(s)ds\right) - \boldsymbol{t}(s)\cos\left(\int_c^s \kappa_\beta(s)ds\right)$, then

$$\iota(s) = \frac{\displaystyle\int_c^s m(s)ds}{\cos\left(\displaystyle\int_c^s \kappa_\beta(s)ds\right)}.$$

Continue taking the derivative of $\boldsymbol{T}(s)$, and we get

$$\frac{d\boldsymbol{T}}{ds} = \kappa(s)\boldsymbol{N}(s) = \boldsymbol{\delta}(s)\cos\left(\int_c^s \kappa_\beta(s)ds\right).$$

Therefore, $\boldsymbol{f}(s,u)$ is the principal normal surface of $\boldsymbol{\beta}(s)$.

Put $\boldsymbol{N}(s) = \boldsymbol{\delta}(s)$, then $\kappa(s) = \cos\left(\int_c^s \kappa_\beta(s)ds\right)$. Since the cross-product of $\boldsymbol{T}(s)$ and $\boldsymbol{N}(s)$ is $\boldsymbol{B}(s)$, then we have

$$\frac{d\boldsymbol{B}}{ds} = -\tau(s)\boldsymbol{N}(s) = \boldsymbol{\delta}(s)\sin\left(\int_c^s \kappa_\beta(s)ds\right).$$

Then $\tau(s) = -\sin\left(\int_c^s \kappa_\beta(s)ds\right)$. Hence, the condition

$$\begin{cases}\displaystyle\int_c^s m(s)ds = \zeta_1\cos^2\left[\int_c^s \kappa_\beta(s)ds\right] - \frac{\zeta_2}{2}\sin 2\left[\int_c^s \kappa_\beta(s)ds\right],\\ \zeta_1\cos\left[\displaystyle\int_c^s \kappa_\beta(s)ds\right] - \zeta_2\sin\left[\displaystyle\int_c^s \kappa_\beta(s)ds\right] \neq 0,\end{cases}$$

is equal to

$$\iota(s) = \zeta_1\kappa(s) + \zeta_2\tau(s),\ \ \zeta_2\kappa(s) - \zeta_1\tau(s) \neq 0.$$

Then we complete the proof. □

From the proof, we know that if s_0 is the singular point of a Bertrand curve of an FTFB curve, then $\int_c^{s_0}\kappa_\beta(s)ds = 0$ and $n(s_0) = 0$. Using the same method, we can describe the binormal surface of the Mannheim curve by using $\{\kappa_\beta(s), m(s), n(s)\}$.

Theorem 7. *Let $f(s,u) = \gamma(s) + u\delta(s)$ be a non-pitched ruled surface with a structure equation. If structure functions satisfy*

$$n(s) = \left[\int_c^s m(s)ds\right]\left\{tan\left[\int_c^s \kappa_\beta(s)ds\right]\right\}, \ \boldsymbol{\beta}(s) = \gamma(s) - \left(\int_c^s m(s)ds\right)\delta(s),$$

then $f(s,u)$ is the binormal surface of a Mannheim curve $\boldsymbol{\beta}(s)$ of an FTFB curve.

Therefore, we know that if $\int_c^{s_0} \kappa_\beta(s)ds = 0$, then s_0 is the singular point of the Mannheim mate of $\boldsymbol{\beta}(s)$.

6. Example

We give an example of the principal normal and binormal surface of an FTFB curve in Euclidean 3-space. Then we can observe the singularity type on these ruled surfaces.

Example 1. *Let $\boldsymbol{\beta} : (\frac{\pi}{4}, \frac{5\pi}{4}) \to \mathbb{R}^3$ be*

$$\boldsymbol{\beta}(t) = \left(\int \sqrt{3}\sin(y+\frac{\pi}{4})\sin(y)\boldsymbol{a}(y)dy\right) - \left(\int \sqrt{3}\sin(y+\frac{\pi}{4})\sin(y)\boldsymbol{b}(y)dy\right),$$

where

$$\begin{cases} \boldsymbol{a}(y) = (\frac{3}{4}\cos y - \frac{1}{4}\cos 3y, \frac{3}{4}\sin y - \frac{1}{4}\sin 3y, \frac{\sqrt{3}}{2}\cos y), \\ \boldsymbol{b}(y) = (\frac{3}{4}\sin y - \frac{1}{4}\sin 3y, -\frac{3}{4}\cos y - \frac{1}{4}\cos 3y, -\frac{\sqrt{3}}{2}\sin y). \end{cases}$$

By straighting calculations, we have $\iota(y) = \sqrt{6}\sin(y+\frac{\pi}{4})\sin(y)$, $\kappa(y) = \sqrt{3}\sin(y-\frac{\pi}{4})$, $\tau(y) = \sqrt{3}\sin(y+\frac{\pi}{4})$, and

$$\begin{aligned} \boldsymbol{T}(y) &= \frac{\sqrt{2}}{2}(\mathbf{a}(y) - \mathbf{b}(y)), \\ \boldsymbol{N}(y) &= (\frac{\sqrt{3}}{2}\cos 2y, \frac{\sqrt{3}}{2}\sin 2y, -\frac{1}{2}), \\ \boldsymbol{B}(y) &= -\frac{\sqrt{2}}{2}(\mathbf{a}(y) + \mathbf{b}(y)). \end{aligned}$$

Let $\boldsymbol{f}_{(\beta,N)}(y,u)$ and $\boldsymbol{f}_{(\beta,B)}(y,u)$ be the principal normal and binormal surfaces of the FTFB curve $\boldsymbol{\beta}$, respectively. Then the sets of singularities of $\boldsymbol{f}_{(\beta,N)}(y,u)$ and $\boldsymbol{f}_{(\beta,B)}(y,u)$ are

$$\begin{aligned} S_{(\beta,N)} &= \{(\frac{3\pi}{4},0),(\pi,0)\}, \\ S_{(\beta,B)} &= \{(\frac{3\pi}{4},u),(\pi,0)\}, u \in \mathbb{R}. \end{aligned}$$

By Theorem 1, $\boldsymbol{f}_{(\beta,N)}(y,u)$ has the cross-cap singularity at $(\pi,0)$ (Figure 1). By Theorems 3 and 4, $\boldsymbol{f}_{(\beta,B)}(y,u)$ has the cross-cap singularity at $(\pi,0)$ and has cuspidal edge singularities at $(\frac{3\pi}{4},u), u \neq 0$ (Figure 2).

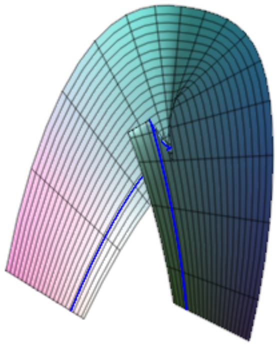

Figure 1. $\beta(y)$ and $f_{(\beta,N)}(y,u)$.

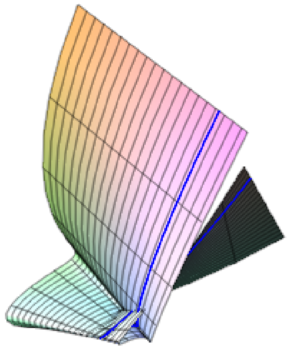

Figure 2. $\beta(y)$ and $f_{(\beta,B)}(y,u)$.

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References

1. Papaioannou, S.G.; Kiritsis, D. An application of Bertrand curves and surfaces to CAD/CAM. *Comput. Aided Des.* **1985**, *17*, 348–352. [CrossRef]
2. Ravani, B.; Ku, T.S. Bertrand offsets of ruled and developable surfaces. *Comput. Aided Des.* **1991**, *23*, 145–152. [CrossRef]
3. Cleave, J.P. The form of the tangent-developable at points of zero torsion on space curves. *Math. Proc. Camb. Philos. Soc.* **1980**, *88*, 403–407. [CrossRef]
4. Ishikawa, G. Determinacy of the envelope of the osculating hyperplanes to a curve. *Bull. Lond. Math. Soc.* **1993**, *25*, 603–610. [CrossRef]
5. Hananoi, S.; Izumiya, S. Normal developable surfaces of surfaces along curves. *Proc. R. Soc. Edinb. Sect. A* **2017**, *147*, 177–203. [CrossRef]
6. Tanaka, J.; Ohmoto, T. Geometric algebra and singularities of ruled and developable surfaces. *arXiv* **2018**, arXiv:1808.10587.
7. Zhao, Q.; Pei, D.; Wang, Y. Singularities for One-Parameter Developable Surfaces of Curves. *Symmetry* **2019**, *11*, 108. [CrossRef]
8. Izumiya, S.; Takeuchi, N. Singularities of ruled surfaces in $\mathbf{R}^3$. *Math. Proc. Camb. Philos. Soc.* **2001**, *130*, 1–11. [CrossRef]

9. Izumiya, S.; Takeuchi, N. Geometry of ruled surfaces. In *Applicable Mathematics in the Golden Age*; Misra, J.C., Ed.; Narosa Publishing House: New Delhi, India, 2003; pp. 305–338.
10. Müller, H.R. Über geschlossene Bewegungsvorgänge. *Monatsh. Math.* **1951**, *55*, 206–214. [CrossRef]
11. Liu, H.; Yuan, Y. Pitch functions of ruled surfaces and B-scrolls in Minkowski 3-space. *J. Geom. Phys.* **2012**, *62*, 47–52. [CrossRef]
12. Liu, H.; Yu, Y.; Jung, S.D. Invariants of non-developable ruled surfaces in Euclidean 3-space. *Beitr. Algebra Geom.* **2014**, *55*, 189–199. [CrossRef]
13. Yu, Y.; Liu, H.; Jung, S.D. Structure and characterization of ruled surfaces in Euclidean 3-space. *Appl. Math. Comput.* **2014**, *233*, 252–259. [CrossRef]
14. Liu, H.; Liu, Y.; Jung, S.D. Ruled invariants and associated ruled surfaces of a space curve. *Appl. Math. Comput.* **2019**, *348*, 479–486. [CrossRef]
15. Fukunaga, T.; Takahashi, M. Framed surfaces in the Euclidean space. *Bull. Braz. Math. Soc. New Ser.* **2019**, *50*, 37–65. [CrossRef]
16. Honda, S. Rectifying developable surfaces of framed base curves and framed helices. *Adv. Stud. Pure Math.* **2018**, *78*, 273–292.
17. Huang, J.; Pei, D. Singularities of special curves in 3-space forms. **2019**, In Preprint.
18. Izumiya, S.; Romero Fuster, M.C.; Ruas, M.A.S.; Tari, F. *Differential Geometry from a Singularity Theory Viewpoint*; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2016; pp. 15–21.
19. Huang, J.; Chen, L.; Izumiya, S.; Pei, D. Geometry of special curves and surfaces in 3-space form. *J. Geom. Phys.* **2019**, *136*, 31–38. [CrossRef]