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Fundamental Questions and New Counterexamples for *b*-Metric Spaces and Fatou Property

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Abstract: In this paper, we give new examples to show that the continuity actually strictly stronger than the Fatou property in *b*-metric spaces. We establish a new fixed point theorem for new essential and fundamental sufficient conditions such that a Ćirić type contraction with contraction constant $\lambda \in [\frac{1}{s}, 1)$ in a complete *b*-metric space with $s > 1$ have a unique fixed point. Many new examples illustrating our results are also given. Our new results extend and improve many recent results and they are completely original and quite different from the well known results on the topic in the literature.

Keywords: *b*-metric space; Fatou property; Ćirić fixed point theorem; Banach contraction principle; Kannan's fixed point theorem; Chatterjea's fixed point theorem

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1. Introduction

The notion of *b*-metric space, introduced by Bakhtin [1] (see also Czerwik [2]), is one of interesting generalizations of standard metric spaces. Later, in 1998, Czerwik improved and generalized this notion in [3] from the constant $s = 2$ to a constant $s \geq 1$. In the last years, a lot of fixed point results in the framework of *b*-metric space were studied by many authors, see e.g., [4–12] and references therein.

What follows we recall the notion of *b*-metric space.

Definition 1 (see [3]). *Let W be a nonempty set and $s \geq 1$, a given real number. A map $\rho : W \times W \to [0, \infty)$ is called a b-metric on W, if for any $x, y, z \in W$,*

(b1) $\rho(x, y) = 0$ *if and only if* $x = y$;
(b2) $\rho(x, y) = \rho(y, x)$;
(b3) $\rho(x, y) \leq s[\rho(x, z) + \rho(z, y)]$.

In this case, the pair (W, ρ) is called a b-metric space.

Clearly, every metric space is a *b*-metric space, but the converse is not true, see [3]. The basic topological properties (convergence, completeness, continuity, etc.) in *b*-metric spaces have been observed by the mimic of the standard metric versions as follows.

Definition 2 (see [3]). *Let (W, ρ) be a b-metric space and $\{z_n\}$ be a sequence in W. Then*

(i) $\{z_n\}$ *is said to converge to* $z \in W$ *if* $\lim_{n\to\infty} \rho(z_n, z) = 0$;
(ii) $\{z_n\}$ *is called Cauchy if* $\lim_{m,n\to\infty} \rho(z_m, z_n) = 0$;
(iii) (W, ρ) *is said to be complete if every Cauchy sequence converges.*

Definition 3. *Let (W, d_W) and (M, d_M) be two b-metric spaces. A map $T : W \to M$ is called continuous at $z \in W$ if*

$$\lim_{n\to\infty} d_M(Tz_n, Tz) = 0,$$

whenever $\{z_n\} \subset W$ with $\lim_{n\to\infty} d_W(z_n, z) = 0$.

T is called *continuous on* W if T is continuous at every point of W.

It is known that the continuity of a metric plays a crucial role in metric fixed point theory. However, it is worth mentioning that a b-metric fail to be continuous in general. So the continuity can be deemed as one of the main differences between a metric and a b-metric. In the past, some examples of b-metric spaces with discontinuous b-metrics were given, but these examples are similar to each other; see e.g., [4–6,10,13]. In 2014, Amini-Harandi [4] introduced the following notion of Fatou property for studying new fixed point results in b-metric spaces.

Definition 4 (see [4])**.** *Let (W, ρ) be a b-metric space. We say that ρ has the Fatou property if*

$$\rho(z, y) \leq \liminf_{n\to\infty} \rho(z_n, y), \tag{1}$$

whenever $\{z_n\} \subset W$ with $\lim_{n\to\infty} \rho(z_n, z) = 0$ and any $y \in W$.

It is obvious that the continuity implies the Fatou property. Some examples of b-metric spaces with Fatou property were provided, see [4] (Examples 2.5 and 2.6). However these given b-metrics are still continuous. In fact, ref [4] (Example 2.3) does not enjoy Fatou property, so it certainly fails to have continuity. Thus the following problem arises from the relationship between Fatou property and continuity.

Question 1. *Is the continuity actually strictly stronger than the Fatou property? In other words, does there exist an example that a b-metric is discontinuous as well as satisfying Fatou property?*

Let (W, ρ) be a b-metric space and $T : W \to W$ be a selfmap. A point x in W is a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively. Recall that a selfmap $T : W \to W$ is called

(i) a *Banach type contraction*, if there exists a nonnegative number $\lambda < 1$ such that

$$\rho(Tx, Ty) \leq \lambda\rho(x, y) \text{ for all } x, y \in W.$$

In this case, λ is called the contraction constant of T.

(ii) a *Kannan type contraction*, if there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that

$$\rho(Tx, Ty) \leq \lambda(\rho(x, Tx) + \rho(y, Ty)) \text{ for all } x, y \in W.$$

In this case, λ is called the contraction constant of T.

(iii) *a Chatterjea type contraction*, if there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that

$$\rho(Tx, Ty) \leq \gamma(\rho(x, Ty) + \rho(y, Tx)) \text{ for all } x, y \in W.$$

In this case, λ is called the contraction constant of T.

(iv) *a Ćirić type contraction*, if there exists a nonnegative number $\lambda < 1$ such that

$$\rho(Tx, Ty) \leq \lambda \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(Tx, y)\}$$

for all $x, y \in W$. In this case, λ is called the contraction constant of T.

It is worth mentioning that Banach type contraction, Kannan type contraction and Chatterjea type contraction are independent and different from each other in general and they are all Ćirić type contractions. In 1974, Ćirić established the following famous fixed point theorem (so-called Ćirić fixed point theorem [14]) in the setting of metric spaces (i.e., b-metric space with $s = 1$).

Theorem 1 (see Ćirić [14])**.** *Let (W, ρ) be a complete metric space (i.e., b-metric space with $s = 1$) and $T : W \to W$ be a Ćirić type contraction with contraction constant $\lambda \in [0, 1)$. Then T admits a unique fixed point in W.*

Clearly, Ćirić fixed point theorem is an actually generalization of the Banach contraction principle [15], Kannan's fixed point theorem [16] and Chatterjea's fixed point theorem [17]. Due to the importance and application potential to quantitative sciences, the generalizations of Ćirić fixed point theorem have been investigated heavily by many authors in various distinct directions over the past 20 years; see, e.g., [4,5,8,12,18–21] and the related references therein. Recently, Amini-Harandi [4] proved a generalization of Ćirić fixed point theorem in the setting of b-metric spaces with Fatou property. Later, He et al. [8] and Zhao et al. [12] respectively improved the results of Amini-Harandi without Fatou property assumption.

Theorem 2 (see [8,12])**.** *Let (W, ρ) be a complete b-metric space with $s \geq 1$, $T : W \to W$ be a Ćirić type contraction with contraction constant $\lambda \in \left[0, \frac{1}{s}\right)$. Then T admits a unique fixed point in W.*

In fact, the ranges of the contraction constants are almost limited to $[0, \frac{1}{s})$ used in all known fixed point results for Ćirić type contractions in the setting of b-metric spaces with $s \geq 1$; see, e.g., [4,8,9,12,13]. In [6], Dung and Hang successfully generalized the Banach contraction principle from metric spaces to b-metric spaces with contraction constant $\lambda \in [0, 1)$. Unfortunately, Theorem 2 is not always true if $s > 1$ and the contraction constant $\lambda \in \left[\frac{1}{s}, 1\right)$, see [6] (Theorem 2.6) and [22] (Remark 3.7). Motivated by that reason, the following question arises naturally.

Question 2. *Can we give some new essential and fundamental sufficient conditions such that a Ćirić type contraction with contraction constant $\lambda \in [\frac{1}{s}, 1)$ in a complete b-metric space with $s > 1$ have a unique fixed point?*

In this work, our questions will be answered affirmatively. In Section 2, we successfully establish one new example to show that there exists a b-metric such that it has the Fatou property as well as is discontinuous. So we prove that the continuity is strictly stronger than the Fatou property, that is a positive answer to Question 1. In Section 3, we first construct a new simple counterexample to show that Theorem 2 is not always true for $\lambda \in [\frac{1}{s}, 1)$. Furthermore, we give three sufficient conditions to demonstrate that a Ćirić type contraction with contraction constant $\lambda \in [\frac{1}{s}, 1)$ in a complete b-metric space with $s > 1$ have a unique fixed point. From this, we successfully give a complete answer to Question 2. Finally, we give three examples to show that three sufficient conditions are independent of each other. Our new results extend and improve many recent results and they are completely original and quite different from the well known results on the topic in the literature.

2. Some New Counterexamples to Answer Question 1

Now we construct two new examples that every b-metric is discontinuous. The first example tell us that there exists a b-metric such that it is discontinuous but fails to have the Fatou property. The second example shows that there exists a b-metric such that it has the Fatou property as well as is discontinuous. The two examples are completely original and quite different from these known examples in [4–6,10,13]. On the basis of these examples, we can construct many examples to answer some questions and establish new results in fixed point theory and nonlinear analysis.

Example 1. *Let $\alpha > 1$ be given. Let $W = \mathbb{R}^+ := [0, \infty)$ and*

$$\rho(x,y) = \begin{cases} |x-y|, & xy \neq 0, \\ \alpha|x-y|, & xy = 0. \end{cases} \quad \text{for any } x, y \in W.$$

Then the following hold:

(a) (W, ρ) *is a complete b-metric space with $s = \alpha$;*
(b) ρ *does not satisfy the Fatou property;*
(c) ρ *is discontinuous on W.*

Proof.

(a) It is obvious that (b1) and (b2) of Definition 1 are satisfied. Now we prove that (b3) holds. For any $x, y, z \in W$, let us consider the following possible cases:

Case 1. Assume that $xy \neq 0$. So $x \neq 0$ and $y \neq 0$.

- If $z \neq 0$, then

$$\rho(x,y) = |x-y| \leq |x-z| + |z-y| = \rho(x,z) + \rho(z,y) \leq \alpha[\rho(x,z) + \rho(z,y)].$$

- If $z = 0$, then

$$\rho(x,y) = |x-y| \leq |x-z| + |z-y| = \frac{1}{\alpha}[\rho(x,z) + \rho(z,y)] \leq \alpha[\rho(x,z) + \rho(z,y)].$$

Case 2. Suppose that $xy = 0$. Without loss of generality, we may assume that $x = 0$.

- If $z = 0$, then $x = z$ and (b3) holds immediately.
- If $z \neq 0$, then

$$\rho(x,y) = \alpha|x-y| \leq \alpha|x-z| + \alpha|z-y| \leq \rho(x,z) + \alpha\rho(z,y) \leq \alpha[\rho(x,z) + \rho(z,y)].$$

Hence, by Cases 1 and 2, we prove that (W, ρ) is a b-metric space with $s = \alpha$. Next, we verify the completeness of W. Let $d(x,y) = |x-y|$ for all $x, y \in W$. Then (W, d) is a complete metric space. Due to the fact that $d(x,y) \leq \rho(x,y) \leq \alpha d(x,y)$ for all $x, y \in W$, we can easily prove that (W, ρ) is complete.

(b) Let $a_n = \frac{1}{n}$, $n \in \mathbb{N}$ and $b = 1$. So, we have $a_n \neq 0$ for any $n \in \mathbb{N}$ and $a_n \to 0$ as $n \to \infty$. Since

$$\liminf_{n\to\infty} \rho(a_n, b) = \liminf_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1 < \alpha = \lim_{n\to\infty} \rho(0, b),$$

we show that ρ does not satisfy the Fatou property.

(c) The conclusion is an immediate consequence of (b).

□

The following example gives a positive answer to Question 1.

Example 2. *Let $\beta > 1$ be given. Let $W = \mathbb{R}^+$ and*

$$\rho(x,y) = \begin{cases} |x-y|, & xy \neq 0, \\ \frac{1}{\beta}|x-y|, & xy = 0. \end{cases} \quad \text{for any } x, y \in W.$$

Then the following hold:

(a) (W, ρ) *is a complete b-metric space with $s = \beta$;*
(b) ρ *has the Fatou property;*
(c) ρ *is discontinuous on W.*

Proof.

(a) Clearly, (b1) and (b2) of Definition 1 hold. To see (b3), let $x, y, z \in W$ be given. We consider the following possible cases:

Case A1. Assume that $xy \neq 0$. So $x \neq 0$ and $y \neq 0$.

- If $z \neq 0$, then
$$\rho(x,y) = |x-y| \leq |x-z| + |z-y| = \rho(x,z) + \rho(z,y) \leq \beta[\rho(x,z) + \rho(z,y)].$$
- If $z = 0$, then
$$\rho(x,y) = |x-y| \leq |x-z| + |z-y| = \beta\left[\rho(x,z) + \rho(z,y)\right].$$

Case A2. Suppose that $xy = 0$. Without loss of generality, we may assume that $x = 0$.

- If $z = 0$, then $x = z$ and (b3) holds immediately.
- If $z \neq 0$, then
$$\rho(x,y) = \frac{1}{\beta}|x-y| \leq \frac{1}{\beta}|x-z| + \frac{1}{\beta}|z-y| \leq \rho(x,z) + \rho(z,y) \leq \beta[\rho(x,z) + \rho(z,y)].$$

Hence, by Cases A1 and A2, we prove that (W, ρ) is a b-metric space with $s = \beta$. Following a similar argument as in the proof of Example 1, we can show that (W, ρ) is complete.

(b) Let $z, y \in W$ be given. Let $\{z_n\}$ be a sequence in W such that z_n converges to z. If there exists $\widehat{n} \in \mathbb{N}$ such that $z_n = z$ for all $n \geq \widehat{n}$, then $\rho(z,y) = \liminf_{n\to\infty} \rho(z_n, y)$ and (1) holds. Otherwise, we may assume that $z_n \neq z$ for all $n \in \mathbb{N}$. We consider the following three cases.

Case B1. Suppose that $zy \neq 0$. In this case, since $z_n \to z \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $z_n \neq 0$ for all $n \geq n_0$. So we have
$$\rho(z,y) = |z-y| \leq |z-z_n| + |z_n - y| = \rho(z,z_n) + \rho(z_n,y) \quad \text{for all } n \geq n_0.$$
Passing to the limit as $n \to \infty$ in the above inequality, we obtain (1).

Case B2. Assume that $z \neq 0$ and $y = 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $z_n \neq 0$ for all $n \geq n_0$. Hence we obtain
$$\rho(z,y) = \frac{1}{\beta}|z-y| \leq \frac{1}{\beta}|z-z_n| + \frac{1}{\beta}|z_n - y| = \frac{1}{\beta}\rho(z,z_n) + \rho(z_n,y) \quad \text{for all } n \geq n_0.$$
Passing to the limit as $n \to \infty$ in the above inequality, we get (1).

Case B3. Assume that $z = 0$. In this case, if $y = 0$, then $z = y$ and (1) always holds. So we suppose $y \neq 0$. Since $z_n \neq z$ for all $n \in \mathbb{N}$, we have
$$\rho(z,y) = \frac{1}{\beta}|z-y| \leq \frac{1}{\beta}|z-z_n| + \frac{1}{\beta}|z_n - y| = \rho(z,z_n) + \frac{1}{\beta}\rho(z_n,y) \quad \text{for all } n \in \mathbb{N}.$$
Passing to the limit as $n \to \infty$ in the above inequality, we obtain
$$\rho(z,y) \leq \frac{1}{\beta}\liminf_{n\to\infty} \rho(z_n,y) \leq \liminf_{n\to\infty} \rho(z_n,y).$$

Therefore, by the above three cases, we prove that ρ has the Fatou property.

(c) Let $x = 0$, $y = 1$, and $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Thus we have
$$\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1 > \rho(x,y).$$
This shows that ρ is not continuous.

□

Remark 1. *In Example 1, the b-metric is discontinuous and the b-metric space does not enjoy Fatou property. In Example 2, the b-metric space enjoys Fatou property but the b-metric ρ is not continuous. It shows that continuity is strictly stronger than Fatou property.*

3. Some New Results for Ćirić Type Contraction and Answers to Question 2

In this section, we first give a new simply counterexample to show that Theorem 2 is not always true if $s > 1$ and the contraction constant $\lambda \in \left[\frac{1}{s}, 1\right)$.

Example 3. *Let $W = \mathbb{R}^+$ and*

$$\rho(x,y) = \begin{cases} |x-y|, & xy \neq 0, \\ 2|x-y|, & xy = 0. \end{cases} \quad \text{for any } x,y \in W.$$

Let $T : W \to W$ be a map defined by

$$Tx = \begin{cases} \frac{2}{3}x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then the following hold:

(a) (W,ρ) *is a complete b-metric space with $s = 2$;*

(b) T *is a Ćirić type contraction with contraction constant $\lambda = \frac{2}{3}$ (note: $\lambda \in \left(\frac{1}{s}, 1\right)$);*

(c) T *has no fixed point in W.*

Proof. Clearly, the conclusion (c) is true and the conclusion (a) immediately follows from Example 1 with $\alpha = 2$. To see (b), we first note that $Tx \neq 0$ for any $x \in W$. So we always have

$$\rho(Tx,Ty) = |Tx - Ty| \quad \text{for all } x,y \in W.$$

Let $x,y \in W$ be given. We consider the following two possible cases.
Case 1. If $xy \neq 0$, then $x \neq 0$ and $y \neq 0$. So we have

$$\begin{aligned} \rho(Tx,Ty) =& |Tx-Ty| = \frac{2}{3}|x-y| = \lambda\rho(x,y) \\ \leq& \lambda \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \rho(x,Ty), \rho(Tx,y)\}. \end{aligned}$$

Case 2. Suppose that $xy = 0$. Then, without loss of generality, we may assume $x = 0$.

- If $y < \frac{3}{2}$, then

$$\begin{aligned} \rho(Tx,Ty) =& 1 - Ty \leq \frac{1}{2} \cdot 2(Tx - x) = \frac{1}{2}\rho(x,Tx) \\ \leq& \lambda \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \rho(x,Ty), \rho(Tx,y)\}. \end{aligned}$$

- If $y \geq \frac{3}{2}$, then

$$\begin{aligned} \rho(Tx,Ty) =& Ty - 1 \leq \frac{1}{2} \cdot 2y = \frac{1}{2}\rho(x,y) \\ \leq& \lambda \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \rho(x,Ty), \rho(Tx,y)\}. \end{aligned}$$

Hence, by Cases 1 and 2, we prove that T is a Ćirić's type contraction with contraction constant $\lambda = \frac{2}{3}$. □

The following lemmas are crucial in this paper.

Lemma 1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If $\{a_n\}$ converges, then*

$$\liminf_{n\to\infty} \max\{a_n, b_n\} \leq \max\left\{\lim_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\right\}.$$

Proof. It is obvious that if $\{x_n\}$ and $\{y_n\}$ are two convergent sequences of real numbers, then

$$\lim_{n\to\infty} \max\{x_n, y_n\} = \max\left\{\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right\}.$$

Take a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that $\lim_{k\to\infty} b_{n_k} = \liminf_{n\to\infty} b_n$. Hence we obtain

$$\begin{aligned}\max\left\{\lim_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\right\} &= \max\left\{\lim_{k\to\infty} a_{n_k}, \lim_{k\to\infty} b_{n_k}\right\}\\ &= \lim_{k\to\infty} \max\{a_{n_k}, b_{n_k}\}\\ &\geq \liminf_{n\to\infty} \max\{a_n, b_n\}.\end{aligned}$$

□

Lemma 2. *Let (W, ρ) be a b-metric space with $s \geq 1$, $T : W \to W$ be a map and $z_0 \in W$. Let $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence by $z_n = z_{n-1} = T^n z_0$ for all $n \in \mathbb{N}$. Define $\mathcal{H}$ be a subset of $\mathbb{N} \times \mathbb{N}$ by*

$$\mathcal{H} = \{(m, n) : m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \text{ and } m < n\}.$$

Define $L : \mathcal{H} \to [0, \infty)$ by

$$L(m, n) = \max\{\rho(z_i, z_j) : m \leq i < j \leq n\}.$$

If T is a Ćirić type contraction with contraction constant $\lambda \in [0, 1)$, then the following hold:

(a) $L(m+1, n) \leq \lambda L(m, n)$ *for any* $(m, n) \in \mathcal{H}$ *with* $n - m > 1$,
(b) $L(m, n) = \max\{\rho(z_m, z_p) : m < p \leq n\}$ *for any* $(m, n) \in \mathcal{H}$,
(c) There exists $M > 0$ *such that* $L(0, n) \leq M$ *for all* $n \in \mathbb{N}$.

Proof. First, we prove (a). Let $(m, n) \in \mathcal{H}$ with $n - m > 1$ be given. For any $i, j \in \mathbb{N}$ with $m + 1 \leq i < j \leq n$, since T is a Ćirić type contraction with contraction constant $\lambda \in [0, 1)$, we have

$$\begin{aligned}\rho(z_i, z_j) &= \rho(Tz_{i-1}, Tz_{j-1})\\ &\leq \lambda \max\{\rho(z_{i-1}, z_{j-1}), \rho(z_{i-1}, z_i), \rho(z_{j-1}, z_j), \rho(z_{i-1}, z_j), \rho(z_i, z_{j-1})\}\\ &\leq \lambda L(m, n),\end{aligned}$$

which implies $L(m+1, n) \leq \lambda L(m, n)$.

Next, we verify (b). Let $(m, n) \in \mathcal{H}$ be given. If $n - m = 1$, then $n = m + 1$ and hence

$$L(m, n) = \rho(z_m, z_n) = \max\{\rho(z_m, z_p) : m < p \leq n\}.$$

We now suppose that $n - m > 1$. For any $i, j \in \mathbb{N}$ with $m + 1 \leq i < j \leq n$, by (a), we obtain

$$\rho(z_i, z_j) \leq L(m+1, n) \leq \lambda L(m, n) < L(m, n).$$

Due to the last inequality, we conclude the fact that

$$L(m, n) = \max\{\rho(z_m, z_p) : m < p \leq n\}.$$

Therefore, as shown above, we prove that (b) holds.

Finally, we show the conclusion (c). Since $\lambda < 1$, there exists $q \in \mathbb{N}$ such that $\lambda^q < \frac{1}{s}$. If $L(0, n) \leq L(0, q)$ for all $n \in \mathbb{N}$, then (c) holds. Otherwise, if $L(0, n_q) > L(0, q)$ for some $n_q \in \mathbb{N}$, then there exists integer $p \leq n_q$ such that $L(0, n_q) = \rho(z_0, z_p)$ and $p > q$. Using (a), we have

$$\begin{aligned}\rho(z_0, z_p) &\leq s\rho(z_0, z_q) + s\rho(z_q, z_p)\\ &\leq s\rho(z_0, z_q) + sL(q, n_q)\\ &\leq s\rho(z_0, z_q) + s\lambda L(q-1, n_q)\\ &\leq \cdots\\ &\leq s\rho(z_0, z_q) + s\lambda^q L(0, n_q),\end{aligned}$$

which deduces $L(0,n_q) \leq \frac{s}{1-s\lambda^q}\rho(z_0,z_q)$. Let

$$M := \max\left\{L(0,q), \frac{s}{1-s\lambda^q}\rho(z_0,z_q)\right\}.$$

Hence, we obtain $L(0,n) \leq M$ for all $n \in \mathbb{N}$. The proof is completed. □

Lemma 3. *Let (W,ρ) be a b-metric space with $s \geq 1$ and $T : W \to W$ be a Ćirić type contraction with contraction constant $\lambda \in [0,1)$. Then for each $x \in W$, $\{T^n x\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in W (here, $T^0 = I$ is the identity map).*

Proof. Let $z_0 \in W$ be given. Let $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence defined by $z_n = Tz_{n-1} = T^n z_0$ for all $n \in \mathbb{N}$. We claim that $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is Cauchy in W. Define $L : \mathcal{H} \to [0,\infty)$ by

$$L(m,n) = \max\{\rho(z_i,z_j) : m \leq i < j \leq n\},$$

where

$$\mathcal{H} = \{(m,n) : m \in \mathbb{N}\cup\{0\}, n \in \mathbb{N} \text{ and } m < n\}.$$

For any $m,n \in \mathbb{N}$ and $m < n$, by applying Lemma 2, we have

$$\begin{aligned}
\rho(z_m,z_n) &\leq L(m,n) \\
&\leq \lambda L(m-1,n) \\
&\leq \cdots \\
&\leq \lambda^m L(0,n) \\
&\leq \lambda^m M \qquad \text{for some } M > 0.
\end{aligned}$$

Since $\lambda < 1$, the last inequalities imply that

$$\lim_{m,n\to\infty} \rho(z_m,z_n) = 0,$$

which show that $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is Cauchy in W. □

By Lemmas 1 and 3, we establish the following new fixed point theorem for Ćirić type contractions in a complete b-metric space. This new fixed point theorem gives a positive answer to Question 2. Notice that the conclusion (a) in Theorem 3 is actually the original Ćirić fixed point theorem (i.e., Theorem 1), but we give a new proof by using Lemma 3 for the sake of completeness and the readers convenience.

Theorem 3. *Let (W,ρ) be a complete b-metric space with $s \geq 1$ and $T : W \to W$ be a Ćirić type contraction with contraction constant $\lambda \in [0,1)$. Then the following hold:*

(a) *If $s = 1$, then T admits a unique fixed point v in W and the sequence $\{T^n z\}_{n\in\mathbb{N}\cup\{0\}}$ converges to v for all $z \in W$.*
(b) *If $s > 1$ and one of the following conditions is satisfied:*
   - $(D1)$ *T is continuous,*
   - $(D2)$ *ρ satisfies the Fatou property,*
   - $(D3)$ $\lambda \in \left[0, \frac{1}{s}\right)$,

   then T admits a unique fixed point v in W and the sequence $\{T^n z\}_{n\in\mathbb{N}\cup\{0\}}$ converges to v for all $z \in W$.

Proof. Given $z_0 \in W$ and let $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence defined by $z_n = Tz_{n-1} = T^n z$ for all $n \in \mathbb{N}$. Applying Lemma 3, $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in W. By the completeness of W, there exists $v \in W$ such that $z_n \to v$ as $n \to \infty$.

(a). Assume that $s=1$. Then (W,ρ) is a complete metric space. Since T is a Ćirić type contraction with contraction constant $\lambda\in[0,1)$, we have

$$\rho(z_{n+1},Tv)\leq\lambda\max\{\rho(z_n,v),\rho(z_n,z_{n+1}),\rho(v,Tv),\rho(z_n,Tv),\rho(z_{n+1},v)\}\quad\text{for all } n\in\mathbb{N}.$$

Since $z_n\to v$ as $n\to\infty$ and ρ is continuous, by taking the limit as $n\to\infty$ in the last inequality, we get

$$\rho(v,Tv)\leq\lambda\rho(v,Tv),$$

which implies $v\in\mathcal{F}(T)\neq\emptyset$. Next, we verify that $\mathcal{F}(T)$ is a singleton set. If $u\in\mathcal{F}(T)$, then we have $\rho(Tu,v)=\rho(u,Tv)=\rho(u,v)$ and $\rho(u,Tv)=\rho(v,Tv)=0$. Since

$$\begin{aligned}\rho(u,v)=&\rho(Tu,Tv)\\ \leq&\lambda\max\{\rho(u,v),\rho(u,Tu),\rho(v,Tv),\rho(u,Tv),\rho(Tu,v)\}\\ =&\lambda\rho(u,v),\end{aligned}$$

which implies $\rho(u,v)=0$ and hence $u=v$. So $\mathcal{F}(T)=\{v\}$ is a singleton set which means that T has the unique fixed point v in W. Since $z\in W$ is arbitrary given, the sequence $\{T^nz\}_{n\in\mathbb{N}\cup\{0\}}$ must converge to v.

(b). Assume that $s>1$. If (D1) holds, then, by the continuity of T, we have

$$v=\lim_{n\to\infty}z_n=\lim_{n\to\infty}Tz_{n-1}=T\left(\lim_{n\to\infty}z_{n-1}\right)=Tv.$$

If (D2) holds, since T is a Ćirić type contraction with contraction constant $\lambda\in[0,1)$, we have

$$\begin{aligned}\rho(z_{n+1},Tv)&\leq\lambda\max\{\rho(z_n,v),\rho(z_n,z_{n+1}),\rho(v,Tv),\rho(z_n,Tv),\rho(z_{n+1},v)\}\\ &\leq\lambda\max\{\max\{\rho(z_n,v),\rho(z_n,z_{n+1}),\rho(v,Tv),\rho(z_{n+1},v)\},\rho(z_n,Tv)\}\end{aligned}\tag{2}$$

for all $n\in\mathbb{N}$. Since $z_n\to v$ as $n\to\infty$and ρ has the Fatou property, we get

$$\rho(v,Tv)\leq\liminf_{n\to\infty}\rho(z_n,Tv)=\liminf_{n\to\infty}\rho(z_{n+1},Tv).\tag{3}$$

On the other hand, since

$$\lim_{n\to\infty}\rho(z_n,v)=\lim_{n\to\infty}\rho(z_n,z_{n+1})=\lim_{n\to\infty}\rho(z_{n+1},v)=0,$$

we obtain

$$\lim_{n\to\infty}\max\{\rho(z_n,v),\rho(z_n,z_{z+1}),\rho(v,Tv),\rho(z_{n+1},v)\}=\rho(v,Tv).\tag{4}$$

By applying Lemma 1 and taking into account (2)–(4), we obtain

$$\begin{aligned}\rho(v,Tv)&\leq\liminf_{n\to\infty}\rho(z_{n+1},Tv)\\ &\leq\lambda\max\left\{\lim_{n\to\infty}\max\{\rho(z_n,v),\rho(z_n,z_{n+1}),\rho(v,Tv),\rho(z_{n+1},v)\},\liminf_{n\to\infty}\rho(z_n,Tv)\right\}\\ &\leq\lambda\max\left\{\rho(v,Tv),\liminf_{n\to\infty}\rho(z_n,Tv)\right\}\\ &=\lambda\liminf_{n\to\infty}\rho(z_n,Tv).\end{aligned}$$

Hence, we know from the last inequalities that

$$\liminf_{n\to\infty}\rho(z_n,Tv)=\liminf_{n\to\infty}\rho(z_{n+1},Tv)\leq\lambda\liminf_{n\to\infty}\rho(z_n,Tv)$$

and hence

$$\rho(v,Tv)\leq\liminf_{n\to\infty}\rho(z_{n+1},Tv)=0.$$

Therefore $v\in\mathcal{F}(T)$.

Finally, suppose that (D3) holds. We first claim the inequality (5) holds, where

$$\begin{aligned}&\rho(z_{n+1},Tv)\\ \leq&\lambda\max\left\{\rho(z_n,v),\rho(z_n,z_{n+1}),\frac{\lambda s}{1-\lambda s}\rho(z_{n+1},v),\frac{\lambda s}{1-\lambda s}\rho(z_n,z_{n+1}),\rho(z_{n+1},v)\right\}\end{aligned}\tag{5}$$

for all $n \in \mathbb{N}$. Since T is a Ćirić type contraction with contraction constant $\lambda \in [0,1)$, we have

$$\begin{aligned}\rho(z_{n+1}, Tv) &= \rho(z_n, Tv)\\ &\leq \lambda \max\{\rho(z_n, v), \rho(z_n, z_{n+1}), \rho(v, Tv), \rho(z_n, Tv), \rho(z_{n+1}, v)\}\end{aligned}$$

for all $n \in \mathbb{N}$. Note that for any $n \in \mathbb{N}$, if $\rho(z_{n+1}, Tv) \leq \lambda\rho(v, Tv)$, since

$$\rho(z_{n+1}, Tv) \leq \lambda s\rho(z_{n+1}, v) + \lambda s\rho(z_{n+1}, Tv),$$

we get

$$\rho(z_{n+1}, Tv) \leq \frac{\lambda s}{1-\lambda s}\rho(z_{n+1}, v).$$

If $\rho(z_{n+1}, Tv) \leq \lambda\rho(z_n, Tv)$, then

$$\rho(z_{n+1}, Tv) \leq \lambda s\rho(z_n, z_{n+1}) + \lambda s\rho(z_{n+1}, Tv),$$

which deduces

$$\rho(z_{n+1}, Tv) \leq \frac{\lambda s}{1-\lambda s}\rho(z_n, z_{n+1}).$$

Therefore, by above, we prove (5) holds. Since $z_n \to v$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \max\left\{\rho(z_n, v), \rho(z_n, z_{n+1}), \frac{\lambda s}{1-\lambda s}\rho(z_{n+1}, v), \frac{\lambda s}{1-\lambda s}\rho(z_n, z_{n+1}), \rho(z_{n+1}, v)\right\} = 0.$$

So, by passing to the limit as $n \to \infty$ in (5), we obtain $\lim\limits_{n\to\infty} \rho(z_{n+1}, Tv) = 0$. Due to the uniqueness of the limit, we get $Tv = v$.

Following the same argument as the proof of (a), we can show that T has the unique fixed point v in W and the sequence $\{T^n z\}_{n\in\mathbb{N}\cup\{0\}}$ converges to v. The proof is completed. □

In [23] (Definition 12.7), the notion of strong b-metric space was introduced. Now we recall this notion as follows.

Definition 5 (see [23])**.** *Let W be a nonempty set and $s \geq 1$, a given real number. A mapping $\rho : W \times W \to [0,\infty)$ is called a strong b-metric on W, if for all $x, y, z \in W$,*

(b1) $\rho(x, y) = 0$ *if and only if* $x = y$;
(b2) $\rho(x, y) = \rho(y, x)$;
(b3)$_S$ $\rho(x, y) \leq \rho(x, z) + s\rho(z, y)$.

In this case, the pair (W, ρ) is called a strong b-metric space.

It is obvious that every strong b-metric space is a b-metric space and the strong b-metric is continuous (see [6] (Remark 1.7)). From Theorem 3 we obtain the following result immediately.

Corollary 1 (see [22])**.** *Let (W, ρ) be a complete strong b-metric space with $s \geq 1$, $T : W \to W$ be a map such that for some $\lambda \in [0,1)$ and all $x, y \in W$,*

$$\rho(Tx, Ty) \leq \lambda \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(Tx, y)\}.$$

Then T has a unique fixed point.

Remark 2.

(a) *From Theorem 3, we see that the ranges of contractive constants in [4] (Theorem 2.8) and [5] (Theorem 3.1) can be fully extended to $[0,1)$;*
(b) *In [5] (Theorem 2.1), Aleksic et al. studied the contraction map T satisfying*

$$\rho(Tx, Ty) \leq \lambda \max\left\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{\rho(x, Ty)}{2s}, \frac{\rho(Tx, y)}{2s}\right\}$$

for some $\lambda \in [\frac{1}{s}, 1)$ in b-metric spaces. In [6] (Theorem 2.1 and Corollary 2.4), Dung and Hang studied Banach type contractions with contraction constant $\lambda \in [0,1)$ in b-metric spaces and another kind of contraction map T satisfying

$$\rho(Tx, Ty) \le \lambda \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty)\}$$

for some $\lambda \in [0,1)$ in strong b-metric spaces. Note that the map T in [6] (Theorem 2.1) is continuous. It is easy to see that Theorem 1, [5] (Theorem 2.1) and [6] (Theorem 2.1 and Corollary 2.4) are all special cases of Theorem 3 and Corollary 1.

We now construct three examples to illustrate Theorem 3 and show the complete independence of these three conditions in Theorem 3. First, we give a example which satisfies (D2) in Theorem 3, but neither of (D1) nor (D3) is satisfied.

Example 4. *Let $W = \mathbb{R}^+$ and*

$$\rho(x,y) = \begin{cases} |x-y|, & xy \neq 0; \\ \frac{1}{2}|x-y|, & xy = 0. \end{cases} \quad \text{for any } x, y \in W.$$

Let $T : W \to W$ be a map defined by

$$Tx = \begin{cases} \frac{2}{3}x, & x \neq 2; \\ 1, & x = 2. \end{cases} \quad \text{for any } x \in W.$$

Then the following hold:

(a) *(W, ρ) is a complete b-metric space with $s = 2$;*
(b) *T has a unique fixed point in W;*
(c) *T is a Ćirić type contraction with contraction constant $\lambda = \frac{2}{3} > \frac{1}{s}$;*
(d) *ρ satisfies the Fatou property;*
(e) *T is discontinuous on W.*

Proof. By using Example 2 with $\beta = 2$, we can show (a) and (d). It is obvious that $x = 0$ is the unique fixed point for T. To see (c), we claim that

$$\rho(Tx, Ty) \le \frac{2}{3}C(x,y) \quad \text{for all } x, y \in W \tag{6}$$

where

$$C(x,y) = \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \rho(x,Ty), \rho(Tx,y)\} \text{ for } x, y \in W.$$

Let $x, y \in W$ be given. It is obvious that (6) hold if $x = y$. Then, without loss of generality, we always assume $x < y$. We consider the following four cases.

Case 1. Assume that $x = 0$ and $y > 0$. if $y \neq 2$, then $Ty = \frac{2}{3}y$. if $y = 2$, then $Ty = 1 < \frac{2}{3}y$. So in this case, we always have $Ty \le \frac{2}{3}y$. Then,

$$\rho(Tx, Ty) = \frac{1}{2}Ty \le \frac{1}{2} \cdot \frac{2}{3}y = \frac{2}{3}\rho(x,y) \le \frac{2}{3}C(x,y).$$

Case 2. If $x \in (0, \frac{3}{2})$ and $y = 2$. In this case, $Tx < \frac{2}{3} \cdot \frac{3}{2} = 1$ and $Ty = 1$. Then, we have

$$\rho(Tx, Ty) = 1 - Tx \le \frac{4}{3} - Tx = \frac{2}{3}\rho(x,y) \le \frac{2}{3}C(x,y).$$

Case 3. Suppose that $x = 2$, $y \ge \frac{3}{2}$, and $y \neq 2$. In this case, $Ty \ge 1$ and $Tx = 1$. Then, we have

$$\rho(Tx, Ty) = Ty - 1 > \frac{2}{3}y - \frac{2}{3} = \frac{2}{3}\rho(Tx, y) \le \frac{2}{3}C(x,y).$$

Case 4. If $xy > 0$, $x \neq 2$, and $y \neq 2$. In this case, we have

$$\rho(Tx, Ty) = \frac{2}{3}y - \frac{2}{3}x = \frac{2}{3}\rho(x,y) \le \frac{2}{3}C(x,y).$$

From the above two cases, we see that (6) holds and hence (c) is proved. In fact, we can also show that T has a unique fixed point by using Theorem 3. Finally, since $T2 = 1 \neq \frac{4}{3}$, we know that T is discontinuous on W. □

In the following example, (D1) in Theorem 3 holds, but neither of (D2) nor (D3) is satisfied.

Example 5. *Let $W = \mathbb{R}^+$ and*

$$\rho(x,y) = \begin{cases} |x-y|, & xy \neq 0; \\ 2|x-y|, & xy = 0. \end{cases} \quad \text{for any } x,y \in W.$$

Let $T : W \to W$ be a map defined by $Tx = \frac{2}{3}x$ for any $x \in W$. Then the following hold:

(a) (W,ρ) is a complete b-metric space with $s = 2$;
(b) T has a unique fixed point in W;
(c) T is a Ćirić type contraction with contraction constant $\lambda = \frac{2}{3} > \frac{1}{s}$;
(d) ρ fails to have the Fatou property;
(e) T is continuous on W.

Proof. By using Example 1 with $\alpha = 2$, we can show (a) and (d). Clearly, $x = 0$ is the unique fixed point for T. Let us verify (c). We claim that

$$\rho(Tx,Ty) \leq \frac{2}{3}C(x,y) \text{ for all } x,y \in W \tag{7}$$

where $C(x,y)$ is defined as in Example 4. Let $x,y \in W$ be given. We consider the following two cases.
Case a1. If $x \neq 0$ and $y \neq 0$, then $\rho(x,y) = |x-y|$. So we have

$$\rho(Tx,Ty) = \frac{2}{3}|x-y| = \frac{2}{3}\rho(x,y) \leq \frac{2}{3}C(x,y).$$

Case a2. Assume that $x = 0$ or $y = 0$. In this case, we have $\rho(x,y) = 2|x-y|$ and $\rho(Tx,Ty) = 2|Tx-Ty|$. Without loss of generality, we may assume $x = 0$. Then we obtain

$$\rho(Tx,Ty) = 2|Tx-Ty| = \frac{2}{3}\cdot 2y = \frac{2}{3}\rho(x,y) \leq \frac{2}{3}C(x,y).$$

From the above two cases, we prove (7) and hence (c) holds. Finally, we prove the continuity of T. For any $z \in W$ and sequence $\{z_n\} \subset W$ converging to z, we consider the following two cases.
Case b1. Assume that $z > 0$. In this case, since $z_n \to z$ as $n \to \infty$, there exists a integer n_0 such that $z_n > 0$ for any integer $n \geq n_0$. Thus

$$\lim_{n\to\infty} \rho(Tz_{n+n_0},Tz) = \lim_{n\to\infty} \frac{2}{3}\,|z_{n+n_0} - z| = 0.$$

Case b2. If $z = 0$, then we have $\rho(z_n,z) = 2|z_n - z|$ and $\rho(Tz_n,Tz) = 2|Tz_n - Tz|$. Hence, we get

$$\lim_{n\to\infty} \rho(Tz_n,Tz) = \lim_{n\to\infty} 2|Tz_n - Tz| = \lim_{n\to\infty} \frac{4}{3}|z_n - z| = 0.$$

From the above two cases, we show (e). □

Finally, we present an example which satisfies (D3) in Theorem 3, but neither of (D1) nor (D2) is satisfied.

Example 6. *Let $W = [0,4]$ and*

$$\rho(x,y) = \begin{cases} |x-y|, & xy \neq 0; \\ 2|x-y|, & xy = 0. \end{cases} \quad \text{for any } x,y \in W.$$

Let $T : W \to W$ be a map defined by

$$Tx = \begin{cases} \frac{1}{3}x, & x \neq 4, \\ 1, & x = 4. \end{cases} \quad \textit{for any } x \in W.$$

Then the following hold:

(a) *(W, ρ) is a complete b-metric space with $s = 2$;*
(b) *T has a unique fixed point in W;*
(c) *T is a Ćirić type contraction with contraction constant $\lambda = \frac{1}{3} < \frac{1}{s}$;*
(d) *ρ fails to have the Fatou property;*
(e) *T is discontinuous on W.*

Proof. By applying Example 1 with $\alpha = 2$, we can prove (a) and (d). Clearly, $x = 0$ is the unique fixed point for T. Let us verify (c). We claim that

$$\rho(Tx, Ty) \leq \frac{1}{3}C(x, y) \text{ for all } x, y \in W \tag{8}$$

where $C(x, y)$ is defined as in Example 4. Let $x, y \in W$ be given. We consider the following five cases.
Case 1. Following a similar argument as in Case a2 of Example 5 with replacing $\frac{2}{3}$ by $\frac{1}{3}$ and $\lambda = \frac{1}{3}$ instead of $\lambda = \frac{2}{3}$, we can prove (8) holds for $x = 0$ and $y \in [0, 4)$.
Case 2. If $x = 0$ and $y = 4$, then $Tx = 0$ and $Ty = 1$. So, we have

$$\rho(Tx, Ty) = 2(1 - 0) < \frac{8}{3} \leq \frac{1}{3}\rho(x, y) \leq \frac{1}{3}C(x, y).$$

Case 3. Following a similar argument as in Case a1 of Example 5 with replacing $\frac{2}{3}$ by $\frac{1}{3}$ and $\lambda = \frac{1}{3}$ instead of $\lambda = \frac{2}{3}$, we can show (8) holds for $x, y \in (0, 4)$.
Case 4. If $x \in (0, 3)$ and $y = 4$, then $Tx < 1$. Hence, we obtain

$$\rho(Tx, Ty) = \left(1 - \frac{1}{3}x\right) < \frac{1}{3}(4 - x) = \frac{1}{3}\rho(x, y) \leq \frac{1}{3}C(x, y).$$

Case 5. Assume that $x \in [3, 4]$ and $y = 4$. Clearly, (8) is true for $x = 4$. Now we suppose $x < 4$, which implies $1 \leq Tx < \frac{4}{3}$. So, we get

$$\rho(Tx, Ty) = \left(\frac{1}{3}x - 1\right) < \frac{1}{3}(4 - 1) = \frac{1}{3}\rho(y, Ty) \leq \frac{1}{3}C(x, y).$$

From the above five cases, we prove (8) and hence (c) holds. It is obvious that T is not continuous at $x = 4$. Hence (e) is proved. □

Remark 3. *In each of the above three examples, only one of three conditions (D1), (D2), and (D3) in Theorem 3 is satisfied. Thus, we conclude that three conditions of Theorem 3 are independent of each other.*

4. Conclusions

In this paper, we study two questions in b-metric spaces as follows:

Question 1. *Is the continuity actually strictly stronger than the Fatou property? In other words, does there exist an example that a b-metric is discontinuous as well as satisfying Fatou property?*

Question 2. *Can we give some new essential and fundamental sufficient conditions such that a Ćirić type contraction with contraction constant $\lambda \in [\frac{1}{s}, 1)$ in a complete b-metric space with $s > 1$ have a unique fixed point?*

We give new examples to show that the continuity actually strictly stronger than the Fatou property in b-metric spaces. We establish a new fixed point theorem for new essential and fundamental sufficient conditions such that a Ćirić type contraction with contraction constant $\lambda \in [\frac{1}{s}, 1)$ in a complete b-metric space with $s > 1$ have a unique fixed point. Many new examples illustrating our results are also given. Our new results extend and improve many recent results and they are completely original and quite different from the well known results on the topic in the literature.

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