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# Nonlinear Impulsive Multi-Order Caputo-Type Generalized Fractional Differential Equations with Infinite Delay

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**Abstract:** We establish sufficient conditions for the existence of solutions for a nonlinear impulsive multi-order Caputo-type generalized fractional differential equation with infinite delay and nonlocal generalized integro-initial value conditions. The existence result is proved by means of Krasnoselskii's fixed point theorem, while the contraction mapping principle is employed to obtain the uniqueness of solutions for the problem at hand. The paper concludes with illustrative examples.

**Keywords:** multi-orders fractional derivatives; impulse; caputo-type generalized fractional derivative; delay; fractional integral; existence; fixed point

## 1. Introduction

Impulsive fractional differential equation is found to serve in a number of practical applications, for example, fractal porous media [1,2], fractal petroleum [3,4], neural networks [5,6], and physiology [7–9].

Delay differential equations appear in the mathematical modeling of several real world phenomena occurring in various disciplines such as immunology [10], population dynamics [11], physiology and epidemiology [12], ecological models [13], and neural networks [14–16]. The concept of time delay relates to the duration of certain hidden processes like the time between the infection of a cell and the production of new viruses. In fact, the evolution of a delay differential system is more complex than the classical one as it relies on its current time as well as on its past stages. For further details, see [17,18].

Impulsive fractional differential equations constitute an important field of study in view of their diverse applications. These equations model the phenomena experiencing abrupt changes. Agarwal et al. [19] discussed iterative techniques for Caputo fractional differential equations with non-instantaneous impulses. Benchohra et al. [20] studied impulsive differential inclusions via a variational method. In [21], the authors investigated optimal controls involving impulsive Hilfer fractional delay evolution inclusions. Li et al. [22] derived a comparison principle for impulsive functional differential equations with infinite delays. The optimal control problem for non-instantaneous impulsive differential equations was studied in [23]. In [24], the authors discussed the approximate controllability of impulsive fractional integro-differential equation with state-dependent delay in Hilbert spaces. Zhang et al. [25] obtained extremal solutions for nonlinear multi-orders fractional impulsive differential equations. In [26], the authors introduced and investigated a nonlinear impulsive multi-order Caputo-type generalized fractional differential equation with nonlocal integro-initial conditions.

Motivated by [25,26], the objective of the present work is to derive the existence and uniqueness results for a nonlinear impulsive multi-order Caputo-type generalized fractional differential equation complemented with nonlocal generalized integro-initial value conditions and infinite delay. Precisely, we investigate the following problem:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha_k}_{t_k^+}y(t) = f(t, y_t), \ \ 1 < \alpha_k \leq 2, \ \ k = 0, 1, 2, \ldots, p, \ \ t \in J', \\ \triangle y(t_k) = S_k(y(t_k)), \quad \triangle\, \delta y(t_k) = S_k^*(y(t_k)) \ \ k = 1, 2, \ldots, p, \\ \delta y(t) = \phi(t), \ \ t \in (-\infty, 0], \\ y(0) = \sum_{k=0}^{p} \lambda_k {}^{\rho} I^{\beta_k}_{t_k^+} y(\xi_k) + \eta, \ \ t_k < \xi_k < t_{k+1}, \end{cases} \tag{1}$$

where ${}^{\rho}_{c}D^{\alpha_k}_{t_k^+}$ is the Caputo-type generalized fractional derivative of order $\alpha_k$, $\rho > 0$, ${}^{\rho} I^{\beta_k}_{t_k^+}$ is the generalized fractional integral of order $\beta_k > 0, \rho > 0$, $f \in C(J \times \mathcal{B}_T, \mathbb{R})$, $\phi \in \mathcal{B}$, $\phi(0) = 0$, $-\int_t^0 s^{\rho-1}\phi(s)ds \in \mathcal{B}$, $\mathcal{B}$ is a phase space to be defined in Section 2, $S_k$, $S_k^* \in C(\mathbb{R}, \mathbb{R})$, $\lambda_k$, $\xi_k$ are positive constants, $J = [0, T]$, $T > 0$, $\eta \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, and $\triangle y(t_k) = y(t_k^+) - y(t_k^-)$. Here $y(t_k^+)$ and $y(t_k^-)$ denote the right and left limits of $y(t)$ at $t = t_k$ $(k = 1, 2, \ldots, p)$, respectively, and $\triangle \delta y(t_k)$ have a similar meaning for $\delta y(t)$, $\delta = t^{1-\rho}\frac{d}{dt}$. We assume that $y_t : (-\infty, 0] \to \mathbb{R}$, $y_t(s) = y(t + s)$, $s \leq 0$, belong to the abstract phase space $\mathcal{B}$ and $y_t(.)$ represents the history of the state from time $-\infty$ up to the present time $t$. Here we emphasize that our problem is the delay-variant of the one studied in [26].

The rest of the content is arranged as follows. In Section 2, we recall some preliminary concepts and prove an auxiliary lemma. Section 3 is devoted to our main results and illustrative examples.

## 2. Preliminaries

Let $(\mathcal{B}, \|.\|_{\mathcal{B}})$ denote the seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following axioms due to Hale and Kato [27]:

$(B_0)$ For $y : (-\infty, T] \to \mathbb{R}$, $y_0 \in \mathcal{B}$ and for every $t \in [0, T]$, the following conditions hold:

$(i)$ $y_t$ is in $\mathcal{B}$;
$(ii)$ $\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}$;
$(iii)$ $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$, where $H \geq 0$ is a constant, $K : [0, T] \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and $H$, $K$, $M$ are independent of $y(.)$ and

$$K_T = \sup\{|K(t)| : t \in [0, T]\}, \ M_T = \sup\{|M(t)| : t \in [0, T]\} \tag{2}$$

$(B_1)$ For the function $y(.)$ in $(B_0)$, $y_t$ is a $\mathcal{B}-$valued continuous function on $[0, T]$.
$(B_2)$ The space $\mathcal{B}$ is complete.

Let us fix $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, p$ with $t_{p+1} = T$, and consider the Banach space $PC(J, \mathbb{R}) = \{y : J \to \mathbb{R} : y \in C(J_k, \mathbb{R}), k = 0, 1, \ldots, p$ and $y(t_k^+)$ and $y(t_k^-)$ exist with $y(t_k^-) = y(t_k)$, $k = 1, 2, \ldots, p\}$ with the norm $\|y\| = \sup_{t \in J} |y(t)|$, where $C(J, \mathbb{R})$ denotes the space of all continuous real valued functions on $J$, and $PC^1_{\delta}(J, \mathbb{R}) = \{y : J \to \mathbb{R} : \delta y \in PC(J, \mathbb{R})$; $\delta y(t_k^+), \delta y(t_k^-)$ exist and $\delta y$ is left continuous at $t_k$ for $k = 1, 2, \ldots, p$, $\delta = t^{1-\rho}\frac{d}{dt}\}$ is endowed with the norm $\|y\| = \sup_{t \in J}\{|y(t)|_{PC}, |\delta y(t)|_{PC}\}$.

Let the space $\mathcal{B}_T = \{y : (-\infty, T] \to \mathbb{R} : y|_{(-\infty, 0]} \in \mathcal{B}$ and $y|_{[0,T]} \in PC(J, \mathbb{R})\}$ be equipped with the seminorm defined by: $\|y\|_{\mathcal{B}_T} = \|\phi\|_{\mathcal{B}} + \sup_{s \in J} |y(s)|$, $y \in \mathcal{B}_T$.

**Definition 1** ([28]). *For $\alpha > 0$ and $\rho > 0$, the generalized fractional integral of $f \in X_c^q(a,b)$ for $-\infty < a < t < b < \infty$, is defined by*

$$({}^{\rho}I_{a+}^{\alpha} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\alpha}} f(s) ds, \tag{3}$$

*where $X_c^q(a,b)$ denotes the space of all complex-valued Lebesgue measurable functions $\varphi$ on $(a,b)$ equipped with the norm:*

$$\|\varphi\|_{X_c^q} = \Big( \int_a^b |x^c \varphi(x)|^p \frac{dx}{x} \Big)^{1/q} < \infty, \quad c \in \mathbb{R}, 1 \le q \le \infty.$$

Note that the integral in Equation (3) is called the left-sided fractional integral. Similarly we can define the right-sided fractional integral ${}^{\rho}I_{b-}^{\beta} f$ as:

$$({}^{\rho}I_{b-}^{\alpha} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{s^{\rho-1}}{(s^{\rho} - t^{\rho})^{1-\alpha}} f(s) ds. \tag{4}$$

**Definition 2** ([29]). *For $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$, the generalized fractional derivatives, associated with the generalized fractional integrals (3) and (4), are defined, for $0 \le a < x < b < \infty$, by:*

$$\begin{aligned} ({}^{\rho}D_{a+}^{\alpha} f)(t) &= \Big(t^{1-\rho} \frac{d}{dt}\Big)^n ({}^{\rho}I_{a+}^{n-\beta} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \Big(t^{1-\rho} \frac{d}{dt}\Big)^n \int_a^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{\alpha-n+1}} f(s) ds, \end{aligned}$$

*and,*

$$\begin{aligned} ({}^{\rho}D_{b-}^{\alpha} f)(t) &= \Big(-t^{1-\rho} \frac{d}{dt}\Big)^n ({}^{\rho}I_{b-}^{n-\alpha} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \Big(-t^{1-\rho} \frac{d}{dt}\Big)^n \int_t^b \frac{s^{\rho-1}}{(s^{\rho} - t^{\rho})^{\alpha-n+1}} f(s) ds, \end{aligned}$$

*if the integrals exist. In particular, when $\alpha = n$, then:*

$${}^{\rho}D_{a+}^{\alpha} f(t) = \Big(t^{1-\rho} \frac{d}{dt}\Big)^n f(t), \; {}^{\rho}D_{b-}^{\alpha} f(t) = \Big(t^{1-\rho} \frac{d}{dt}\Big)^n f(t).$$

**Definition 3** ([30]). *For $\alpha > 0, n = [\alpha] + 1$ and $f \in AC_{\delta}^n[a,b]$, the Caputo-type generalized fractional derivative ${}_c^{\rho}D_{a+}^{\alpha}$ is defined via the above generalized fractional derivative by:*

$${}_c^{\rho}D_{a+}^{\alpha} f(x) = {}^{\rho}D_{a+}^{\alpha} \Big[ f(t) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} \Big( \frac{t^{\rho} - a^{\rho}}{\rho} \Big)^k \Big](x), \; \delta = x^{1-\rho} \frac{d}{dx}.$$

*Similarly we have,*

$${}_c^{\rho}D_{b-}^{\alpha} f(x) = {}^{\rho}D_{b-}^{\alpha} \Big[ f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k f(b)}{k!} \Big( \frac{b^{\rho} - t^{\rho}}{\rho} \Big)^k \Big](x), \; \delta = x^{1-\rho} \frac{d}{dx},$$

*where $AC_{\delta}^n[a,b]$ denotes the class of all functions $f$ that have absolutely continuous $\delta^{n-1}$-derivative ($\delta^{n-1} f \in AC([a,b], \mathbb{R})$), which is equipped with the norm $\|f\|_{AC_{\delta}^n} = \sum_{k=0}^{n-1} \|\delta^k f\|_C$.*

**Remark 1** ([30]). *For $\alpha \ge 0$ and $f \in AC_{\delta}^n[a,b]$, the left and right generalized Caputo derivatives of $f$ are defined as:*

$${}_c^{\rho}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \Big( \frac{t^{\rho} - s^{\rho}}{\rho} \Big)^{n-\alpha-1} \frac{(\delta^n g)(s) ds}{s^{1-\rho}},$$

$$ {}^{\rho}_{c}D^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b \Big(\frac{s^\rho - t^\rho}{\rho}\Big)^{n-\alpha-1}\frac{(-1)^n(\delta^n g)(s)ds}{s^{1-\rho}}. $$

*when* $\alpha \notin \mathbb{N}_0$, *and,*

$$ {}^{\rho}_{c}D^{\alpha}_{a+}f(t) = \Big(t^{1-\rho}\frac{d}{dt}\Big)^n f(t),\ {}^{\rho}_{c}D^{\alpha}_{b-}f(t) = \Big(t^{1-\rho}\frac{d}{dt}\Big)^n f(t). \tag{5} $$

*for* $\alpha \in \mathbb{N}_0$. *In particular,*

$$ {}^{\rho}_{c}D^{0}_{a+}f(t) = f(t),\ {}^{\rho}_{c}D^{0}_{b-}f(t) = f(t). $$

**Lemma 1** ([30]). *Let* $f \in AC^n_\delta[a,b]$ *or* $C^n_\delta[a,b]$ *and* $\alpha \in \mathbb{R}$. *Then,*

$$ {}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}_{c}D^{\alpha}_{a+}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{(\delta^k f)(a)}{k!}\Big(\frac{x^\rho - a^\rho}{\rho}\Big)^k, $$

$$ {}^{\rho}I^{\alpha}_{b-}\ {}^{\rho}_{c}D^{\alpha}_{b-}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{(-1)^k(\delta^k f)(a)}{k!}\Big(\frac{b^\rho - x^\rho}{\rho}\Big)^k. $$

*In particular, for* $0 < \alpha \le 1$, *we have,*

$$ {}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}_{c}D^{\alpha}_{a+}f(x) = f(x) - f(a),\ {}^{\rho}I^{\alpha}_{b-}\ {}^{\rho}_{c}D^{\alpha}_{b-}f(x) = f(x) - f(b). $$

**Definition 4.** *A function* $y \in \mathcal{B}_T$ *is said to be a solution of the problem (1) if y satisfies the differential equation* ${}^{\rho}_{c}D^{\alpha}_{t_k^+}y(t) = f(t, y_t)$ *on* $J \setminus \{t_1, \dots, t_p\}$ *and the following conditions:*

$$ \begin{cases} \triangle y(t_k) = S_k(y(t_k)),\quad \triangle\,\delta y(t_k) = S^*_k(y(t_k))\ \ k = 1,2,\dots,p, \\ \delta y(t) = \phi(t),\ \ t \in (-\infty, 0], \\ y(0) = \sum_{k=0}^{p}\lambda_k\,{}^{\rho}I^{\beta_k}_{t_k^+}y(\xi_k) + \eta,\ \ t_k < \xi_k < t_{k+1}, \end{cases} $$

**Lemma 2.** *Let* $h \in C([0,T],\mathbb{R})$, $y \in PC^1_\delta(J,\mathbb{R}) \cap AC^2_\delta(J_k)$, $S_k$, $S^*_k (k = 1,2,\dots,p)$ *be constants and,*

$$ \Omega = 1 - \sum_{k=0}^{p}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} \neq 0, \tag{6} $$

*then the following impulsive integro-initial value problem with infinite delay:*

$$ \begin{cases} {}^{\rho}_{c}D^{\alpha_k}_{t_k^+}y(t) = h(t),\ \ 0 < \alpha_k \le 2,\ \ k = 0,1,2,\dots,p,\ \ t \in J', \\ \triangle y(t_k) = S_k,\quad \triangle\,\delta y(t_k) = S^*_k,\ \ k = 1,2,\dots,p, \\ \delta y(t) = \phi(t),\ \ t \in (-\infty, 0], \\ y(0) = \sum_{k=0}^{p}\lambda_k\,{}^{\rho}I^{\beta_k}_{t_k^+}y(\xi_k) + \eta,\ \ t_k < \xi_k < t_{k+1}, \end{cases} \tag{7} $$

*can be transformed into its equivalent system of integral equations:*

$$ y(t) = \begin{cases} \psi(t) + A, \quad t \in (-\infty, 0], \\ {}^{\rho}I^{\alpha_0}_{0^+}h(t) + A, \quad t \in J_0, \\ {}^{\rho}I^{\alpha_k}_{t_k^+}h(t) + \sum_{i=1}^{k}\Big[{}^{\rho}I^{\alpha_{i-1}}_{t_{i-1}^+}h(t_i) + S_i\Big] + \sum_{i=1}^{k-1}\Big(\frac{t_k^\rho - t_i^\rho}{\rho}\Big)\Big[{}^{\rho}I^{\alpha_{i-1}-1}_{t_{i-1}^+}h(t_i) + S^*_i\Big] \\ + \sum_{i=1}^{k}\Big(\frac{t^\rho - t_k^\rho}{\rho}\Big)\Big[{}^{\rho}I^{\alpha_{i-1}-1}_{t_{i-1}^+}h(t_i) + S^*_i\Big] + A, \\ t \in J_k, k = 1,2,\dots,p, \end{cases} \tag{8} $$

*where,*

$$\begin{aligned} A &= \frac{1}{\Omega}\Bigg\{ \sum_{k=0}^{p} \lambda_k {}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k} h(\xi_k) + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} h(t_i) + S_i \right] \\ &+ \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k} (t_k^{\rho} - t_i^{\rho})}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] \\ &+ \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta+2)} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] + \eta \Bigg\}, \end{aligned} \tag{9}$$

*and* $\psi(t) = -\int_t^0 s^{\rho-1}\phi(s)ds$.

**Proof.** In view of lemma 2.7 in [26] and by the given condition $\phi(0) = 0$, the solution of Equation (7) on the interval $J_k$, $k = 0, 1, \ldots, p$ is:

$$y(t) = \begin{cases} {}^{\rho}I_{0^+}^{\alpha_0} h(t) + A, \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k} h(t) + \sum_{i=1}^{k} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} h(t_i) + S_i \right] + \sum_{i=1}^{k-1} \left( \frac{t_k^{\rho} - t_i^{\rho}}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] \\ + \sum_{i=1}^{k} \left( \frac{t^{\rho} - t_k^{\rho}}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] + A, \\ t \in J_k, k = 1, 2, \ldots, p, \end{cases} \tag{10}$$

Now, we extend the solution of Equation (7) to $(-\infty, 0]$. Solving the differential equation $\delta y = \phi(t)$ and using the definition of $y$ at zero, we get,

$$y(t) = -\int_t^0 s^{\rho-1}\phi(s)ds + A,$$

which together with Equation (10) yields the solution (8). The converse follows by direct computation. This completes the proof. □

Further we introduce the following assumptions to establish our results.

$(A_1)$ There exists a constant $\mathcal{L}$, such that:

$$|f(t,\phi) - f(t,\psi)| \leq \mathcal{L}\|\phi - \psi\|_{\mathcal{B}}, \quad \text{for } t \in J \ \phi, \psi \in \mathcal{B}.$$

$(A_2)$ For each $k = 1, \ldots, p$, there exists $\mathcal{K}_1$, $\mathcal{K}_2 > 0$, such that:

$$\|S_k(x) - S_k(y)\| \leq \mathcal{K}_1\|x - y\|, \ \|S_k^*(x) - S_k^*(y)\| \leq \mathcal{K}_2\|x - y\|, \ \forall x, y \in \mathbb{R}.$$

$(A_3)$ The function $f : J \times \mathcal{B}_T \to \mathbb{R}$ is continuous and there exists a continuous function $\mu : J \to (0, \infty)$ such that $|f(t,\psi)| \leq \mu(t)$ and $\mu^* = \sup_{t\in[0,T]} \mu(t)$.

$(A_4)$ The functions $S_k : \mathbb{R} \to \mathbb{R}$, $S_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, p$ are continuous and there exists constants $M_1, M_2$ such that $\|S_k(x)\| \leq M_1$ and $\|S_k^*(x)\| \leq M_2$.

## 3. Existence and Uniqueness Results

By Lemma 2, we transform problem (1) into a fixed point problem by defining an operator $F : \mathcal{B}_T \to \mathcal{B}_T$ as:

$$(Fy)(t) = \begin{cases} \psi(t) + A, \quad t \in (-\infty, 0], \\ {}^{\rho}I_{0^+}^{\alpha_0} f(t, y_t) + A, \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k} f(t, y_t) + \sum_{i=1}^{k} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, y_{t_i}) + S_i(y(t_i)) \right] \\ + \sum_{i=1}^{k-1} \left( \frac{t_k^{\rho} - t_i^{\rho}}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y_{t_i}) + S_i^*(y(t_i)) \right] \\ + \sum_{i=1}^{k} \left( \frac{t^{\rho} - t_k^{\rho}}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y_{t_i}) + S_i^*(y(t_i)) \right] + A, \\ t \in J_k, k = 1, 2, \ldots, p, \end{cases} \tag{11}$$

where $A$ is defined by Equation (9) with $f(t, y_t)$ instead of $h(t)$. Let $x(.) : (-\infty, T] \to \mathbb{R}$ be the function defined by:

$$x(t) = \begin{cases} \psi(t) + A, \quad t \in (-\infty, 0], \\ A, \quad t \in J, \end{cases} \tag{12}$$

then $x_0 = \psi + A$. For each $z \in C([0, T], \mathbb{R})$ with $z(0) = 0$, we denote:

$$\bar{z}(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ z(t), \quad t \in J, \end{cases} \tag{13}$$

If $y(.)$ satisfies Equation (1) then we can decompose $y(.)$ as $y(t) = x(t) + \bar{z}(t)$, which implies $y_t = x_t + \bar{z}_t$ for $t \in J$ and the function $z(.)$ satisfies:

$$z(t) = \begin{cases} {}^{\rho}I_{0^+}^{\alpha_0} f(t, x_t + \bar{z}_t), \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k} f(t, x_t + \bar{z}_t) + \sum_{i=1}^{k} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, x_{t_i} + \bar{z}_{t_i}) + S_i(x(t_i) + \bar{z}(t_i)) \right] \\ + \sum_{i=1}^{k-1} \left( \frac{t_k^{\rho} - t_i^{\rho}}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, x_{t_i} + \bar{z}_{t_i}) + S_i^*(x(t_i) + \bar{z}(t_i)) \right] \\ + \sum_{i=1}^{k} \left( \frac{t^{\rho} - t_k^{\rho}}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, x_{t_i} + \bar{z}_{t_i}) + S_i^*(x(t_i) + \bar{z}(t_i)) \right], \\ t \in J_k, \, k = 1, 2, \ldots, p. \end{cases} \tag{14}$$

Set $\mathcal{B}'_T = \{z \in \mathcal{B}_T \text{ such that } z_0 = 0\}$ and let $\|.\|_{\mathcal{B}'_T}$ be a seminorm in $\mathcal{B}'_T$ defined by:

$$\|z\|_{\mathcal{B}'_T} = \sup_{t \in J} |z(t)| + \|z_0\|_{\mathcal{B}_T} = \sup_{t \in J} |z(t)|, \; z \in \mathcal{B}'_T.$$

Thus $(\mathcal{B}'_T, \|.\|_{\mathcal{B}'_T})$ is a Banach space. Next we introduce an operator $N : \mathcal{B}'_T \to \mathcal{B}'_T$ by:

$$Nz(t) = \begin{cases} {}^{\rho}I_{0^+}^{\alpha_0} f(t, x_t + \bar{z}_t), \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k} f(t, x_t + \bar{z}_t) + \sum_{i=1}^{k} \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, x_{t_i} + \bar{z}_{t_i}) + S_i(x(t_i) + \bar{z}(t_i)) \right] \\ + \sum_{i=1}^{k-1} \left( \frac{t_k^\rho - t_i^\rho}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, x_{t_i} + \bar{z}_{t_i}) + S_i^*(x(t_i) + \bar{z}(t_i)) \right] \\ + \sum_{i=1}^{k} \left( \frac{t^\rho - t_k^\rho}{\rho} \right) \left[ {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, x_{t_i} + \bar{z}_{t_i}) + S_i^*(x(t_i) + \bar{z}(t_i)) \right], \\ t \in J_k, \; k = 1, 2, \ldots, p. \end{cases} \tag{15}$$

It is clear that the operator $F$ has a fixed point if and only if $N$ has a fixed point. For $p \geq 1$, we set:

$$\Lambda_1 = (1+p) \frac{\max_{0 \leq i \leq p} T^{\alpha_i \rho}}{\min_{0 \leq i \leq p} \{\rho^{\alpha_i} \Gamma(\alpha_i + 1)\}} + (2p-1) \frac{\max_{0 \leq i \leq p} T^{\alpha_i \rho}}{\min_{0 \leq i \leq p} \{\rho^{\alpha_i} \Gamma(\alpha_i)\}}, \tag{16}$$

$$\begin{aligned} \Lambda_2 \quad = & \frac{1}{|\Omega|} \Big\{ \sum_{k=0}^{p} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\alpha_k + \beta_k}}{\rho^{\alpha_k + \beta_k} \Gamma(\alpha_k + \beta_k + 1)} + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k} (t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}}}{\rho^{\beta_k + \alpha_{i-1}} \Gamma(\beta_k + 1) \Gamma(\alpha_{i-1} + 1)} \\ & + \sum_{k=2}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k} (t_k^\rho - t_i^\rho)(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k + \alpha_{i-1}} \Gamma(\beta_k + 1) \Gamma(\alpha_{i-1})} + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k + 1} (t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k + \alpha_{i-1}} \Gamma(\beta_k + 2) \Gamma(\alpha_{i-1})} \Big\}, \end{aligned} \tag{17}$$

$$\Lambda_3 = (2p-1) \frac{T^\rho}{\rho}, \tag{18}$$

$$\Lambda_4 = \frac{1}{|\Omega|} \Big\{ \sum_{k=1}^{p} k \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k + 1)} \Big\}, \tag{19}$$

and,

$$\Lambda_5 \quad = \quad \frac{1}{|\Omega|} \Big\{ \sum_{k=2}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k} (t_k^\rho - t_i^\rho)}{\rho^{\beta_k + 1} \Gamma(\beta_k + 1)} + \sum_{k=1}^{p} k \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k + 1}}{\rho^{\beta_k + 1} \Gamma(\beta_k + 2)} \Big\}. \tag{20}$$

In the following theorem, we prove the existence of solutions for problem (1) by applying Krasnoselskii's fixed point theorem [31].

**Lemma 3.** *(Krasnoselskii's fixed point theorem). Let $\mathcal{S}$ be a bounded, closed convex, and nonempty subset of a Banach space $X$. Let $\mathcal{P}, \mathcal{Q}$ be the operators from $\mathcal{S}$ to $X$ such that $(i)$ $\mathcal{P}x + \mathcal{Q}y \in \mathcal{S}$ whenever $x, y \in \mathcal{S}$, $(ii)$ $\mathcal{P}$ is compact and continuous, and $(iii)$ $\mathcal{Q}$ is a contraction mapping. Then there exists $\zeta \in \mathcal{S}$ such that $\zeta = \mathcal{P}\zeta + \mathcal{Q}\zeta$.*

**Theorem 1.** *Assume that the assumptions $(A_2)$, $(A_3)$, and $(A_4)$ are satisfied. Then problem (1) has at least one solution on $(-\infty, T]$, provided that:*

$$p\mathcal{K}_1 + \mathcal{K}_2 \Lambda_3 < 1, \tag{21}$$

*where $\mathcal{K}_1, \mathcal{K}_2$ are given in $(A_2)$ and $\Lambda_3$ is defined by (18).*

**Proof.** Consider $B_r = \{z \in \mathcal{B}'_T : \|z\|_{\mathcal{B}'_T} \leq r\}$ with $r > \mu^* \Lambda_1 + pM_2 + M_3 \Lambda_3$, where $\mu^*$ and $M_2$, $M_3$ are given in $(A_3)$ and $(A_4)$ respectively, and $\Lambda_1$ is defined by Equation (16). Next we define operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_r$ as follows:

$$(\mathcal{P}z)(t) = \begin{cases} {}^{\rho}I_{0^+}^{\alpha_0} f(t, x_t + \bar{z}_t), \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k} f(t, x_t + \bar{z}_t) + \sum_{i=1}^{k} {}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, x_{t_i} + \bar{z}_{t_i}) + \sum_{i=1}^{k-1} \Big(\frac{t_k^\rho - t_i^\rho}{\rho}\Big)^{\rho} I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, x_{t_i} + \bar{z}_{t_i}) \\ + \sum_{i=1}^{k} \Big(\frac{t^\rho - t_k^\rho}{\rho}\Big)^{\rho} I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, x_{t_i} + \bar{z}_{t_i}) \quad t \in J_k, k = 1, 2, \ldots, p, \end{cases}$$

and,

$$(\mathcal{Q}z)(t) = \begin{cases} 0, \quad t \in J_0, \\ \sum_{i=1}^{k} S_i(x(t_i) + \bar{z}(t_i)) + \sum_{i=1}^{k-1} \Big(\frac{t_k^\rho - t_i^\rho}{\rho}\Big) S_i^*(x(t_i) + \bar{z}(t_i)) + \sum_{i=1}^{k} \Big(\frac{t^\rho - t_k^\rho}{\rho}\Big) S_i^*(x(t_i) + \bar{z}(t_i)) \\ \quad t \in J_k, k = 1, 2, \ldots, p. \end{cases}$$

Observe that $\mathcal{P} + \mathcal{Q} = N$, where the operator $N : \mathcal{B'}_T \to \mathcal{B'}_T$ is defined by Equation (15). For $z$, $z^* \in B_r$ and $t \in J_0$, we have:

$$|\mathcal{P}z(t) + \mathcal{Q}z^*(t)| \leq \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_0 - 1} |f(t, x_s + \bar{z}_s)| ds \leq \mu^* \Big(\frac{t_1^{\rho\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0 + 1)}\Big) \leq \mu^* \Lambda_1.$$

Next, for $z$, $z^* \in B_r$ and $t \in J_k$, $k = 1, 2, \ldots, p$, we obtain:

$$\begin{aligned} & |\mathcal{P}z(t) + \mathcal{Q}z^*(t)| \\ \leq\ & \frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)} \int_{t_k}^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_k - 1} |f(t, x_s + \bar{z}_s)| ds \\ +\ & \sum_{i=1}^{k} \Big[\frac{\rho^{1-\alpha_{i-1}}}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t_i^\rho - s^\rho)^{\alpha_{i-1}-1} |f(s, x_s + \bar{z}_s)| ds + |S_i(x(t_i) + \bar{z}^*(t_i))|\Big] \\ +\ & \sum_{i=1}^{k-1} \Big(\frac{t_k^\rho - t_i^\rho}{\rho}\Big) \Big[\frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1} - 1)} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t_i^\rho - s^\rho)^{\alpha_{i-1}-2} |f(s, x_s + \bar{z}_s)| ds + |S_i^*(x(t_i) + \bar{z}^*(t_i))|\Big] \\ +\ & \sum_{i=1}^{k} \Big(\frac{t^\rho - t_k^\rho}{\rho}\Big) \Big[\frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1} - 1)} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t_i^\rho - s^\rho)^{\alpha_{i-1}-2} |f(s, x_s + \bar{z}_s)| ds + |S_i^*(x(t_i) + \bar{z}^*(t_i))|\Big] \\ \leq\ & \mu^* \Lambda_1 + pM_2 + M_3\Lambda_3. \end{aligned}$$

Thus, for $z$, $z^* \in B_r$ and $t \in J_k$, $k = 0, 1, 2, \ldots, p$, we have:

$$\|\mathcal{P}z + \mathcal{Q}z^*\|_{\mathcal{B'}_T} = \sup_{t \in J} |\mathcal{P}z(t) + \mathcal{Q}z^*(t)| \leq \mu^* \Lambda_1 + pM_2 + M_3\Lambda_4 < r,$$

which implies that $\mathcal{P}x + \mathcal{Q}y \in B_r$. Using the assumptions $(A_2)$ and Equation (21), we now show that $\mathcal{Q}$ is a contraction. For $z$, $z^* \in B_r$ and $t \in J_0$, it is clear that $\mathcal{Q}$ is contraction, where $\mathcal{Q}z(t) = 0$ for each $z \in B_r$ and $t \in J_0$. Furthermore, for $z, z^* \in B_r$ and $t \in J_k$, one can obtain:

$$\begin{aligned} \sup_{t \in J} |\mathcal{Q}z(t) - \mathcal{Q}z^*(t)| \leq\ & \sup_{t \in J} \Big\{ \sum_{i=1}^{k} |S_i(x(t_i) + \bar{z}(t_i)) - S_i(x(t_i) + \bar{z}^*(t_i))| \\ & + \sum_{i=1}^{k-1} \Big(\frac{t_k^\rho - t_i^\rho}{\rho}\Big) |S_i^*(x(t_i) + \bar{z}(t_i)) - S_i^*(x(t_i) + \bar{z}^*(t_i))| \\ & + \sum_{i=1}^{k} \Big(\frac{t^\rho - t_k^\rho}{\rho}\Big) |S_i^*(x(t_i) + \bar{z}(t_i)) - S_i^*(x(t_i) + \bar{z}^*(t_i))| \Big\} \\ \leq\ & \mathcal{K}_1 p \sup_{t \in J} |z(t) - z^*(t)| + \mathcal{K}_2(2p - 1) \frac{T^\rho}{\rho} \sup_{t \in J} |z(t) - z^*(t)| \end{aligned}$$

$$= \Big(p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3\Big) \sup_{t\in J} |z(t) - z^*(t)|.$$

Consequently, for $z, z^* \in B_r$ and $t \in J_k$, $k = 0, 1, \ldots, p$, we have:

$$\|\mathcal{Q}z - \mathcal{Q}z^*\|_{\mathcal{B}'_T} = \sup_{t\in J} |\mathcal{Q}z(t) - \mathcal{Q}z^*(t)| \leq \Big(p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3\Big)\|z - z^*\|_{\mathcal{B}'_T}.$$

Continuity of $f$ implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_r$ as:

$$\|\mathcal{P}y\| \leq \mu^*\Lambda_1.$$

In order to prove the compactness of the operator $\mathcal{P}$, let $z \in B_r$. Then, by the assumption $(A_3)$, for $\tau_1, \tau_2 \in J_0$ with $\tau_1 < \tau_2$, we have:

$$\begin{aligned} |(\mathcal{P}z)(\tau_2) - (\mathcal{P}z)(\tau_1)| &= \Big|\frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\Big[\int_0^{\tau_1} s^{\rho-1}[(\tau_2^\rho - s^\rho)^{\alpha_0-1} - (\tau_1^\rho - s^\rho)^{\alpha_0-1}]f(s, x_s + \bar{z}_s)ds \\ &+ \int_{\tau_1}^{\tau_2} s^{\rho-1}(\tau_2^\rho - s^\rho)^{\alpha_0-1} f(s, x_s + \bar{z}_s)ds\Big]\Big| \\ &\leq \frac{\mu^*}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}\Big\{2(\tau_2^\rho - \tau_1^\rho)^{\alpha_0} + |\tau_2^{\rho\alpha_0} - \tau_1^{\rho\alpha_0}|\Big\}. \end{aligned}$$

Also, for $\tau_1, \tau_2 \in J_k$, $k = 1, 2, \ldots, p$ $(\tau_1 < \tau_2)$, we get:

$$\begin{aligned} &|(\mathcal{P}z)(\tau_2) - (\mathcal{P}z)(\tau_1)| \\ = &\Big|\frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)}\Big[\int_{t_k}^{\tau_1} s^{\rho-1}[(\tau_2^\rho - s^\rho)^{\alpha_k-1} - (\tau_1^\rho - s^\rho)^{\alpha_k-1}]f(s, x_s + \bar{z}_s)ds \\ + &\int_{\tau_1}^{\tau_2} s^{\rho-1}(\tau_2^\rho - s^\rho)^{\alpha_k-1} f(s, x_s + \bar{z}_s)ds\Big] \\ &+ \sum_{i=1}^{k}\Big(\frac{(\tau_2^\rho - \tau_1^\rho)}{\rho}\Big)\Big(\frac{\rho^{2-\alpha_i}}{\Gamma(\alpha_i - 1)}\int_{t_{i-1}}^{t_i} s^{\rho-1}(t_i^\rho - s^\rho)^{\alpha_i-2} f(s, x_s + \bar{z}_s)ds\Big)\Big| \\ \leq &\frac{\mu^*}{\rho^{\alpha_k}\Gamma(\alpha_k+1)}\Big\{2(\tau_2^\rho - \tau_1^\rho)^{\alpha_k} + |(\tau_2^\rho - t_k^\rho)^{\alpha_k} - (\tau_1^\rho - t_k^\rho)^{\alpha_k}|\Big\} \\ &+\mu^*\sum_{i=1}^{k}\Big(\frac{(\tau_2^\rho - \tau_1^\rho)(t_i^\rho - t_{i-1}^\rho)^{\alpha_i-1}}{\rho^{\alpha_i}\Gamma(\alpha_i)}\Big). \end{aligned}$$

From the above inequalities, it follows that $|(\mathcal{P}y)(\tau_2) - (\mathcal{P}y)(\tau_1)| \to 0$ as $\tau_2 - \tau_1 \to 0, \forall \tau_1, \tau_2 \in J_k, k = 0, 1, \ldots, p$, independent of $z \in B_r$. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on $B_r$. Hence, by the Arzelá-Ascoli theorem, $\mathcal{P}$ is compact on $B_r$. Thus all the assumptions of Lemma 3 (Krasnoselskii's fixed point theorem) are satisfied. Therefore, by the conclusion of Lemma 3, problem (1) has at least one solution on $(-\infty, T]$. □

Our second result deals with the uniqueness of solutions of Equation (1) and relies on Banach contraction mapping principle.

**Theorem 2.** *Let $f \in C(J \times \mathcal{B}, \mathbb{R})$ and the assumptions $(A_1)$, $(A_2)$, and $(A_4)$ are satisfied. Then there exists a unique solution for problem (1) on $(-\infty, T]$ if:*

$$\mathcal{L}K_T\Lambda_1 + p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3 < 1, \tag{22}$$

*and,*

$$\mathcal{L}K_T + \mathcal{L}\mathcal{J}\Lambda_2 < 1, \tag{23}$$

*where $\mathcal{L}$ is given in $(A_1)$, $\mathcal{K}_1, \mathcal{K}_2$ are given in $(A_2)$, $\Lambda_1$, $\Lambda_2, \Lambda_3$ are respectively defined by Equations (16)–(18), and $\mathcal{J} = K_T + M_T$ ($K_T, M_T$ are given in Equation (2)).*

**Proof.** Setting $\sup_{t\in J}|f(t,0)| = M_1$, we consider the set:

$$B_{\bar{r}} = \{z \in \mathcal{B}'_T : \|z\|_{\mathcal{B}'_T} \leq \bar{r}\}$$

with:

$$\bar{r} > \frac{\mathcal{L}\Big[\sigma + M_T\|\psi\|_{\mathcal{B}}\Big]\Lambda_1 + (M_1\Lambda_1 + pM_2 + M_3\Lambda_3)(1 - \mathcal{L}\mathcal{J}\Lambda_2)}{1 - (\mathcal{L}K_T + \mathcal{L}\mathcal{J}\Lambda_2)},$$

where $\sigma = M_1\Lambda_2 + M_2\Lambda_4 + M_3\Lambda_5 + \frac{|\eta|}{|\Omega|}$, $M_2$, $M_3$ are given in $(A_4)$, $\Lambda_4$, $\Lambda_5$ are defined by Equation (19) and Equation (20), respectively, and show that $NB_{\bar{r}} \subset B_{\bar{r}}$. For $z \in B_{\bar{r}}$ and $t \in J_0$, we have:

$$\begin{aligned}
|(Nz)(t)| &= \Big|{}^{\rho}I_{0^+}^{\alpha_0} f(t, x_t + \bar{z}_t)\Big| \\
&\leq \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_0-1}[|f(s, x_s + \bar{z}_s) - f(s,0)| + |f(s,0)|]ds \\
&\leq \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_0-1}(\mathcal{L}\|x_s + \bar{z}_s\|_{\mathcal{B}} + M_1)ds \\
&\leq \Big[\mathcal{L}\Big[\bar{r}\Big(\frac{K_T}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big) + \frac{\sigma + M_T\|\psi\|_{\mathcal{B}}}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big] + M_1\Big]\Big\{\frac{t_1^{\rho\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}\Big\} \\
&\leq \mathcal{L}\Big[\bar{r}\Big(\frac{K_T}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big) + \frac{\sigma + M_T\|\psi\|_{\mathcal{B}}}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big]\Lambda_1 + M_1\Lambda_1 < \bar{r},
\end{aligned}$$

which, on taking norm for $t \in J_0$, implies that $\|Nz\| < \bar{r}$. For $t \in [0,T]$, we have:

$$\begin{aligned}
\|x_t + \bar{z}_t\|_{\mathcal{B}} &\leq \|x_t\|_{\mathcal{B}} + \|\bar{z}_t\|_{\mathcal{B}} \\
&\leq (K_T + M_T)\|A\| + M_T\|\psi\|_{\mathcal{B}} + K_T\sup\{|z(s)| : s \in [0,t]\} \\
&\leq \mathcal{J}\|A\| + M_T\|\psi\|_{\mathcal{B}} + K_T\bar{r} \\
&\leq \mathcal{J}\frac{\mathcal{L}(M_T\|\psi\|_{\mathcal{B}} + K_T\bar{r})\Lambda_2 + \sigma}{1-\mathcal{L}\mathcal{J}\Lambda_2} + M_T\|\psi\|_{\mathcal{B}} + K_T\bar{r} \\
&= \bar{r}\Big(\frac{K_T}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big) + \frac{\sigma + M_T\|\psi\|_{\mathcal{B}}}{1-\mathcal{L}\mathcal{J}\Lambda_2},
\end{aligned}$$

and,

$$\begin{aligned}
\|A\| \leq\ & \frac{1}{|\Omega|}\Bigg\{\sum_{k=0}^{p}\lambda_k{}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k}|f(\xi_k, x_{\xi_k} + \bar{z}_{\xi_k})| + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\Big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}|f(t_i, x_{t_i} + \bar{z}_{t_i})| \\
&+|S_i(x(t_i) + \bar{z}(t_i))|\Big] \\
&+\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\Big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}|f(t_i, x_{t_i} + \bar{z}_{t_i})| + |S_i^*(x(t_i) + \bar{z}(t_i))|\Big] \\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}|f(t_i, x_{t_i} + \bar{z}_{t_i})| + |S_i^*(x(t_i) + \bar{z}(t_i))|\Big] + |\eta|\Bigg\} \\
\leq\ & (\mathcal{L}\|x_t + z_t\| + M_1)\Big\{\frac{1}{|\Omega|}\Big\{\sum_{k=0}^{p}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\alpha_k+\beta_k}}{\rho^{\alpha_k+\beta_k}\Gamma(\alpha_k+\beta_k+1)} \\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1}+1)} + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1})} \\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+2)\Gamma(\alpha_{i-1})}\Big\}\Big\} + \frac{M_2}{|\Omega|}\Big\{\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{M_3}{|\Omega|}\Big\{\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k}(t_k^\rho-t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Big\}+\frac{|\eta|}{|\Omega|}\\
\leq\ &\Big[\mathcal{L}\Big((K_T+M_T)\|A\|+M_T\|\psi\|_{\mathcal{B}}+K_T\bar{r}\Big)+M_1\Big]\Lambda_2+M_2\Lambda_4+M_3\Lambda_5+\frac{|\eta|}{|\Omega|}\\
\leq\ &\mathcal{L}\mathcal{J}\Lambda_2\|A\|+\mathcal{L}(M_T\|\psi\|_{\mathcal{B}}+K_T\bar{r})\Lambda_2+\sigma\\
\leq\ &\frac{\mathcal{L}(M_T\|\psi\|_{\mathcal{B}}+K_T\bar{r})\Lambda_2+\sigma}{1-\mathcal{L}\mathcal{J}\Lambda_2}.
\end{aligned}
$$

For $z\in B_r$ and $t\in J_k$, we have:

$$
\begin{aligned}
|(Nz)(t)| &= \Big|{}^{\rho}I_{t_k^+}^{\alpha_k}f(t,x_t+\bar{z}_t)+\sum_{i=1}^{k}\Big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}f(t_i,x_{t_i}+\bar{z}_{t_i})+S_i(x(t_i)+\bar{z}(t_i))\Big]\\
&\quad+\sum_{i=1}^{k-1}\Big(\frac{t_k^\rho-t_i^\rho}{\rho}\Big)\Big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}f(t_i,x_{t_i}+\bar{z}_{t_i})+S_i^*(x(t_i)+\bar{z}(t_i))\Big]\\
&\quad+\sum_{i=1}^{k}\Big(\frac{t^\rho-t_k^\rho}{\rho}\Big)\Big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}f(t_i,x_{t_i}+\bar{z}_{t_i})+S_i^*(x(t_i)+\bar{z}(t_i))\Big]\Big|\\
&\leq \frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)}\int_{t_k}^{t}s^{\rho-1}(t^\rho-s^\rho)^{\alpha_k-1}[|f(s,x_s+\bar{z}_s)-f(s,0)|+|f(s,0)|]ds\\
&\quad+\sum_{i=1}^{k}\Big[\frac{\rho^{1-\alpha_{i-1}}}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}s^{\rho-1}(t^\rho-s^\rho)^{\alpha_{i-1}-1}[|f(s,x_s+\bar{z}_s)-f(s,0)|+|f(s,0)|]ds\\
&\quad+|S_i(x(t_i)+\bar{z}(t_i))|\Big]\\
&\quad+\sum_{i=1}^{k-1}\Big(\frac{t_k^\rho-t_i^\rho}{\rho}\Big)\Big[\frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1}-1)}\int_{t_{i-1}}^{t_i}s^{\rho-1}(t^\rho-s^\rho)^{\alpha_{i-1}-2}[|f(s,x_s+\bar{z}_s)-f(s,0)|+|f(s,0)|]ds\\
&\quad+|S_i^*(x(t_i)+\bar{z}(t_i))|\Big]\\
&\quad+\sum_{i=1}^{k}\Big(\frac{t^\rho-t_k^\rho}{\rho}\Big)\Big[\frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1}-1)}\int_{t_{i-1}}^{t_i}s^{\rho-1}(t^\rho-s^\rho)^{\alpha_{i-1}-2}[|f(s,x_s+\bar{z}_s)-f(s,0)|+|f(s,0)|]ds\\
&\quad+|S_i^*(x(t_i)+\bar{z}(t_i))|\Big]\\
&\leq \Big[\mathcal{L}\bar{r}\Big(\frac{K_T}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big)+\frac{\sigma+M_T\|\psi\|_{\mathcal{B}}}{1-\mathcal{L}\mathcal{J}\Lambda_2}+M_1\Big]\Big\{\frac{(t_{k+1}^\rho-t_k^\rho)^{\alpha_k}}{\rho^{\alpha_k}\Gamma(\alpha_k+1)}\\
&\quad+\sum_{i=1}^{k}\frac{(t_i^\rho-t_{i-1}^\rho)^{\alpha_{i-1}}}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\sum_{i=1}^{k-1}\Big(\frac{t_k^\rho-t_i^\rho}{\rho}\Big)\Big(\frac{(t_i^\rho-t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\alpha_{i-1}-1}\Gamma(\alpha_{i-1})}\Big)\\
&\quad+\sum_{i=1}^{k}\Big(\frac{t_{k+1}^\rho-t_k^\rho}{\rho}\Big)\Big(\frac{(t_i^\rho-t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\alpha_{i-1}-1}\Gamma(\alpha_{i-1})}\Big)\Big\}+kM_2+M_3\Big\{\sum_{i=1}^{k-1}\Big(\frac{t_k^\rho-t_i^\rho}{\rho}\Big)+\sum_{i=1}^{k}\Big(\frac{t_{k+1}^\rho-t_k^\rho}{\rho}\Big)\Big\}\\
&\leq \mathcal{L}\Big[\bar{r}\Big(\frac{K_T}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big)+\frac{\sigma+M_T\|\psi\|_{\mathcal{B}}}{1-\mathcal{L}\mathcal{J}\Lambda_2}\Big]\Lambda_1+M_1\Lambda_1+pM_2+M_3\Lambda_3<\bar{r}.
\end{aligned}
$$

Consequently, we get $\|Nz\|<\bar{r}$ for $t\in J_k, k=0,1,\ldots,p$. Thus $NB_{\bar{r}}\subset B_{\bar{r}}$.
Now, for $z$, $z^*\in\mathcal{B}'_T$ and $t\in J_0$, we have:

$$
\begin{aligned}
|(Nz)(t)-(Nz^*)(t)| &\leq \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_0-1}|f(t,x_s+\bar{z}_s)-f(t,x_s+\bar{z}_s^*)|ds\\
&\leq \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_0-1}\mathcal{L}\|\bar{z}_s-\bar{z}_s^*\|_{\mathcal{B}}ds\\
&\leq \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_0-1}\mathcal{L}K_T\sup_{s\in[0,t]}|z(s)-z^*(s)|ds\\
&\leq \mathcal{L}K_T\Lambda_1\sup_{s\in[0,T]}|z(s)-z^*(s)|.
\end{aligned}
$$

In a similar manner, for $t \in J_k$ , we obtain:

$$\begin{aligned}
& |(Nz)(t) - (Nz^*)(t)| \\
\leq\ & \frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)} \int_{t_k}^{t} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_k-1} |f(t, x_s + \bar{z}_s) - f(t, x_s + \bar{z}_s^*)| ds \\
& + \sum_{i=1}^{k} \Big[ \frac{\rho^{1-\alpha_{i-1}}}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_{i-1}-1} |f(s, x_s + \bar{z}_s) - f(s, x_s + \bar{z}_s^*)| ds \\
& + |S_i(x(t_i) + \bar{z}(t_i)) - S_i(x(t_i) + \bar{z}^*(t_i))| \Big] \\
& + \sum_{i=1}^{k-1} \Big( \frac{t_k^\rho - t_i^\rho}{\rho} \Big) \Big[ \frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1}-1)} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_{i-1}-2} |f(s, x_s + \bar{z}_s) - f(s, x_s + \bar{z}_s^*)| ds \\
& + |S_i^*(x(t_i) + \bar{z}(t_i)) - S_i^*(x(t_i) + \bar{z}^*(t_i))| \Big] \\
& + \sum_{i=1}^{k} \Big( \frac{t^\rho - t_k^\rho}{\rho} \Big) \Big[ \frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1}-1)} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_{i-1}-2} |f(s, x_s + \bar{z}_s) - f(s, x_s + \bar{z}_s^*)| ds \\
& + |S_i^*(x(t_i) + \bar{z}(t_i)) - S_i^*(x(t_i) + \bar{z}^*(t_i))| \Big] \\
\leq\ & \frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)} \int_{t_k}^{t} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_k-1} \mathcal{L} \|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}} ds \\
& + \sum_{i=1}^{k} \Big[ \frac{\rho^{1-\alpha_{i-1}}}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_{i-1}-1} \mathcal{L} \|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}} ds + \mathcal{K}_1 \sup_{t\in[0,T]} |z(t) - z^*(t)| \Big] \Big] \\
& + \sum_{i=1}^{k-1} \Big( \frac{t_k^\rho - t_i^\rho}{\rho} \Big) \Big[ \frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1}-1)} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_{i-1}-2} \mathcal{L} \|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}} ds + \mathcal{K}_2 \sup_{t\in[0,T]} |z(t) - z^*(t)| \Big] \\
& + \sum_{i=1}^{k} \Big( \frac{t^\rho - t_k^\rho}{\rho} \Big) \Big[ \frac{\rho^{2-\alpha_{i-1}}}{\Gamma(\alpha_{i-1}-1)} \int_{t_{i-1}}^{t_i} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_{i-1}-2} \mathcal{L} \|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}} ds + \mathcal{K}_2 \sup_{t\in[0,T]} |z(t) - z^*(t)| \Big] \\
\leq\ & \Big( \mathcal{L} K_T \Lambda_1 + p\mathcal{K}_1 + \mathcal{K}_2 \Lambda_3 \Big) \sup_{s\in[0,T]} |z(t) - z^*(t)|.
\end{aligned}$$

In consequence, for $t \in J_k, k = 0, 1, 2, \ldots, p$, we deduce that:

$$\|Nz - Nz^*\|_{\mathcal{B}'_T} = \sup_{t\in[0,T]} |(Nz)(t) - (Nz^*)(t)| \quad \leq \quad \Big( \mathcal{L} K_T \Lambda_1 + p\mathcal{K}_1 + \mathcal{K}_2 \Lambda_3 \Big) \|z - z^*\|_{\mathcal{B}'_T},$$

which, in view of Equation (23), implies that $N$ is a contraction. Thus the conclusion of the theorem follows by contraction mapping principle. □

*Examples*

(a) Let us consider the following problem:

$$\begin{cases}
{}^{1/3}_{c}D^{\alpha_k} y(t) = f(t, y_t), \quad t \in [0, 2],\ t \neq 3/2,\ \ k = 0, 1, \\
\triangle y(3/2) = \frac{1}{4} \tan^{-1} y(3/2), \quad \triangle\, \delta y(3/2) = \frac{|y(3/2)|}{12 + |y(3/2)|}, \\
\delta y(t) = \phi(t), \quad t \in (-\infty, 0], \\
y(0) = \sum_{k=0}^{1} \lambda_k \rho I_{t_k^+}^{\beta_k} y(\xi_k) + 2/3,
\end{cases} \tag{24}$$

where $\rho = 1/3$, $\alpha_0 = 6/5$, $\alpha_1 = 8/5$, $\beta_0 = 2/5$, $\beta_1 = 3/7$, $\lambda_0 = 1/4$, $\lambda_1 = 1$, $\xi_0 = 3/4$, $\xi_1 = 7/4$, $t_1 = 3/2$, $\eta = 2/3$, $p = 1$, $T = 2$, $S_1(y) = \frac{1}{4} \tan^{-1} y$, $S_1^*(y) = \frac{|y|}{12+|y|}$ and $f(t, y_t)$, $\phi(t)$ will be fixed later.

Let us define $\mathcal{B}_\omega = \{y \in C((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\omega\theta}y(\theta) \text{ exists in } \mathbb{R}\}$, where $\omega$ is a positive real constant. Clearly the space $\mathcal{B}_\omega$ satisfies the axioms of phase space with the norm $\|y\|_\omega = \sup_{-\infty<\theta\le 0} e^{\omega\theta}|y(\theta)|$, and $K = M = H = 1$.

Let $\phi(t)$ be a continuous function such that $\phi(0) = 0$ and $\lim_{t\to-\infty} e^{\omega t}\phi(t) < \infty$, $\lim_{t\to-\infty} e^{\omega t}\psi(t) < \infty$. Thus $\phi, \psi \in \mathcal{B}_\omega$. For example, one can take $\phi(t) = e^{\sqrt[3]{t}} - e^{\sqrt[3]{t}/3}$ which yields $\psi(t) = -\int_t^0 s^{-2/3}\phi(s)ds = 3(2 + e^{\sqrt[3]{t}} - 3e^{\sqrt[3]{t}/3})$. Obviously $\phi, \psi \in \mathcal{B}_\omega$ and $\phi(0) = 0$.

Using the given data, we find that $|\Omega| \approx 0.0366186289$, $\Lambda_1 \approx 21.13605055$, $\Lambda_2 \approx 85.89633432$, $\Lambda_3 \approx 3.77976315$, $\Lambda_4 \approx 14.81704598$, and $\Lambda_5 = 1.878052710$, where $\Omega$, $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, and $\Lambda_4$ are given by Equation (6) and Equations (16)–(20) respectively.

In order to illustrate Theorem 1, we consider:

$$f(t, y_t) = \frac{e^{-\omega t}}{\sqrt{400+t}}\Big(\frac{|y_t|}{|y_t|+1} + 1/2\cos t\Big), \quad (t, y_t) \in [0,2] \times \mathcal{B}_\omega, \tag{25}$$

and note that the assumptions $(A_2)$, $(A_3)$, and $(A_4)$ are satisfied with $\mathcal{K}_1 = 1/4$, $\mathcal{K}_2 = 1/12$, $M_1 = \pi/8$, $M_2 = 1/12$, and $\mu(t) = \frac{e^{-\omega t}(1+1/2\cos t)}{\sqrt{400+t}}$. Furthermore, $p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3 \approx 0.5649802625 < 1$. Thus all the conditions of Theorem 1 hold true and consequently the problem (24) with $f(t, y_t)$ given by Equation (25) has at least one solution on $(-\infty, 2]$.

Next, for illustrating Theorem 2, we take:

$$f(t, y_t) = \frac{e^{-\omega t}}{(t+15)^2}\Big(\tan^{-1} y_t + 1/8\Big), \quad (t, y_t) \in [0,2] \times \mathcal{B}_\omega. \tag{26}$$

Notice that $f$ is continuous and the conditions $(A_1)$, $(A_2)$, and $(A_4)$ are satisfied with $\mathcal{L} = 1/225$, $\mathcal{K}_1 = 1/4$, $\mathcal{K}_2 = 1/12$, $M_1 = \pi/8$, $M_2 = 1/12$. Also $\mathcal{L}K_T\Lambda_1 + p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3 \approx 0.6589182649 < 1$, and $\mathcal{L}K_T + \mathcal{L}\mathcal{J}\Lambda_2 \approx 0.7679674161 < 1$. Since the hypothesis of Theorem 2 holds true, therefore the problem (24) with $f(t, y_t)$ given by Equation (26) has a unique solution on $(-\infty, 2]$.

(b) Fixing $\alpha_k = \alpha = 2$ and $\beta_1 = 1, \beta_2 = 2$, the differential equation in Equation (24) will take the form: $\Big(t^{2/3}\frac{d}{dt}\Big)^2 y(t) = f(t, y_t)$ (see Equation (5)) and the integro-initial condition in Equation (24) will become:

$$y(0) = \frac{1}{4}\int_0^{3/4} s^{-2/3}y(s)ds + 3\int_{3/2}^{7/4} s^{-2/3}((7/4)^{1/3} - s^{1/3})y(s)ds + 2/3.$$

In this case, we have $|\Omega| \approx 0.3021864840$, $\Lambda_1 \approx 72$, $\Lambda_2 \approx 3.123430595$, $\Lambda_3 \approx 3.779763150$, $\Lambda_4 \approx 0.05424893115$, $\Lambda_5 \approx 0.003274296770$, and the conditions in Equations (21)–(23) are satisfied, that is, $p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3 \approx 0.5649802625 < 1$, $\mathcal{L}K_T\Lambda_1 + p\mathcal{K}_1 + \mathcal{K}_2\Lambda_3 \approx 0.8849802625 < 1$, $\mathcal{L}K_T + \mathcal{L}\mathcal{J}\Lambda_2 \approx 0.03220827195 < 1$. Clearly the hypotheses of Theorems 1 and 2 are satisfied with the functions defined by Equations (25) and (26) respectively. In consequence, the conclusions of Theorems 1 and 2 apply to the problem at hand.

## 4. Conclusions

We have presented the sufficient criteria for the existence and uniqueness of solutions for a nonlinear impulsive multi-order Caputo-type generalized fractional differential equation equipped with infinite delay and nonlocal generalized integro-initial value conditions. The results obtained in this paper may have potential applications in diffraction-free and self-healing optoelectronic devices. Examples include propagation properties of the fractional Schrodinger equation [32,33], parity-time symmetry in a fractional Schrodinger equation [34], light beam in a fractional Schrodinger equation [35], etc. It is imperative to note that our results specialize to new ones for an appropriate choice of the parameters involved in the problem at hand, for example, the results for a nonlinear single order

Caputo-type generalized fractional differential equation with generalized fractional integral boundary conditions can be found by taking $\alpha_k = \alpha$. Moreover, our results reduce to the ones for the infinite-delay case of the problem considered in [25] by taking $\rho = 1$. We can also extend our discussion to a 'short-memory' case as argued in [36,37].

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