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The Topological Transversality Theorem for Multivalued Maps with Continuous Selections

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Abstract: This paper considers a topological transversality theorem for multivalued maps with continuous, compact selections. Basically, this says, if we have two maps F and G with continuous compact selections and $F \cong G$, then one map being essential guarantees the essentiality of the other map.

Keywords: essential maps; homotopy; selections

MSC: 47H10; 54H25

1. Introduction

In this paper, we consider multivalued maps F and G with continuous, compact selections and $F \cong G$ in this setting. The topological transversality theorem will state that F is essential if and only if G is essential (essential maps were introduced by Granas [1] and extended by Precup [2], Gabor, Gorniewicz, and Slosarsk [3], and O'Regan [4,5]). For an approach to other classes of maps, we refer the reader to O'Regan [6], where one sees that $\cong$ in the appropriate class can be challenging. However, the topological transversality theorem for multivalued maps with continuous compact selections has not been considered in detail. In this paper, we present a simple result that immediately yields a topological transversality theorem in this setting. In particular, we show that, for two maps F and G with continuous compact selections and $F \cong G$, then one map being essential (or d–essential) guarantees that the other is essential (or d–essential). We also discuss these maps in the weak topology setting.

2. Topological Transversality Theorem

We will consider a class $\mathbf{A}$ of maps. Let E be a completely regular space (i.e., a Tychonoff space) and U an open subset of E.

Definition 1. *We say $f \in D(\overline{U}, E)$ if $f : \overline{U} \to E$ is a continuous, compact map; here, $\overline{U}$ denotes the closure of U in E.*

Definition 2. *We say $f \in D_{\partial U}(\overline{U}, E)$ if $f \in D(\overline{U}, E)$ and $x \neq f(x)$ for $x \in \partial U$; here, ∂U denotes the boundary of U in E.*

Definition 3. *We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ with $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $f \in D(\overline{U}, E)$ of F; here, 2^E denotes the family of nonempty subsets of E.*

Remark 1. *Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in PK(Z, W)$ if W is convex and there exists a map $S : Z \to W$ with $Z = \cup\{int\, S^{-1}(w) : w \in W\}$, $co\,(S(x)) \subseteq F(x)$ for $x \in Z$ and $S(x) \neq \emptyset$ for each $x \in Z$; here, $S^{-1}(w) = \{z : w \in S(z)\}$. Let E be* *a Hausdorff topological vector space (note topological vector spaces are completely regular), U an open subset of E and $\overline{U}$ paracompact. In this case, we say $F \in \mathbf{A}(\overline{U}, E)$ if $F \in PK(\overline{U}, E)$ is a compact map. Now, [7] guarantees that there exists a continuous, compact selection $f : \overline{U} \to E$ of F.*

Definition 4. *We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$.*

Definition 5. *We say $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ if for any selection $f \in D(\overline{U}, E)$ of F and any map $g \in D_{\partial U}(\overline{U}, E)$ with $f|_{\partial U} = g|_{\partial U}$ there exists a $x \in U$ with $x = g(x)$.*

Remark 2. *If $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ and if $f \in D(\overline{U}, E)$ is any selection of F then there exists a $x \in U$ with $x = f(x)$ (take $g = f$ in Definition 5), so in particular there exists a $x \in U$ with $x \in F(x)$.*

Definition 6. *Let $f, g \in D_{\partial U}(\overline{U}, E)$. We say $f \cong g$ in $D_{\partial U}(\overline{U}, E)$ if there exists a continuous, compact map $h : \overline{U} \times [0, 1] \to E$ with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0, 1)$ (here $h_t(x) = h(x, t)$), $h_0 = f$ and $h_1 = g$.*

Remark 3. *A standard argument guarantees that $\cong$ in $D_{\partial U}(\overline{U}, E)$ is an equivalence relation.*

Definition 7. *Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if for any selection $f \in D_{\partial U}(\overline{U}, E)$ (respectively, $g \in D_{\partial U}(\overline{U}, E)$) of F (respectively, of G) we have $f \cong g$ in $D_{\partial U}(\overline{U}, E)$.*

Theorem 1. *Let E be a completely regular topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$. In addition, suppose*

$$\begin{cases} \text{for any selection } f \in D_{\partial U}(\overline{U}, E) \ \ (\text{respectively, } g \in D_{\partial U}(\overline{U}, E)) \\ \text{of } F \ \ (\text{respectively, of } G) \text{ and any map } \theta \in D_{\partial U}(\overline{U}, E) \\ \text{with } \theta|_{\partial U} = f|_{\partial U} \text{ we have } g \cong \theta \text{ in } D_{\partial U}(\overline{U}, E). \end{cases} \quad (1)$$

Then, F is essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $f \in D_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = f|_{\partial U}$. We must show that there exists a $x \in U$ with $x = \theta(x)$. Let $g \in D_{\partial U}(\overline{U}, E)$ be any selection of G. Now, (1) guarantees that there exists a continuous, compact map $h : \overline{U} \times [0, 1] \to E$ with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0, 1)$ (here, $h_t(x) = h(x, t)$), $h_0 = g$ and $h_1 = \theta$. Let

$$\Omega = \left\{ x \in \overline{U} : \ x = h(x, t) \ \text{ for some } \ t \in [0, 1] \right\}.$$

Now, $\Omega \neq \emptyset$ (note G is essential in $A_{\partial U}(\overline{U}, E)$) and Ω is closed (note h is continuous) and so Ω is compact (note h is a compact map). In addition, note $\Omega \cap \partial U = \emptyset$ since $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in [0, 1]$. Then, since E is Tychonoff, there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map r by $r(x) = h(x, \mu(x)) = h \circ g(x)$, where $g : \overline{U} \to \overline{U} \times [0, 1]$ is given by $g(x) = (x, \mu(x))$. Note that $r \in D_{\partial U}(\overline{U}, E)$ (i.e., r is a continuous compact map) with $r|_{\partial U} = g|_{\partial U}$ (note if $x \in \partial U$ then $r(x) = h(x, 0) = g(x)$) so since G is essential in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $x = r(x)$ (i.e., $x = h_{\mu(x)}(x)$). Thus, $x \in \Omega$ so $\mu(x) = 1$ and thus $x = h_1(x) = \theta(x)$. □

Let E be a topological vector space. Before we prove the topological transversality theorem, we note the following:

(a) If $f, g \in D_{\partial U}(\overline{U}, E)$ with $f|_{\partial U} = g|_{\partial U}$, then $f \cong g$ in $D_{\partial U}(\overline{U}, E)$. To see this, let $h(x, t) = (1 - t) f(x) + t g(x)$ and note $h : \overline{U} \times [0, 1] \to E$ is a continuous, compact map with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0, 1)$ (note $f|_{\partial U} = g|_{\partial U}$).

Theorem 2. *Let E be a topological vector space and U an open subset of E. Suppose that F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Now, F is essential in $A_{\partial U}(\overline{U}, E)$ if and only if G is essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Assume G is essential in $A_{\partial U}(\overline{U}, E)$. We will use Theorem 1 to show F is essential in $A_{\partial U}(\overline{U}, E)$. Let $f \in D_{\partial U}(\overline{U}, E)$ be any selection of F, $g \in D_{\partial U}(\overline{U}, E)$ be any selection of G and consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = f|_{\partial U}$. Now, (a) above guarantees that $f \cong \theta$ in $D_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (so $f \cong g$ in $D_{\partial U}(\overline{U}, E)$) and Remark 3 guarantees that $g \cong \theta$ in $D_{\partial U}(\overline{U}, E)$. Thus, (1) holds so Theorem 1 guarantees that F is essential in $A_{\partial U}(\overline{U}, E)$. A similar argument shows that, if F is essential in $A_{\partial U}(\overline{U}, E)$, then G is essential in $A_{\partial U}(\overline{U}, E)$. □

Theorem 3. *Let E be a Hausdorff locally convex topological vector space, U an open subset of E and $0 \in U$. Assume the zero map is in $\mathbf{A}(\overline{U}, E)$. Then, the zero map is essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Note $F(x) = \{0\}$ for $x \in \overline{U}$ (i.e., F is the zero map) and let $f \in D_{\partial U}(\overline{U}, E)$ be any selection of F. Note $f(x) = 0$ for $x \in \overline{U}$. Consider any map $g \in D_{\partial U}(\overline{U}, E)$ with $g|_{\partial U} = f|_{\partial U} = \{0\}$. We must show there exists a $x \in U$ with $x = g(x)$. Let

$$r(x) = \begin{cases} g(x), & x \in \overline{U}, \\ 0, & x \in E \backslash \overline{U}. \end{cases}$$

Note $r : E \to E$ is a continuous, compact map so [8] guarantees that there exists a $x \in E$ with $x = r(x)$. If $x \in E \backslash U$, then $r(x) = 0$, a contradiction since $0 \in U$. Thus, $x \in U$ and so $x = g(x)$. □

Now, we consider the above in the weak topology setting. Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of C where C is a closed convex subset of X. Again, we consider a class $\mathbf{A}$ of maps.

Definition 8. *We say $f \in WD(\overline{U^w}, C)$ if $f : \overline{U^w} \to C$ is a weakly continuous, weakly compact map; here, $\overline{U^w}$ denotes the weak closure of U in C.*

Definition 9. *We say $f \in WD_{\partial U}(\overline{U^w}, C)$ if $f \in WD(\overline{U^w}, C)$ and $x \neq f(x)$ for $x \in \partial U$; here, ∂U denotes the weak boundary of U in C.*

Definition 10. *We say $F \in WA(\overline{U^w}, C)$ if $F : \overline{U^w} \to 2^C$ with $F \in \mathbf{A}(\overline{U^w}, C)$ and there exists a selection $f \in WD(\overline{U^w}, C)$ of F.*

Definition 11. *We say $F \in WA_{\partial U}(\overline{U^w}, C)$ if $F \in WA(\overline{U^w}, C)$ and $x \notin F(x)$ for $x \in \partial U$.*

Definition 12. *We say $F \in WA_{\partial U}(\overline{U^w}, C)$ is essential in $WA_{\partial U}(\overline{U^w}, C)$ if for any selection $f \in WD(\overline{U^w}, C)$ of F and any map $g \in WD_{\partial U}(\overline{U^w}, C)$ with $f|_{\partial U} = g|_{\partial U}$ there exists a $x \in U$ with $x = g(x)$.*

Definition 13. *Let $f, g \in WD_{\partial U}(\overline{U^w}, C)$. We say $f \cong g$ in $WD_{\partial U}(\overline{U^w}, C)$ if there exists a weakly continuous, weakly compact map $h : \overline{U^w} \times [0,1] \to C$ with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $h_t(x) = h(x,t)$), $h_0 = f$ and $h_1 = g$.*

Definition 14. *Let $F, G \in WA_{\partial U}(\overline{U^w}, C)$. We say $F \cong G$ in $WA_{\partial U}(\overline{U^w}, C)$ if for any selection $f \in WD_{\partial U}(\overline{U^w}, C)$ (respectively, $g \in WD_{\partial U}(\overline{U^w}, C)$) of F (respectively, of G) we have $f \cong g$ in $WD_{\partial U}(\overline{U^w}, C)$.*

Theorem 4. *Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of C, where C is a closed convex subset of X. Suppose $F \in WA_{\partial U}(\overline{U^w}, C)$ and $G \in WA_{\partial U}(\overline{U^w}, C)$ is essential in $WA_{\partial U}(\overline{U^w}, C)$ and*

$$\begin{cases} \text{for any selection } f \in WD_{\partial U}(\overline{U^w}, C) \text{ (respectively, } g \in WD_{\partial U}(\overline{U^w}, C)) \\ \text{of } F \text{ (respectively, of } G) \text{ and any map } \theta \in WD_{\partial U}(\overline{U^w}, C) \\ \text{with } \theta|_{\partial U} = f|_{\partial U} \text{ we have } g \cong \theta \text{ in } WD_{\partial U}(\overline{U^w}, C). \end{cases} \quad (2)$$

Then, F is essential in $WA_{\partial U}(\overline{U^w}, C)$.

Proof. Let $f \in WD_{\partial U}(\overline{U^w}, C)$ be any selection of F and consider any map $\theta \in WD_{\partial U}(\overline{U^w}, C)$ with $\theta|_{\partial U} = f|_{\partial U}$. Let $g \in WD_{\partial U}(\overline{U^w}, C)$ be any selection of G. Now, (2) guarantees that there exists a weakly continuous, weakly compact map $h : \overline{U^w} \times [0,1] \to C$ with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $h_t(x) = h(x,t)$), $h_0 = g$ and $h_1 = \theta$. Let

$$\Omega = \left\{ x \in \overline{U^w} : \; x = h(x,t) \; \text{ for some } \; t \in [0,1] \right\}.$$

Recall that $X = (X, w)$, the space X endowed with the weak topology, is completely regular. Now, $\Omega \neq \emptyset$ is weakly closed and is in fact weakly compact with $\Omega \cap \partial U = \emptyset$. Thus, there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map r by $r(x) = h(x, \mu(x))$ and note $r \in WD_{\partial U}(\overline{U^w}, C)$ with $r|_{\partial U} = g|_{\partial U}$. Since G is essential in $WA_{\partial U}(\overline{U^w}, C)$, there exists a $x \in U$ with $x = r(x)$. Thus, $x \in \Omega$ so $x = h_1(x) = \theta(x)$. □

An obvious modification of the argument in Theorem 2 immediately yields the following result.

Theorem 5. *Let X be a Hausdorff locally convex topological vector space and U a weakly open subset of C, where C is a closed convex subset of X. Suppose F and G are two maps in $WA_{\partial U}(\overline{U}, C)$ with $F \cong G$ in $WA_{\partial U}(\overline{U}, C)$. Now, F is essential in $WA_{\partial U}(\overline{U}, C)$ if and only if G is essential in $WA_{\partial U}(\overline{U}, C)$.*

Now, we consider a generalization of essential maps, namely the d–essential maps [2]. Let E be a completely regular topological space and U an open subset of E. For any map $f \in D(\overline{U}, E)$, let $f^\star = I \times f : \overline{U} \to \overline{U} \times E$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (f^\star)^{-1}\,(B) \right\} \cup \{\emptyset\} \;\to\; K \tag{3}$$

be any map with values in the nonempty set K; here, $B = \left\{ (x,x) : \; x \in \overline{U} \right\}$.

Definition 15. *Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d–essential if, for any selection $f \in D(\overline{U}, E)$ of F and any map $g \in D_{\partial U}(\overline{U}, E)$ with $f|_{\partial U} = g|_{\partial U}$, we have that $d\left((f^\star)^{-1}\,(B) \right) = d\left((g^\star)^{-1}\,(B) \right) \neq d(\emptyset)$; here, $f^\star = I \times f$ and $g^\star = I \times g$.*

Remark 4. *If $F^\star$ is d–essential, then, for any selection $f \in D(\overline{U}, E)$ of F (with $f^\star = I \times f$), we have*

$$\emptyset \neq (f^\star)^{-1}\,(B) = \{ x \in \overline{U} : \; (x, f(x)) \in B \},$$

so there exists a $x \in U$ with $x = f(x)$ (so, in particular, $x \in F(x)$).

Theorem 6. *Let E be a completely regular topological space, U an open subset of E, $B = \left\{ (x,x) : \; x \in \overline{U} \right\}$, d is defined in (3), $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$ and $G^\star = I \times G$. Suppose $G^\star$ is d–essential and*

$$\begin{cases} \text{for any selection } \; f \in D_{\partial U}(\overline{U}, E) \;\; (\text{respectively, } \; g \in D_{\partial U}(\overline{U}, E)) \\ \text{of } \; F \;\; (\text{respectively, of } \; G) \text{ and any map } \; \theta \in D_{\partial U}(\overline{U}, E) \\ \text{with } \; \theta|_{\partial U} = f|_{\partial U} \;\; \text{we have } \; g \cong \theta \;\; \text{in } \; D_{\partial U}(\overline{U}, E) \;\; \text{and} \\ d\left((f^\star)^{-1}\,(B) \right) = d\left((g^\star)^{-1}\,(B) \right); \text{ here } \; f^\star = I \times f \;\; \text{and} \;\; g^\star = I \times g. \end{cases} \tag{4}$$

Then, $F^\star$ is d–essential.

Proof. Let $f \in D_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = f|_{\partial U}$. We must show $d\left((f^\star)^{-1}\,(B)\right) = d\left((\theta^\star)^{-1}\,(B)\right) \neq d(\emptyset)$; here, $f^\star = I \times f$ and $\theta^\star = I \times \theta$. Let $g \in D_{\partial U}(\overline{U}, E)$ be any selection of G. Now, (4) guarantees that there exists a continuous, compact map $h : \overline{U} \times [0,1] \to E$ with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $h_t(x) = h(x,t)$), $h_0 = g$, $h_1 = \theta$ and $d\left((f^\star)^{-1}\,(B)\right) = d\left((g^\star)^{-1}\,(B)\right)$; here, $g^\star = I \times g$. Let $h^\star : \overline{U} \times [0,1] \to \overline{U} \times E$ be given by $h^\star(x,t) = (x, h(x,t))$ and let

$$\Omega = \left\{x \in \overline{U} : \; h^\star(x,t) \in B \;\text{ for some }\; t \in [0,1]\right\}.$$

Now, $\Omega \neq \emptyset$ is closed, compact and $\Omega \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map r by $r(x) = h(x, \mu(x))$ and $r^\star = I \times r$. Now, $r \in D_{\partial U}(\overline{U}, E)$ with $r|_{\partial U} = g|_{\partial U}$. Since $G^\star$ is d–essential, then

$$d\left((g^\star)^{-1}\,(B)\right) = d\left((r^\star)^{-1}\,(B)\right) \neq d(\emptyset). \tag{5}$$

Now, since $\mu(\Omega) = 1$, we have

$$\begin{aligned} (r^\star)^{-1}\,(B) &= \left\{x \in \overline{U} : \; (x, h(x, \mu(x))) \in B\right\} = \left\{x \in \overline{U} : \; (x, h(x,1)) \in B\right\} \\ &= (\theta^\star)^{-1}\,(B), \end{aligned}$$

so, from the above and Equation (5), we have $d\left((f^\star)^{-1}\,(B)\right) = d\left((\theta^\star)^{-1}\,(B)\right) \neq d(\emptyset)$. □

Theorem 7. *Let E be a completely regular topological space, U an open subset of E, $B = \left\{(x,x) : \; x \in \overline{U}\right\}$ and d is defined in (3). Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$, $G^\star = I \times G$ and $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then, $F^\star$ is d–essential if and only if $G^\star$ is d–essential.*

Proof. Assume $G^\star$ is d–essential. Let $f \in D_{\partial U}(\overline{U}, E)$ be any selection of F, $g \in D_{\partial U}(\overline{U}, E)$ be any selection of G and consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = f|_{\partial U}$. If we show (4), then $F^\star$ is d–essential from Theorem 6. Now, $f \cong \theta$ in $D_{\partial U}(\overline{U}, E)$ together with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (so $f \cong g$ in $D_{\partial U}(\overline{U}, E)$) guarantees that $g \cong \theta$ in $D_{\partial U}(\overline{U}, E)$. To complete (4), we need to show $d\left((f^\star)^{-1}\,(B)\right) = d\left((g^\star)^{-1}\,(B)\right)$; here, $f^\star = I \times f$ and $g^\star = I \times g$. We will show this by following the argument in Theorem 6. Note $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ and let $h : \overline{U} \times [0,1] \to E$ be a continuous, compact map with $x \neq h_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $h_t(x) = h(x,t)$), $h_0 = g$ and $h_1 = f$. Let $h^\star : \overline{U} \times [0,1] \to \overline{U} \times E$ be given by $h^\star(x,t) = (x, h(x,t))$ and let

$$\Omega = \left\{x \in \overline{U} : \; h^\star(x,t) \in B \;\text{ for some }\; t \in [0,1]\right\}.$$

Now, $\Omega \neq \emptyset$ and there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map r by $r(x) = h(x, \mu(x))$ and $r^\star = I \times r$. Now, $r \in D_{\partial U}(\overline{U}, E)$ with $r|_{\partial U} = g|_{\partial U}$ so, since $G^\star$ is d–essential, then $d\left((g^\star)^{-1}\,(B)\right) = d\left((r^\star)^{-1}\,(B)\right) \neq d(\emptyset)$. Now, since $\mu(\Omega) = 1$, we have (see the argument in Theorem 6) $(r^\star)^{-1}\,(B) = (f^\star)^{-1}\,(B)$ and, as a result, we have $d\left((f^\star)^{-1}\,(B)\right) = d\left((g^\star)^{-1}\,(B)\right)$. □

Remark 5. *It is also easy to extend the above ideas to other natural situations. Let E be a (Hausdorff) topological vector space (so automatically completely regular), Y a topological vector space, and U an open subset of E. In addition, let $L : dom\,L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here, $dom\,L$ is a vector subspace of E. Finally, $T : E \to Y$ will be a linear, continuous single valued map with $L + T : dom\,L \to Y$ an isomorphism (i.e., a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$.*

We say $F \in A(\overline{U}, Y; L, T)$ *if* $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ *and we could discuss essential and d–essential in this situation.*

Now, we present an example to illustrate our theory.

Example 1. *Let* E *be a Hausdorff locally convex topological vector space,* U *an open subset of* E, $0 \in U$ *and* $\overline{U}$ *paracompact. In this case, we say that* $F \in \mathbf{A}(\overline{U}, E)$ *if* $F \in PK(\overline{U}, E)$ *(see Remark 1) is a compact map. Let* $F \in A_{\partial U}(\overline{U}, E)$ *and assume* $x \notin \lambda F(x)$ *for* $x \in \partial U$ *and* $\lambda \in (0,1)$. *Then,* $F \cong 0$ *in* $A_{\partial U}(\overline{U}, E)$. *To see this, let* $f \in D_{\partial U}(\overline{U}, E)$ *be any selection of F and let* $h : \overline{U} \times [0,1]$ *be given by* $h(x,t) = t\, f(x)$. *Note that* $h_0 = 0$, $h_1 = f$ *and* $x \notin h_t(x)$ *for* $x \in \partial U$ *and* $\lambda \in (0,1)$ *so* $f \cong 0$ *in* $D_{\partial U}(\overline{U}, E)$. *Now, Theorems 2 and 3 guarantee that F is essential in* $A_{\partial U}(\overline{U}, E)$.

3. Conclusions

In this paper, we prove that, for two set-valued maps F and G with continuous compact selections and $F \cong G$, then one being essential (or d–essential) guarantees that the other is essential (or d–essential).

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