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On ω-Limit Sets of Zadeh's Extension of Nonautonomous Discrete Systems on an Interval

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Abstract: Let $I = [0, 1]$ and f_n be a sequence of continuous self-maps on I which converge uniformly to a self-map f on I. Denote by $\mathcal{F}(I)$ the set of fuzzy numbers on I, and denote by $(\mathcal{F}(I), \widehat{f})$ and $(\mathcal{F}(I), \widehat{f}_n)$ the Zadeh's extensions of (I, f) and (I, f_n), respectively. In this paper, we study the ω-limit sets of $(\mathcal{F}(I), \widehat{f}_n)$ and show that, if all periodic points of f are fixed points, then $\omega(A, \widehat{f}_n) \subset F(\widehat{f})$ for any $A \in \mathcal{F}(I)$, where $\omega(A, \widehat{f}_n)$ is the ω-limit set of A under $(\mathcal{F}(I), \widehat{f}_n)$ and $F(\widehat{f}) = \{A \in \mathcal{F}(I) : \widehat{f}(A) = A\}$.

Keywords: fuzzy number; ω-limit set; periodic point; Zadeh's extension

1. Introduction

Research for the dynamical properties of nonautonomous discrete systems on a metric space is very interesting (see [1–11]). In [12], Kempf investigated the ω-limit sets of a sequence of continuous self-maps f_n on I which converge uniformly to a self-map f on I and showed that, if $P(f) = F(f)$, then $\omega(x, f_n)$ is a closed subset of I with $\omega(x, f_n) \subset F(f)$ for any $x \in I$, where $F(f)$ and $P(f)$ are the set of fixed points of f and the set of periodic points of f, respectively, and $\omega(x, f_n)$ is the set of ω-limit points of x under (X, f_n). Further, Cánovas [13] showed that, if f_n is a sequence of continuous self-maps on I which converge uniformly to a self-map f on I and $P(f) = F(f^{2^s})$ for some $s \in \mathbb{N}$, then $\omega(x, f_n) = \cup_{k=1}^{2^s}[p_k, q_k] \subset F(f^{2^s})$ with $f([p_k, q_k]) = [p_{k+1}, q_{k+1}]$ $(1 \leq k \leq 2^s - 1)$ and $f([p_{2^s}, q_{2^s}]) = [p_1, q_1]$ for any $x \in I$. In [14], we studied the ω-limit sets of a sequence of continuous self-maps f_n on a tree T which converge uniformly to a self-map f on T and showed that, if $P(f) = F(f)$, then $\omega(x, f_n)$ is a closed connected subset of T with $\omega(x, f_n) \subset F(f)$ for any $x \in T$.

It is well known [15] that the discrete dynamical system (X, f) naturally induces a dynamical system $(\mathcal{F}(X), \widehat{f})$, where $\mathcal{F}(X)$ is the set of all fuzzy sets on a metric space X and $\widehat{f}$ is the Zadeh's extension of continuous self-maps f on X. It is natural to ask how the dynamical properties of f is related to the dynamical properties of $\widehat{f}$. Already, there are many results for this question so far; see, e.g., References [16–21] and the related references therein, where different chaotic properties and topological entropies of Zadeh's extensions of continuous seif-maps on metric spaces were considered. Our aim in this paper is to study the ω-limit sets of Zadeh's extensions of nonautonomous discrete systems on intervals.

2. Preliminaries

Throughout this paper, let (X, d) be a metric space, write $I = [0, 1]$, and denote by $\mathbb{N}$ the set of all positive integers. Let $C^0(X)$ be the set of all continuous self-maps on X. For a given $f \in C^0(X)$, let $f^{n+1} = f \circ f^n$ for any $n \in \mathbb{N}$. We call $F(f) = \{x \in X : f(x) = x\}$ the set of fixed points of f and $P(f) = \{x \in X : f^n(x) = x$ for some $n \in \mathbb{N}\}$ the set of periodic points of f.

Let $f_n \in C^0(X)$ $(n \in \mathbb{N})$ and F^0 be the identity map of X, and write

$$F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1 \quad \text{for any} \;\; n \in \mathbb{N}.$$

$y \in X$ is called the ω-limit point of $x(\in X)$ under (X, f_n) if there are $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\lim_{k \longrightarrow \infty} F_{n_k}(x) = y.$$

Denote by $\omega(x, f_n)$ the set of ω-limit points of x under (X, f_n). We write $f_n \Longrightarrow f$ if f_n converges uniformly to f.

Now, let us recall some definitions for fuzzy theory which are from [15].

Definition 1. *Let X be a metric space. A mapping $A : X \longrightarrow [0,1]$ is called a fuzzy set on X. For each fuzzy set A and each $\alpha \in (0,1]$, $A_\alpha = \{t \in X : A(t) \geq \alpha\}$ is called an α-level set of A and $A_0 = \overline{\{t \in X : A(t) > 0\}}$ is called the support of A, where $\overline{B}$ means the closure of subset B of X.*

Definition 2. *A fuzzy set A on I is said to be a fuzzy number if it satisfies the following conditions:*

(1) $A_1 \neq \emptyset$;
(2) *A is an upper semicontinuous function;*
(3) *For any $t_1, t_2 \in I$ and any $\lambda \in [0,1]$,* $A(\lambda t_1 + (1-\lambda)t_2) \geq \min\{A(t_1), A(t_2)\}$;
(4) *A_0 is compact.*

Let $\mathcal{F}(I)$ denote the set of fuzzy numbers on I. It is known that α-level set A_α of A determines the fuzzy number A and that every A_α is a closed connected subset of I. If $A \in I$, then $A \in \mathcal{F}(I)$ with $A_\alpha = [A, A]$ for any $\alpha \in [0,1]$.

For any $A, B \in \mathcal{F}(I)$ with $A_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$ and $B_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$ for any $\alpha \in (0,1]$, we define the metric of A and B as follows:

$$D(A, B) = \sup_{\alpha \in (0,1]} \max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}.$$

Obviously, we have

$$D(A, B) = \sup_{\alpha \in (0,1]} \max\{\sup_{x \in A_\alpha} d(x, B_\alpha), \sup_{y \in B_\alpha} d(y, A_\alpha)\} H(A_\alpha, B_\alpha),$$

where $d(x, J) = \inf_{y \in J} d(x, y)$ for any $x \in I$ and $J \subset I$. It is known that $(\mathcal{F}(I), D)$ is a complete metric space (refer to [15]).

Let $f \in C^0(I)$. We define the Zadeh's extension $\widehat{f} : \mathcal{F}(I) \longrightarrow \mathcal{F}(I)$ of f for any $x \in I$ and $A \in \mathcal{F}(I)$ by

$$(\widehat{f}(A))(x) = \sup_{f(y)=x} A(y).$$

It follows from [15] that f is continuous if and only if $\widehat{f}$ is continuous, and it follows from [22] (Lemma 2.1) that

$$[\widehat{f}(A)]_\alpha = f(A_\alpha)$$

for any $A \in \mathcal{F}(I)$ and $\alpha \in (0,1]$. In this paper, we will show the following theorem.

Theorem 1. *Let f_n be a sequence of continuous self-maps on I with $f_n \Longrightarrow f$. If $P(f) = F(f)$, then $\omega(A, \widehat{f_n}) \subset F(\widehat{f})$ for any $A \in \mathcal{F}(I)$.*

3. Proof of the Main Result

In this section, we let $f \in C^0(I)$ and $\widehat{f}$ is the Zadeh's extension of f. Let $\widehat{F}^0$ be the identity map of $\mathcal{F}(I)$ and for any $n \in \mathbb{N}$, we write

$$\widehat{F}_n = \widehat{f}_n \circ \widehat{f}_{n-1} \circ \cdots \circ \widehat{f}_1.$$

Lemma 1. *Assume that $f_n \in C^0(I)$ for any $n \in \mathbb{N}$ with $f_n \Longrightarrow f$. Then, $\widehat{f}_n \Longrightarrow \widehat{f}$ on $\mathcal{F}(I)$.*

Proof. Since $f_n \Longrightarrow f$ on I, it follows that, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that, when $n \geq N$, we have

$$|f(x) - f_n(x)| < \frac{\varepsilon}{2}$$

for any $x \in I$, which implies that, for any $B \subset I$, we have $d(z, f_n(B)) < \varepsilon/2$ for any $z \in f(B)$ and $d(z, f(B)) < \varepsilon/2$ for any $z \in f_n(B)$. Thus when $n \geq N$, we have

$$D(\widehat{f}(A), \widehat{f}_n(A)) = \sup_{\alpha \in (0,1]} H(f(A_\alpha), f_n(A_\alpha)) \leq \frac{\varepsilon}{2} < \varepsilon$$

for any $A \in \mathcal{F}(I)$. Lemma 1 is proven. □

Lemma 2. *Assume that $f_n \in C^0(I)$ for any $n \in \mathbb{N}$ with $f_n \Longrightarrow f$. If $B \in \omega(A, \widehat{f}_n)$ for some $A \in \mathcal{F}(I)$, then $\omega(x, f_n) \cap B_\alpha \neq \emptyset$ for any $\alpha \in (0,1]$ and $x \in A_\alpha$.*

Proof. Let $B \in \omega(A, \widehat{f}_n)$ and $\alpha \in (0,1]$. Let $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\lim_{k \longrightarrow \infty} D(\widehat{F}n_k(A), B) = 0.$$

Then,

$$\lim_{k \longrightarrow \infty} H(Fn_k(A_\alpha), B_\alpha) = 0.$$

Let $x \in A_\alpha$. By taking a subsequence, we let $\lim_{k \longrightarrow \infty} Fn_k(x) = y \in \omega(x, f_n)$. If $y \notin B_\alpha$, then $\varepsilon = d(y, B_\alpha) > 0$. Since $\lim_{k \longrightarrow \infty} H(Fn_k(A_\alpha), B_\alpha) = 0$, there is an $N \in \mathbb{N}$ such that, when $n_k > N$, we have

$$H(Fn_k(A_\alpha), B_\alpha) < \frac{\varepsilon}{2},$$

which implies $d(Fn_k(x), B_\alpha) < \varepsilon/2$ and $\lim_{k \longrightarrow \infty} Fn_k(x) \neq y$ since $Fn_k(x) \in Fn_k(A_\alpha)$. This is a contradiction. Thus, $y \in B_\alpha$. Lemma 2 is proven. □

Proposition 1. *Assume that $f_n \in C^0(I)$ for any $n \in \mathbb{N}$ with $f_n \Longrightarrow f$ and $P(f) = F(f)$. Then, the following statements hold:*

(1) If $B \in \omega(A, \widehat{f}_n)$ for some $A \in \mathcal{F}(I)$, then $\emptyset \neq f(B_\alpha) \cap B_\alpha \cap \omega(x, f_n) \subset F(f)$ for any $\alpha \in (0,1]$ and $x \in A_\alpha$.

(2) If $B \in \omega(A, \widehat{f}_n)$ for some $A \in \mathcal{F}(I)$, then $\cup_{B \in \omega(A, \widehat{f}_n)} B_\alpha \cup \omega(x, f_n)$ is a connected subset of I for any $\alpha \in (0,1]$ and $x \in A_\alpha$.

Proof. It follows from Theorem 1 and Lemma 2. □

Lemma 3 (See [14] (Lemma 2))**.** *Assume that $f \in C^0(I)$ with $F(f) = P(f)$. Then, for any $x \in I$ and $n \in \mathbb{N}$, $f^n(x) > x$ if $f(x) > x$ and $f^n(x) < x$ if $f(x) < x$.*

Now, we show the main result of this paper.

Proof of Theorem 1. Let $B \in \omega(A, \widehat{f_n})$. For any $\alpha \in (0,1]$, write $B_\alpha = [a_\alpha, b_\alpha]$ and $f(B_\alpha) = [c_\alpha, d_\alpha]$. Let $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$\lim_{k \longrightarrow \infty} D(\widehat{F}n_k(A), B) = 0. \tag{1}$$

By $\widehat{f} \in C^0(\mathcal{F}(I))$, we see that, for any $\varepsilon > 0$, there is an $\delta = \delta(\varepsilon) > 0$ such that, if $D(B, C) < \delta$ with $C \in \mathcal{F}(I)$, then

$$D(\widehat{f}(B), \widehat{f}(C)) < \frac{\varepsilon}{3}.$$

By Lemma 1, we see that there is an $N = N(\varepsilon) \in \mathbb{N}$ such that , when $n \geq N$, we have

$$D(\widehat{f}(W), \widehat{f_n}(W)) \leq \frac{\varepsilon}{3} \tag{2}$$

for any $W \in \mathcal{F}(I)$. Take $r = n_k \geq N$ such that $D(\widehat{F}_r(A), B) < \delta$. Thus,

$$D(\widehat{f}(B), \widehat{F}_{r+1}(A)) \leq D(\widehat{f}(B), \widehat{f}(\widehat{F}_r(A))) + D(\widehat{f}(\widehat{F}_r(A)), \widehat{F}_{r+1}(A)) \leq \frac{2\varepsilon}{3}. \tag{3}$$

In the following, we show that $a_\alpha = c_\alpha$ and $b_\alpha = d_\alpha$. For convenience, write $a_\alpha = a, b_\alpha = b$, $c_\alpha = c$, and $d_\alpha = d$.

(i) We will show $c \leq a$. Assume on the contrary that $c > a$. Then, by Proposition 1, we see $c \leq b$.

We claim that there is an $u \in (a, 1]$ such that $f(u) = a$. Indeed, if $f([a, 1]) \subset (a, 1]$, then let $\varepsilon = \min\{d(a, f([a, 1])), c - a\} > 0$. By Equation (3), we see $F_{r+1}(A_\alpha) \subset [a + \varepsilon/3, 1]$. It follows from Equation (2) that

$$H(f(F_{r+1}(A_\alpha)), F_{r+2}(A_\alpha)) \leq \frac{\varepsilon}{3}.$$

Thus, $F_{r+2}(A_\alpha) \subset [a + \varepsilon/3, 1]$. Continuing in this fashion, we have that $F_n(A_\alpha) \subset [a + \varepsilon/3, 1]$ for any $n \geq r + 1$, which contradicts Equation (1). The claim is proven.

Let $u = \min\{x \in (a, 1] : f(x) = a\}$. Then, $u > b$ since $f([a, b]) = [c, d]$ and $u > d$ (Otherwise, if $b < u \leq d$, then there exists an $u_1 \in [a, b]$ and $u_2 \in [u_1, u]$ such that $u_1 = f(u_2) < u_2 \leq d = f(u_1)$. This contradicts Lemma 3.). By Lemma 3, we see $f([a, u]) \subset [a, u)$. Write

$$\begin{aligned} p &= \max\{b, d, \max f([a, u])\}, \\ \varepsilon_1 &= (u - p)/2, \\ q &= \min\{c, \min f([a, p + \varepsilon_1])\}, \\ \varepsilon &= \min\{(q - a)/2, \varepsilon_1\}. \end{aligned}$$

By Equation (3), we see $F_{r+1}(A_\alpha) \subset [q - \varepsilon, p + \varepsilon_1]$. It follows from Equation (2) that

$$H(f(F_{r+1}(A_\alpha)), F_{r+2}(A_\alpha)) \leq \frac{\varepsilon}{3}.$$

Thus, $F_{r+2}(A_\alpha) \subset [q - \varepsilon, p + \varepsilon_1]$. Continuing in this fashion, we have that $F_n(A_\alpha) \subset [q - \varepsilon, p + \varepsilon_1]$ for any $n \geq r + 1$, which contradicts Equation (1).

(ii) In similar fashion, we can show $d \geq b$.

(iii) We will show that, if $c = a$, then $d = b$. Assume on the contrary that $d > b$. Let $u = \max\{z \in [a, b] : f(z) = d\}$ and $e = \min\{z \in [a, b] : f(z) = a\}$. Then, we have $e < u$ (Otherwise, if $e > u$, then there is an $w \in [u, e]$ satisfying $u = f(w) < w < d = f^2(w)$. This contradicts Lemma 3.) and $f(a) < u$.

We claim that there is an $v \in (u, 1]$ such that $f(v) = u$. Indeed, if $p = \min f([u, 1]) > u$, then let $\varepsilon = \min\{(d-b)/2, d(u, p)\} > 0$. By Equation (3), we see

$$D(\widehat{f}(B), \widehat{F}_{r+1}(A)) \leq \frac{2\varepsilon}{3}.$$

If $F_n(A_\alpha) \cap [a, u] \neq \emptyset$ for any $n \geq r+1$, then we have $d - 2\varepsilon/3 \in F_n(A_\alpha)$ for any $n \geq r+1$, which contradicts Equation (1). If $F_n(A_\alpha) \cap [a, u] = \emptyset$ for some $n \geq r+1$, then let $m = \min\{F_n(A_\alpha) \cap [a, u] = \emptyset : n \geq r+1\}$. Thus, $F_m(A_\alpha) \subset (u, 1]$, and it follows from Equation (2) that

$$H(f(F_m(A_\alpha)), F_{m+1}(A_\alpha)) \leq \frac{\varepsilon}{3},$$

which implies $F_{m+1}(A_\alpha) \subset [p - \varepsilon/3, 1] \subset (u, 1]$. Continuing in this fashion, we obtain that $F_n(A_\alpha) \subset [p - \varepsilon/3, 1] \subset (u, 1]$ for any $n \geq m$, which contradicts Equation (1). The claim is proven.

Let $v = \min\{x \in (u, 1] : f(x) = u\}$. Then, by Lemma 3, we see $d < v$ and $p = \max f([u, v]) < v$.

If there is an $w \in [0, a)$ satisfying $f(w) = u$, then let $w = \max\{x \in [0, a) : f(x) = u\}$. By Lemma 3, we see $q = \min f([w, a]) > w$ and $f([w, v]) = [q, p]$. Write

$$\begin{aligned} \varepsilon_1 &= (v-p)/2, \\ z &= \min f([u, p+\varepsilon_1]) > u, \\ \varepsilon &= \min\{(d-b)/2, (q-w)/2, (z-u)/2, \varepsilon_1\}. \end{aligned}$$

By Equation (3), we see

$$D(\widehat{f}(B), \widehat{F}_{r+1}(A)) \leq \frac{2\varepsilon}{3}.$$

This implies $F_{r+1}(A_\alpha) \subset [q - \varepsilon, p + \varepsilon_1]$. If $F_n(A_\alpha) \cap [a, u] \neq \emptyset$ for any $n \geq r+1$, then we have $d - 2\varepsilon/3 \in F_n(A_\alpha)$ for any $n \geq r+1$, which contradicts Equation (1). If $F_n(A_\alpha) \cap [a, u] = \emptyset$ for some $n \geq r+1$, then let $m = \min\{F_n(A_\alpha) \cap [a, u] = \emptyset : n \geq r+1\}$. Thus, $F_m(A_\alpha) \subset (u, p+\varepsilon_1]$, and it follows from Equation (2) that

$$H(f(F_m(A_\alpha)), F_{m+1}(A_\alpha)) \leq \frac{\varepsilon}{3},$$

which implies $F_{m+1}(A_\alpha) \subset [z - \varepsilon/3, p + \varepsilon_1] \subset (u, p + \varepsilon_1]$. Continuing in this fashion, we obtain that $F_n(A_\alpha) \subset [z - \varepsilon/3, p + \varepsilon_1] \subset (u, p + \varepsilon_1]$ for any $n \geq m$, which contradicts Equation (1).

If $\max f([0, a]) < u$, then $f([0, v]) = [0, p]$. Using the similar arguments as ones developed in the above given proof, we also obtain a conclusion which contradicts Equation (1).

(iv) We will show $c = a$. Assume on the contrary that $c < a$. Then by claim (ii), we see $b \leq d$. Using the similar arguments as ones developed in the proof of claim (iii), we can obtain $b < d$. Let $\varepsilon = \min\{(a-c)/2, (d-b)/2\}$. By Equation (3), we see $F_{r+1}(A_\alpha) \supset [a - \varepsilon, b + \varepsilon]$. It follows from Equation (2) that

$$H(f(F_{r+1}(A_\alpha)), F_{r+2}(A_\alpha)) \leq \frac{\varepsilon}{3}.$$

Thus, $F_{r+2}(A_\alpha) \supset [a - \varepsilon, b + \varepsilon]$. Continuing in this fashion, we have that $F_n(A_\alpha) \supset [a - \varepsilon, b + \varepsilon]$ for any $n \geq r+1$, which contradicts Equation (1).

By claims (iii) and (iv), we see $f(B_\alpha) = B_\alpha$ for any $\alpha \in (0, 1]$, which implies $f(B) = B$. Theorem 1 is proven. □

Using the similar arguments as ones developed in the proofs of Proposition 1.4 of [13] and Theorem 1, we may show the following result.

Corollary 1. *Let $f_n \in C^0(I)$ for any $n \in \mathbb{N}$ with $f_n \Longrightarrow f$. If $P(f) = F(f^{2^s})$ for some $s \in \mathbb{N}$, then $\omega(A, \widehat{f}_n) \subset F(\widehat{f}^{\,2^s})$ for any $A \in \mathcal{F}(I)$.*

The following example illustrates that there are $f_n \in C^0(I)$ for any $n \in \mathbb{N}$ such that $f_n \Longrightarrow f$ with $P(f) = F(f)$ and $\omega(A, \widehat{f_n}) = \emptyset$ for some $A \in \mathcal{F}(I)$.

Example 1. *Let $f \in C^0(I)$ with $f(1) = 1 > 0 = f(0)$ and $x < f(x)$ for any $x \in (0,1)$ and $f_n \equiv f$ for any $n \in \mathbb{N}$. Thus, $f_n \Longrightarrow f$. We define $A \in \mathcal{F}(I)$ for any $x \in I$ by*

$$A(x) = -x + 1.$$

By calculation, we have $A_\alpha = [0, 1-\alpha]$ for any $\alpha \in (0,1]$ and $f^n(A_1) = \{0\}$ for any $n \in \mathbb{N}$. In the following, we assume that $\alpha \in (0,1)$ and let $f^n(A_\alpha) = [a_n(\alpha), b_n(\alpha)]$. Then, $a_n(\alpha) = 0$ for any $n \in \mathbb{N}$ and $b_n(\alpha) \leq b_{n+1}(\alpha) \leq 1$ for any $n \in \mathbb{N}$ and $b_n(\alpha) \longrightarrow 1$. Since $b(\alpha) = 1$ is not left continuous at $\alpha = 1$, by Theorem 2.1 of [23], there is not a $B \in \mathcal{F}(I)$ such that $B_\alpha = [a(\alpha), b(\alpha)] = [0,1]$ for any $\alpha \in (0,1]$. Thus, $\omega(A, \widehat{f}) = \emptyset$.

4. Conclusions

In this paper, we investigated the ω-limit sets of Zadeh's extensions of a nonautonomous discrete system f_n on an interval which converges uniformly to a map f and show that, if $P(f) = F(f)$, then $\omega(A, \widehat{f_n}) \subset F(\widehat{f})$ for any $A \in \mathcal{F}(I)$.

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