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A New Fixed Point Theorem and a New Generalized Hyers-Ulam-Rassias Stability in Incomplete Normed Spaces

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Abstract: In this study, our goal is to apply a new fixed point method to prove the Hyers-Ulam-Rassias stability of a quadratic functional equation in normed spaces which are not necessarily Banach spaces. The results of the present paper improve and extend some previous results.

Keywords: orthogonal set; Hyers-Ulam-Rassias stability; quadratic equation; fixed point; incomplete metric space

MSC: 47H10; 44C60; 46B03; 47H04

1. Introduction

The notion of the stability of functional equations was presented in 1940 by Ulam [1], "Under what conditions does there exist an additive mapping near an approximately additive mapping?" One year later, Hyers [2] found a partial answer to Ulam's question in a Banach space. Since then, the stability of such forms is known as Hyers-Ulam stability. In 1978, Rassias [3] proved the existence of unique linear mapping near approximate additive mapping, which provides a remarkable generalization of the Hyers-Ulam stability. Gavruta [4] investigated a different generalization of the Hyers-Ulam-Rassias theorem. For more details, see References [5–11]. Also, there are several applications of this concept in pure mathematics, sociology, financial and actuarial mathematics and psychology [12].

A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [13] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [14] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. The Hyers-Ulam-Rassias stability of the quadratic functional equation was proved in Reference [15]. Several functional equations have been presented in References [16,17].

There are many forms of the quadratic functional equation, one among them of great interest to us is the following:

$$f(2a+b) + f(2a-b) = f(a+b) + f(a-b) + 6f(a). \tag{1}$$

The fixed point method for studying the stability of functional equations was used for the first time in 1991 by Baker [18]. Yang [19] proved the Hyers-Ulam-Rassias stability of the quadratic functional Equation (1) in F-spaces.

In this paper, with the idea of the fixed point theorem [20], we investigate a new generalized Hyers-Ulam-Rassias stability of the functional Equation (1). Also, we give some examples to show that our results are real extensions of the previous results.

2. Preliminaries

This section consists of some required background for the main results.

Definition 1 ([20,21])**.** *Let X be a nonempty set. If a binary relation $\perp \subseteq X \times X$ satisfies the following*

$$\exists x_0 \in X : (\forall y \in X, y \perp x_0) \ \ or \ \ (\forall y \in X, x_0 \perp y),$$

then $\perp$ is said to be an orthogonal relation and the pair $(X, \perp)$ is called an orthogonal set (briefly O-set).

In the above definition, we say that x_0 is an orthogonal element and elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$.

Definition 2 ([21])**.** *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an O-set $(X, \perp)$ is called a strongly orthogonal sequence (briefly, SO-sequence) if*

$$(\forall n, k; \quad x_n \perp x_{n+k}) \quad or \quad (\forall n, k; \quad x_{n+k} \perp x_n).$$

Definition 3 ([21])**.** *Let $(X, \perp, d)$ be an orthogonal metric space where $(X, \perp)$ is an O-set and (X, d) is a metric space. X is strongly orthogonal complete (briefly, SO-complete) if every Cauchy SO-sequence is convergent.*

It is clear that every complete metric space is SO-complete but it has been proved that the converse does not hold in general [21].

Definition 4 ([21])**.** *Let $(X, \perp, d)$ be an orthogonal metric space. Then $f : X \to X$ is strongly orthogonal continuous (briefly, SO-continuous) in $a \in X$ if for each SO-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is SO-continuous on X if f is SO-continuous in each $a \in X$.*

It is obvious that every continuous mapping is SO-continuous but the converse is not true in general (see Reference [21]).

Definition 5 ([20])**.** *Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(x) \perp f(y)$ whenever $x \perp y$ and $x, y \in X$.*

Recently, Eshaghi et al. [20] have given a real generalization of the Banach fixed point theorem in incomplete metric spaces. The main result of Reference [20] is given as follows:

Theorem 1 ([20])**.** *Let $(X, \perp, d)$ be an O-complete orthogonal metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be O-continuous and $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, namely, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.*

Theorem 2. *Let $(X, \perp, d)$ be an SO-complete orthogonal metric space (not necessarily a complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be SO-continuous, $\perp$-preserving and $\perp$-contraction with Lipschitz constant λ. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.*

Proof. The proof of this result uses the same ideas in Theorem 3.11 of [20] and it suffices to replace the O-sequence by the SO-sequence. □

The reader can find more details on orthogonal metric spaces in References [22,23].

3. A New Hyers-Ulam-Rassias Stability

In this section, we will assume that $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are two normed spaces. We denote by d the induced metric by $\|.\|_Y$ and $\perp$ is an orthogonal relation on Y which is $\mathbb{R}$-preserving.

Theorem 3. *Let $(Y, d, \perp)$ be an SO-complete orthogonal metric space (not necessarily complete metric space). Assume that $f : X \to Y$ is a function such that*

$$\left[\ \forall x \in X, \forall n \in \mathbb{N}, \quad f\Big(\frac{x}{2^n}\Big) \perp \frac{f(x)}{4^n}\ \right] \quad \text{or} \quad \left[\ \forall x \in X, \forall n \in \mathbb{N}, \quad \frac{f(x)}{4^n} \perp f\Big(\frac{x}{2^n}\Big)\ \right] \tag{2}$$

and $\phi : X^2 \to \mathbb{R}^+ := [0, \infty)$ is a mapping satisfying

$$\|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 6f(x)\|_Y \le \phi(x, y) \tag{3}$$

for each $x, y \in X$. Suppose there exists a function $\alpha : [0, \infty) \to [0, 1)$ satisfying the following statements:
$(A1)$ $\limsup_{t\to s^+} \alpha(t) < 1$ *for all* $s \ge 0$;
$(A2)$ $\phi(\frac{x}{2}, \frac{y}{2}) \le \frac{1}{4}\,\alpha(\phi(x, y))\,\phi(x, y)$ *for all* $x, y \in X$;
$(A3)$ $\alpha(\phi(\frac{x}{2}, 0)) \le \alpha(\phi(x, 0)$ *for all* $x \in X$.

Then there exists a quadratic function $F : X \to Y$ and a nonempty subset X^* in X such that for some positive real number $L < 1$ we have

$$\|F(x) - f(x)\|_Y \le \frac{L}{8(1-L)}\,\phi(x, 0) \tag{4}$$

for all $x \in X^*$.

Proof. Consider $S_0 := \{g : X \to Y \mid g(0) = 0\}$ with the following generalized metric,

$$\mathcal{D}(h, g) := \inf\{M > 0 :\ \|h(x) - g(x)\|_Y \le M\phi(x, 0),\ \forall x \in X\}$$

for all $h, g \in S_0$. Taking $x = y = 0$ in (A2), we see that $\phi(0, 0) = 0$ and by using (3) we observe that $f(0) = 0$. Hence $f \in S_0$ and S_0 is a nonempty set. Let $S = \{g \in S_0 \mid \mathcal{D}(g, f) < \infty\}$ and $T : S \to S_0$ be a function given by

$$Tg(x) = 4\,g(\frac{x}{2}) \tag{5}$$

for every $x \in X$. In order to show that $T(S) \subseteq S$, substitute $y = 0$ in (3) we have

$$\|f(2x) - 4f(x)\|_Y \le \frac{1}{2}\phi(x, 0) \tag{6}$$

for all $x \in X$. Replacing x with $\frac{x}{2}$ in the above equation and employing (A2), we have

$$\|f(x) - Tf(x)\|_Y \le \frac{1}{8}\,\alpha(\phi(x, 0))\,\phi(x, 0) \tag{7}$$

for all $x \in X$. This implies that $\mathcal{D}(Tf, f) \le \frac{1}{8}$. Now if $g \in S$, then the definition of $\mathcal{D}$ and the relation (A2) conclude that $\mathcal{D}(Tg, Tf) \le \mathcal{D}(g, f)$ and the triangle inequality results that

$$\mathcal{D}(Tg, f) \le D(Tg, Tf) + \mathcal{D}(Tf, f) < \infty.$$

So $Tg \in S$ and hence T is self-adjoint mapping, that is $T(S) \subseteq S$. Consider

$$O(x) := \{f(x), (Tf)(x), (T^2 f)(x), (T^3 f)(x), \dots\}$$

for all $x \in X$ and for each $g, h \in S$ we define $\perp_S$ on S as follows:

$$g \perp_S h \iff \Big(\{g(x), h(x)\} \cap O(x) \neq \emptyset \quad \text{or} \quad g(x) \perp h(x)\Big); \;\; \forall x \in X.$$

Clearly, $(S, \perp_S)$ is an O-set. We now show that $(S, d, \perp_S)$ is an SO-complete orthogonal metric space, first of all we need to prove that for each $x \in X$, the sequence $\{(T^n f)(x)\}$ is a Cauchy SO-sequence in Y. To see this, since the relation $\perp$ is $\mathbb{R}$-preserving, the definition of $\perp_S$ implies that T is $\perp_S$-preserving. According to the assumptions (2) and $\mathbb{R}$-preserving of $\perp$, we obtain

$$\Big[\,\forall x \in X,\, \forall n \in \mathbb{N}, \quad (T^n f)(x) \perp f(x)\,\Big] \quad \text{or} \quad \Big[\,\forall x \in X,\, \forall n \in \mathbb{N}, \quad f(x) \perp (T^n f)(x)\,\Big].$$

Replacing x by $\frac{x}{2^k}$ and multiplying both sides of the above relations by 4^k, we obtain

$$\Big[\forall x \in X,\, \forall n, k \in \mathbb{N}, \quad (T^{n+k} f)(x) \,\perp\, (T^k f)(x)\,\Big] \quad \text{or} \quad \Big[\forall x \in X,\, \forall n, k \in \mathbb{N}, \quad (T^k f)(x) \,\perp\, (T^{n+k} f)(x)\,\Big].$$

That is, $\{(T^n f)(x)\}$ is an SO-sequence in Y for all $x \in X$.

Also, we need to prove that $\{(T^n f)(x)\}$ is a Cauchy sequence for each $x \in X$. Replacing x by $\frac{x}{2^n}$ and multiplying both sides of the inequality (7) by 4^n and making use of (A2) and (A3), we get

$$\|(T^{n+1} f)(x) - (T^n f)(x)\|_Y \leq \Big[\alpha(\phi(x, 0))\Big]^n \phi(x, 0)$$

for all $x \in X$ and $n \in \mathbb{N}$. Setting $L_x := \alpha(\phi(x, 0))$, we get

$$\begin{aligned} \|(T^m f)(x) - (T^n f)(x)\|_Y &\leq \sum_{i=n}^{m-1} \|(T^{i+1} f)(x) - (T^i f)(x)\|_Y \\ &\leq \sum_{i=n}^{m-1} L_x^i \,\phi(x, 0) = \frac{L_x^n (1 - L_x^{m-1})}{1 - L_x}\, \phi(x, 0) \end{aligned}$$

for all $x \in X$ and $m, n \in \mathbb{N}$. Since $L_x < 1$, taking the limit as $m, n \to \infty$ in the above inequality, we deduce that the sequence $\{(T^n f)(x)\}$ is a Cauchy sequence for each $x \in X$. By SO-completeness of Y, we obtain that for every $x \in X$, there exists an element $F(x) \in Y$ which is a limit point of $\{(T^n f)(x)\}$. That is, $F : X \to Y$ is well-defined and is given by

$$F(x) = \lim_{n \to \infty} (T^n f)(x) = \lim_{n \to \infty} 4^n\, f(\frac{x}{2^n}) \tag{8}$$

for all $x \in X$. Therefore, $\{(T^n f)(x)\}$ is a convergent sequence for each $x \in X$.

Now, take a Cauchy SO-sequence $\{g_n\}$ in S. It follows that

$$\Big(\forall n, k \in \mathbb{N}, \quad g_{n+k} \,\perp_S\, g_k\Big) \quad \text{or} \quad \Big(\forall n, k \in \mathbb{N}, \quad g_k \,\perp_S\, g_{n+k}\Big). \tag{9}$$

Let x_0 be an arbitrary point in X. We can see that the following cases can occur:

Case 1. There exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ for which $g_{n_k}(x_0) \in O(x_0)$ for all $k \in \mathbb{N}$. The convergence of $\{(T^n f)(x_0)\}$ implies the convergence of $\{g_{n_k}(x_0)\}$. On the other hand, since every Cauchy sequence with a convergent subsequence is convergent, the sequence $\{g_n(x_0)\}$ is convergent.

Case 2. $\{g_n(x_0)\}$ is an SO-sequence in Y.

Let $\epsilon > 0$ be given. Since $\{g_n\}$ is a Cauchy sequence in S, then there exists $N \in \mathbb{N}$ such that $\mathcal{D}(g_n, g_m) < \epsilon$ for every $n, m \geq N$ which implies the following inequality:

$$\|g_n(x) - g_m(x)\|_Y \leq \epsilon\, \phi(x, 0) \tag{10}$$

for every $n, m \geq N$ and $x \in X$. This means that for every $x \in X$, $\{g_n(x)\}$ is a Cauchy sequence in Y. The SO-completeness of Y implies that $\{g_n(x_0)\}$ is a convergent sequence.

In the above two cases, there is a point $g(x_0) \in Y$ such that $\lim_{n\to\infty} g_n(x_0) = g(x_0)$. According to the choice of x_0, we can see that $g : X \to Y$ is well-defined and also, $g(x) = \lim_{n\to\infty} g_n(x)$ for each $x \in X$. If we take the limit as $m \to \infty$ in the inequality (10), then

$$\|g_n(x) - g(x)\|_Y \leq \epsilon\, \phi(x, 0)$$

for every $n \geq N$ and $x \in X$. From the definition of $\mathcal{D}$, we gain $\mathcal{D}(g_n, g) \leq \epsilon$ for all $n \geq N$, that is, $g \in S$ and $\{g_n\}$ is a convergent sequence. Therefore, $(S, \mathcal{D}, \perp_S)$ is an SO-complete orthogonal metric space.

On the other hand, since $\limsup_{t\to 0^+} \alpha(t) < 1$, then there exist $r \in (0, \infty]$ and $0 < L < 1$ such that $\alpha(t) \leq L$ for all $t \in [0,\ r)$. Put $X^* = \{x \in X \mid \phi(x, 0) < r\}$. It follows from $\phi(0, 0) = 0$ that $0 \in X^*$. Now, we replace X by X^* in definition of S_0. Note that for all $g, h \in S$

$$\begin{aligned}
\mathcal{D}(g, h) < K \Rightarrow & \|g(x) - h(x)\|_Y \leq K\phi(x, 0), \quad (x \in X^*)\\
\Rightarrow & \left\|4\, g(\frac{x}{2}) - 4\, h(\frac{x}{2})\right\|_Y \leq K\, 4\, \phi(\frac{x}{2}, 0),\\
\Rightarrow & \left\|4\, g(\frac{x}{2}) - 4\, h(\frac{x}{2})\right\|_Y \leq K\, \alpha(\phi(x, 0))\ \ \phi(x, 0),\\
\Rightarrow & \left\|4\, g(\frac{x}{2}) - 4\, h(\frac{x}{2})\right\|_Y \leq K\, L\, \phi(x, 0)),\\
\Rightarrow & \mathcal{D}(Tg, Th) \leq KL.
\end{aligned}$$

Hence we see that $\mathcal{D}(Tg, Th) \leq L\mathcal{D}(g, h)$ for all $g, h \in S$. It follows from $L < 1$ that T is a contraction. Consequently, T is an SO-continuous mapping and is a contraction on $\perp_S$-comparable elements with Lipschitz constant L. Since $(S, \mathcal{D}, \perp_S)$ is SO-complete and T is also $\perp_S$-preserving, then from the fixed point Theorem 2, we conclude that T has a unique fixed point and T is a Picard operator. This means that the sequence $\{T^n f\}$ converges to the fixed point of T. It follows from (8) that F is a unique fixed point of T. Moreover,

$$\begin{aligned}
\mathcal{D}(F, f) &\leq \mathcal{D}(F, TF) + \mathcal{D}(TF, Tf) + \mathcal{D}(Tf, f)\\
&\leq L\mathcal{D}(F, f) + \mathcal{D}(Tf, f).
\end{aligned}$$

Therefore, $\mathcal{D}(F, f) \leq \frac{1}{1-L}\mathcal{D}(Tf, f)$. The relation (7) ensures that the inequality (4) holds.

Finally, we will show that F is a quadratic mapping. To this aim, fix x and y in X. Since $\{\phi(\frac{x}{2^n}, \frac{y}{2^n})\}$ is a non-negative and decreasing sequence, then there is $\tau \geq 0$ for which $\phi(\frac{x}{2^n}, \frac{y}{2^n}) \to \tau$ as $n \to \infty$. Taking into account (A1), we have $\limsup_{t\to\tau^+} \alpha(t) < 1$, so there exist $\delta > 0$ and $\nu < 1$ such that for all $t \in [\tau, \tau + \delta)$, $\alpha(t) < \nu$. Consider the positive integer N such that for all $n \geq N$, $\phi(\frac{x}{2^n}, \frac{y}{2^n}) \in [\tau,\ \tau + \delta)$. By virtue of (3), we obtain

$$\begin{aligned}
&\|F(2x + y) + F(2x - y) - F(x + y) - F(x - y) - 6F(x)\|_Y\\
&= \lim_{n\to\infty} 4^n \left\|f(2\frac{x}{2^n} + \frac{y}{2^n}) + f(2\frac{x}{2^n} - \frac{y}{2^n}) - f(\frac{x}{2^n} + \frac{y}{2^n}) - f(\frac{x}{2^n} - \frac{y}{2^n}) - 6f(\frac{x}{2^n})\right\|_Y\\
&\leq \lim_{n\to\infty} 4^n\, \phi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\\
&\leq \lim_{n\to\infty} 4^n\, \frac{1}{4^n} \prod_{i=0}^{n-1} \alpha\Big(\phi(\frac{x}{2^i}, \frac{y}{2^i})\Big)\ \phi(x, y)\\
&= \lim_{n\to\infty} \nu^n \,.\, \prod_{i=0}^{N-1} \alpha\Big(\phi(\frac{x}{2^i}, \frac{x}{2^i})\Big)\ \phi(x, y) = 0.
\end{aligned}$$

This completes the proof. □

Corollary 1. *Let Y be a Banach space and $f : X \to Y$ be a function such that there exists a function $\phi : X^2 \to [0, \infty)$ satisfying (3). If there exists a positive real number $L < 1$ such that*

$$\phi(\frac{x}{2}, \frac{y}{2}) \leq \frac{1}{4}\, L\, \phi(x, y) \tag{11}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $F : X \to Y$ which satisfies the inequality

$$\|f(x) - F(x)\|_Y \leq \frac{L}{8(1-L)}\, \phi(x, 0)$$

for all $x \in X$.

Proof. For every $y_1, y_2 \in Y$, we define $y_1 \perp y_2$ if and only if $\|y_1\|_Y \leq \|y_2\|_Y$. It is easy to see that $(Y, \perp)$ is an O-set. Moreover, since Y is a Banach space, then $(Y, d, \perp)$ is an SO-complete orthogonal metric space which d is the induced metric by norm. From the definition of $\perp$, it follows that

$$\Big[\ \forall x \in X,\ \forall n \in \mathbb{N},\quad f(\frac{x}{2^n})\ \perp\ \frac{f(x)}{4^n}\ \Big] \ \text{ or } \ \Big[\ \forall x \in X,\ \forall n \in \mathbb{N},\quad \frac{f(x)}{4^n}\ \perp\ f(\frac{x}{2^n})\ \Big].$$

Setting $\alpha(t) = L$ for all $t \in [0, \infty)$, from the proof of Theorem 3 we can see the result. □

Theorem 4. *Let $(Y, d, \perp)$ be an SO-complete orthogonal metric space (not necessarily complete metric space) and $f : X \to Y$ be a mapping such that*

$$\Big[\ \forall x \in X,\ \forall n \in \mathbb{N},\quad f(2^n x)\ \perp\ 4^n f(x)\ \Big] \ \textit{ or } \ \Big[\ \forall x \in X,\ \forall n \in \mathbb{N},\quad 4^n f(x)\ \perp\ f(2^n x)\ \Big]. \tag{12}$$

Assume that there exists a function $\phi : X^2 \to [0, \infty)$ satisfying the Equation (3) of Theorem 3 and the following property,
(B1) $\phi(x, y) = 0$ if and only if $x = y = 0$ and $\Big\{\phi(2^n x, 2^n y)\Big\}$ is an increasing sequence for all $x, y \in X$ such that both are not zero.

If $\alpha : [0, \infty) \to [0, 1)$ is a mapping which satisfies in (A1) of Theorem 3 and the following conditions:
(B2) $\phi(2x, 2y) \leq 4\, \alpha\Big([\phi(x, y)]^{-1}\Big)\, \phi(x, y)$ for all $x, y \in X$ not both being zero;
(B3) $\alpha\Big([\phi(2x, 0)]^{-1}\Big) \leq \alpha\Big([\phi(x, 0)]^{-1}\Big)$ for all $x \in X$ where $x \neq 0$.

Then there exist a quadratic function $F : X \to Y$ and a nonempty subset X^* of X such that for some positive real number $L < 1$ we have

$$\|F(x) - f(x)\|_Y \leq \frac{1}{8(1-L)}\, \phi(x, 0) \tag{13}$$

for all $x \in X^*$.

Proof. By the same reasoning as in the proof of Theorem 3, there are $\lambda \in (0, \infty]$ and $0 < L < 1$, such that $\alpha(t) \leq L$ for each $0 \leq t < \lambda$. Set $X^* := \{x \in X |\ \ x \neq 0,\ \ [\phi(x, 0)]^{-1} < \lambda\} \cup \{0\}$. By the same argument of Theorem 3, one can show that the mapping $T : S \to S$ defined by $Tg(x) = \frac{1}{4}\, g(2x)$ for all $x \in X$, is a $\perp_S$-preserving mapping. Define $F : X \to Y$ by

$$F(x) = \lim_{n \to \infty} \frac{1}{4^n}\, f(2^n x)$$

for all $x \in X$. Replacing X^* by X in definition of S_0 we obtain that T is a contraction with Lipschitz constant L. Applying Theorem 2 we can see F is a unique fixed point of T. Dividing both sides of the inequality (6) by 4, we have

$$\|\frac{f(2x)}{4} - f(x)\|_Y \leq \frac{1}{8}\,\phi(x,0)$$

for all $x \in X$. In fact, $\mathcal{D}(f,Tf) \leq \frac{1}{8}$. It follows that

$$\mathcal{D}(f,F) \leq \mathcal{D}(f,Tf) + \mathcal{D}(Tf,TF) \leq \mathcal{D}(f,Tf) + L\mathcal{D}(f,F)$$

and consequently,

$$\mathcal{D}(f,F) \leq \frac{1}{1-L}\mathcal{D}(f,Tf) \leq \frac{1}{8(1-L)}.$$

That is, the inequality (13) holds.

To show that the function F is quadratic, let us consider x, y are elements in X which not both zero. Since $\left\{[\phi(2^n x, 2^n y)]^{-1}\right\}$ is a non-negative and decreasing sequence in $\mathbb{R}^+$, so the rest of the proof is similar to the proof of Theorem 3. □

Corollary 2. *Let Y be a Banach space and $f : X \to Y$ be a mapping such that there exists a function $\phi : X^2 \to [0,\infty)$ satisfying the condition (B1) and inequality* (3) *of Theorem 4. If there exists a positive real number $L < 1$ such that*

$$\phi(2x, 2y) \leq 4\,L\,\phi(x,y) \tag{14}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $F : X \to Y$ which satisfies the inequality

$$\|f(x) - F(x)\|_Y \leq \frac{1}{1-L}\,\phi(x,0)$$

for all $x \in X$.

Proof. Take the same metric d and the orthogonal relation of Corollary 1. By the same argument of Corollary 1, one can show that $(Y, d, \perp)$ is an SO-complete orthogonal metric space and the relation (12) holds. Putting $\alpha(t) = L$ for all $t \in [0,\infty)$ and applying Theorem 4, we can easily obtain the result. □

Corollary 3. *Suppose that Y is a Banach space and $\theta \geq 0$ and $r \neq 2$ are fixed. Assume that $f : X \to Y$ is a function which satisfies the functional inequality*

$$\|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 6f(x)\|_Y \leq \theta(\|x\|_X^r + \|y\|_X^r) \tag{15}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $F : X \to Y$ such that the inequality

$$\|f(x) - F(x)\|_Y \leq \frac{\theta}{2^{r+1} - 8}\,\|x\|_X^r \tag{16}$$

holds for all $x \in X$, where $r > 2$, or the inequality

$$\|f(x) - F(x)\|_X \leq \frac{\theta}{8 - 2^{r+1}}\,\|x\|_X^r \tag{17}$$

holds for all $x \in X$, where $r < 2$.

Proof. Take the same metric d and the orthogonal relation of Corollary 1. By the same argument of Corollary 1, one can show that $(Y, d, \perp)$ is an SO-complete orthogonal metric space. Moreover, the definition of $\perp$ ensures that the relations (2) and (12) hold. Let $\phi(x,y) = \theta(\|x\|_X^r + \|y\|_X^r)$ for each $x, y \in X$. It follows that

$$\phi(\frac{x}{2}, \frac{y}{2}) \leq \frac{1}{4}\left(\frac{1}{2}\right)^{r-2} \phi(x,y)$$

for all $x, y \in X$ where $r > 2$. Set $\alpha(t) = \frac{1}{2^{r-2}}$ for all $t \in [0,\infty)$. This ensures that $X^* = X$ and the relations (A1) and (A3) of Theorem 3 hold. Applying Theorem 3, we see that inequality (4) holds with $L = \frac{1}{2^{r-2}}$ which yields the inequality (16). On the other hand, the function ϕ satisfies in the properties (B1), (B2) and also,

$$\phi(2x, 2y) \leq 4\, 2^{r-2}\, \phi(x,y)$$

for all $x, y \in X$, where $r < 2$. Putting $\alpha(t) = \frac{1}{2^{2-r}}$ for every $t \in [0,\infty)$, it is easily seen that $X^* = X$ and the conditions (A1) and (B3) hold. Employing Theorem 4, we see that the inequality (13) holds with $L = \frac{1}{2^{2-r}}$. This implies the inequality (17). □

The next example shows that Theorem 3 is a real extension of Corollary 1.

Example 1. *Let Y be a Banach space. Suppose that a function $f : X \to Y$ has the property*

$$\|f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 6f(x)\|_Y \leq \phi(x,y)$$

for all $x, y \in X$, where $\phi : X^2 \to [0,\infty)$ is defined by

$$\phi(x,y) = \begin{cases} m(\|x\|_X + \|y\|_X)\,, & \|2x\|_X + \|2y\|_X - (\|x\|_X + \|y\|_X) > 1, \text{ and } m \text{ is the smallest natural} \\ & \text{number such that } \|x\|_X + \|y\|_X < m < \|2x\|_X + \|2y\|_X \\ 0\,, & \text{otherwise.} \end{cases}$$

We define a function $\alpha : [0,\infty) \to [0,1)$ as

$$\alpha(t) = \begin{cases} \frac{m-1}{m}, & m \text{ is the smallest natural number such that } t \leq m \\ 0\,, & \text{otherwise.} \end{cases}$$

for all $t \in [0,\infty)$. Then the following properties hold:
(C1) The function α satisfies the relations (A1) and (A3) of Theorem 3.
(C2) The function ϕ satisfies the relation (A2) of Theorem 3.
(C3) For every positive real number and r, there exist a constant $L \in (0,1)$ and a quadratic mapping $F : X \to Y$ such that the inequality (4) holds for any $x \in X$ with $\|x\|_X \leq r$.

Proof. Take the same metric d and the orthogonal relation of Corollary 1. By the same argument of Corollary 1, one can show that $(Y, d, \perp)$ is an SO-complete orthogonal metric space and the relation (2) holds. Let us take $x, y \in X$ with

$$\|x\|_X + \|y\|_X - \left(\left\|\frac{x}{2}\right\|_X + \left\|\frac{y}{2}\right\|_X\right) > 1 \tag{18}$$

and m be the smallest natural number such that

$$\left\|\frac{x}{2}\right\|_X + \left\|\frac{y}{2}\right\|_X < m < \|x\|_X + \|y\|_X.$$

Then

$$\phi\Big(\frac{x}{2},\frac{y}{2}\Big) = m\left(\Big\|\frac{x}{2}\Big\|_X + \Big\|\frac{y}{2}\Big\|_X\right)$$
$$= \frac{1}{4}\, m\,(\|x\|_X + \|y\|_X).$$

From the inequality (18), we observe that

$$\|2x\|_X + \|2y\|_X - (\|x\|_X + \|y\|_X) > 2.$$

This follows that there exists $k_0 \in \mathbb{N}$ for which

$$\|x\|_X + \|y\|_X < k_0 < \|2x\|_X + \|2y\|_X.$$

Assume k is the smallest natural number satisfying the above condition. Clearly, $k > m$ and

$$\phi(x,y) = k\,(\|x\|_X + \|y\|_X).$$

Suppose that r is the smallest natural number such that $k\,(\|x\|_X + \|y\|_X) \le r$, then $\alpha(\phi(x,y)) = \frac{r-1}{r}$. Since $\|x\|_X + \|y\|_X > 1$, then $k < r$ and we conclude that

$$\frac{m}{k} \le \frac{m}{m+1} \le \frac{r-1}{r}.$$

This implies that

$$\phi(\frac{x}{2},\frac{y}{2}) = m\left(\Big\|\frac{x}{2}\Big\|_X + \Big\|\frac{y}{2}\Big\|_X\right)$$
$$\le \frac{1}{4}\,\frac{r-1}{r}\,k\,(\|x\|_X + \|y\|_X)$$
$$= \frac{1}{4}\,\alpha(\phi(x,y))\,\phi(x,y).$$

Therefore, the property (C2) holds. From the definition of the function α, it is easily seen that α is an increasing mapping. Finally, it follows from $\limsup_{t\to 0^+}\alpha(t) = 0$ that for every $r > 0$ there exists $L < 1$ such that $\alpha(\phi(x,0)) \le L$ for all $x \in X$ with $\|x\|_X \le r$. By the same proof of Theorem 3, we prove (C3).

Note that there is no $L < 1$ such that the inequality (11) holds and hence the stability of f does not imply by Corollary 1. □

In the following example, we observe that our results go further than the stability on Banach spaces.

Example 2. *Assume that θ and r are two real numbers such that $\theta \ge 0$ and $r \ne 2$. Consider*

$$Y = \{x = \{x_n\} \subset \mathbb{R};\ \ \exists n_1, n_2, ..., n_k;\ \ \forall n \ne n_1, n_2, \dots, n_k,\ \ x_n = 0\}$$

with norm $\|x\|_Y = \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}}$ where $1 < p < \infty$. Suppose $f : X \to Y$ is a mapping satisfying the inequality (7) and the following condition

$$\exists \gamma > 0,\ \forall x \in X, \qquad f(\frac{x}{2}) = \frac{\gamma}{4}\, f(x). \tag{19}$$

Then there exists a unique quadratic mapping $F : X \to Y$ such that the inequality (8) holds for all $x \in X$, where $r > 2$, or the inequality (9) holds for all $x \in X$, where $r < 2$.

Proof. Let q be the conjugate of p; that is, $\frac{1}{p} + \frac{1}{q} = 1$. Note that $(Y, \|.\|_Y)$ is not a Banach space because, $A_n = \{1, \frac{1}{2}, ..., \frac{1}{2^n}, 0, 0, 0, ...\}$, $n \in \mathbb{N}$, is a sequence in Y where the limit of $\{A_n\}$ does not belong to Y. For all $A = \{x_n\}$ and $B = \{y_n\}$ in Y, define

$$A \perp B \iff \sum_{n=1}^{\infty} |x_n y_n| = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} |y_n|^q\Big)^{\frac{1}{q}}$$

and consider $d(A, B) = \|A - B\|_Y$. We claim that $(Y, \perp, d)$ is an SO-complete orthogonal metric space. Indeed, let $\{A_n\}$ be a Cauchy SO-sequence in Y and for all $n, k \in \mathbb{N}$, $A_n \perp A_{n+k}$. The relation $\perp$ ensures that for all $n \in \mathbb{N}$,

$$\exists \lambda_n \neq 0 \quad |A_n|^p = \lambda_n \, |A_{n+1}|^q \quad or \quad |A_{n+1}|^q = \lambda_n \, |A_n|^p \tag{20}$$

where $|A|^p = \{|x_n|^p\}$. We distinguish two cases:

Case 1. There exists a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $A_{n_k} = 0$ for all k. This implies that $A_n \to 0 \in Y$.

Case 2. For all sufficiently large $n \in \mathbb{N}$, $A_n \neq 0$. Take $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $A_n \neq 0$. It follows from (20) that for all $n \geq n_0$ there exists $\lambda_n \neq 0$ for which $A_n = \lambda_n A_{n_0}^{\frac{p}{q}}$. It leads to

$$|\lambda_n - \lambda_m| \, \|A_{n_0}^{\frac{p}{q}}\|_p = \|\lambda_n A_{n_0}^{\frac{p}{q}} - \lambda_m A_{n_0}^{\frac{p}{q}}\|_p = \|A_n - A_m\|_p$$

for each $m, n \geq n_0$. As $n \to \infty$, the right-hand side of the above inequality tends to 0. Therefore, $\{\lambda_n\}$ is a Cauchy sequence in $\mathbb{R}$. Assume that $\lambda_n \to \lambda$ as $n \to \infty$. Put $A = \lambda A_{n_0}^{\frac{p}{q}}$. It follows that $A \in Y$ and for all $n \geq n_0$,

$$\|A_n - A\|_p = \|\lambda_n A_{n_0}^{\frac{p}{q}} - \lambda A_{n_0}^{\frac{p}{q}}\| = |\lambda_n - \lambda| \, \|A_{n_0}^{\frac{p}{q}}\|_p.$$

This implies that $A_n \to A$ as $n \to \infty$. Note that the case $A_{n+k} \perp A_n$ for all $n, k \in \mathbb{N}$ is in a similar way.

By virtue of (19) and the definition of $\perp$, we obtain that the relation (2) holds. Moreover, putting $x := 2x$ in (19), we can also see that the relation (12) holds. The rest of the proof is similar to the proof of Corollary 3. □

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