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Weighted Fractional Iyengar Type Inequalities in the Caputo Direction

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Abstract: Here we present weighted fractional Iyengar type inequalities with respect to L_p norms, with $1 \leq p \leq \infty$. Our employed fractional calculus is of Caputo type defined with respect to another function. Our results provide quantitative estimates for the approximation of the Lebesgue–Stieljes integral of a function, based on its values over a finite set of points including at the endpoints of its interval of definition. Our method relies on the right and left generalized fractional Taylor's formulae. The iterated generalized fractional derivatives case is also studied. We give applications at the end.

Keywords: Iyengar inequality; right and left generalized fractional derivatives; iterated generalized fractional derivatives; generalized fractional Taylor's formulae

MSC: 26A33; 26D10; 26D15

1. Introduction

We are motivated by the following famous Iyengar inequality (1938), [1].

Theorem 1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\, dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \tag{1}$$

We need

Definition 1 ([2]). *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Here $g \in AC([a,b])$ (absolutely continuous functions) and strictly increasing. We assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_\infty([a,b])$. We define the left generalized g-fractional derivative of f of order α as follows:*

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{2}$$

$x \geq a$.

If $\alpha \notin \mathbb{N}$, by [3], pp. 360–361, we have that $D_{a+;g}^{\alpha} f \in C([a,b])$.

We see that

$$\left(I_{a+;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \ x \geq a. \tag{3}$$

We set

$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x), \tag{4}$$

$$D_{a+;g}^{0} f(x) = f(x), \ \forall\, x \in [a,b]. \tag{5}$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \tag{6}$$

the usual left Caputo fractional derivative.

We mention the following g-left fractional generalized Taylor's formula:

Theorem 2 ([2]). *Let g be a strictly increasing function and $g \in AC([a,b])$. We assume that $\left(f \circ g^{-1}\right) \in AC^n([g(a), g(b)])$, i.e., $\left(f \circ g^{-1}\right)^{(n-1)} \in AC([g(a), g(b)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. Also we assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$. Then*

$$f(x) = f(a) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(x) - g(a))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha - 1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt, \quad \forall\, x \in [a,b]. \tag{7}$$

Calling $R_n(a,x)$ the remainder of (7), we find that

$$R_n(a,x) = \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha - 1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{8}$$

We need

Definition 2 ([2]). *Here $g \in AC([a,b])$ and is strictly increasing. We assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$, where $N \ni n = \lceil \alpha \rceil$, $\alpha > 0$. We define the right generalized g-fractional derivative of f of order α as follows:*

$$\left(D_{b-;g}^{\alpha} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)}(g(t))\, dt, \tag{9}$$

all $x \in [a,b]$.

If $\alpha \notin \mathbb{N}$, by [3], p. 378, we find that $\left(D_{b-;g}^{\alpha} f\right) \in C([a,b])$.

We see that

$$I_{b-;g}^{n-\alpha}\left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D_{b-;g}^{\alpha} f\right)(x), \quad a \le x \le b. \tag{10}$$

We set

$$D_{b-;g}^{n} f(x) = (-1)^n \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x), \tag{11}$$

$$D_{b-;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha} f(x) = D_{b-;id}^{\alpha} f(x) = D_{b-}^{\alpha} f, \tag{12}$$

the usual right Caputo fractional derivative.

We mention the g-right generalized fractional Taylor's formula:

Theorem 3 ([2]). *Let g be a strictly increasing function and $g \in AC([a,b])$. We assume that $\left(f \circ g^{-1}\right) \in AC^n([g(a), g(b)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. Also we assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$. Then*

$$f(x) = f(b) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(b))}{k!} (g(x) - g(b))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \ \ all\ a \le x \le b. \tag{13}$$

Calling $R_n(b,x)$ the remainder in (13), we find that

$$R_n(b,x) = \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{14}$$

Denote by

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha} D_{b-;g}^{\alpha} ... D_{b-;g}^{\alpha} \ \ (n\text{-times}),\ n \in \mathbb{N}. \tag{15}$$

We mention the following g-right generalized modified Taylor's formula:

Theorem 4 ([2]). *Suppose that $F_k := D_{b-;g}^{k\alpha} f$, for $k = 0,1,...,n+1$, fulfill: $F_k \circ g^{-1} \in AC([c,d])$, where $c = g(a)$, $d = g(b)$, and $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([a,b])$, where $0 < \alpha \le 1$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g}^{i\alpha} f\right)(b) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha-1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t)\, dt = \tag{16}$$

$$\sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g}^{i\alpha} f\right)(b) + \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha+1)} (g(b) - g(x))^{(n+1)\alpha}, \tag{17}$$

where $\psi_x \in [x,b]$, any $x \in [a,b]$.

Denote by

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha} D_{a+;g}^{\alpha} ... D_{a+;g}^{\alpha} \ \ (n\text{-times}),\ n \in \mathbb{N}. \tag{18}$$

We mention the following g-left generalized modified Taylor's formula:

Theorem 5 ([2]). *Suppose that $F_k := D_{a+;g}^{k\alpha} f$, for $k = 0,1,...,n+1$, fulfill: $F_k \circ g^{-1} \in AC([c,d])$, where $c = g(a)$, $d = g(b)$, and $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([a,b])$, where $0 < \alpha \le 1$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{a+;g}^{i\alpha} f\right)(a) + \tag{19}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha-1} g'(t) \left(D_{a+;g}^{(n+1)\alpha} f\right)(t)\, dt =$$

$$\sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{a+;g}^{i\alpha} f\right)(a) + \frac{\left(D_{a+;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha+1)} (g(x) - g(a))^{(n+1)\alpha}, \tag{20}$$

where $\psi_x \in [a,x]$, any $x \in [a,b]$.

Next we present generalized fractional Iyengar type inequalities.

2. Main Results

We present the following Caputo type generalized g-fractional Iyengar type inequality:

Theorem 6. *Let g be a strictly increasing function and $g \in AC([a,b])$. We assume that $\left(f \circ g^{-1}\right) \in AC^{n}([g(a), g(b)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. We also assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([a,b])$ (clearly here it is $f \in C([a,b])$). Then*

(i)

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f \circ g^{-1}\right)^{(k)} (g(a)) (g(t) - g(a))^{k+1} \right.\right.$$
$$\left.\left. + (-1)^k \left(f \circ g^{-1}\right)^{(k)} (g(b)) (g(b) - g(t))^{k+1} \right] \right| \leq$$
$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\alpha+2)}$$
$$\left[(g(t) - g(a))^{\alpha+1} + (g(b) - g(t))^{\alpha+1} \right], \tag{21}$$

$\forall\, t \in [a,b]$,

(ii) at $g(t) = \frac{g(a)+g(b)}{2}$, the right hand side of (21) is minimized, and we have:

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(g(b) - g(a))^{k+1}}{2^{k+1}} \right.$$
$$\left. \left[\left(f \circ g^{-1}\right)^{(k)} (g(a)) + (-1)^k \left(f \circ g^{-1}\right)^{(k)} (g(b)) \right] \right| \leq$$
$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\alpha+2)} \frac{(g(b) - g(a))^{\alpha+1}}{2^{\alpha}}, \tag{22}$$

(iii) if $\left(f \circ g^{-1}\right)^{(k)} (g(a)) = \left(f \circ g^{-1}\right)^{(k)} (g(b)) = 0$, for $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\, dg(x) \right| \leq$$
$$\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_{\infty}([a,b])} \right\} \frac{(g(b) - g(a))^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}, \tag{23}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{g(b) - g(a)}{N} \right)^{k+1} \right.$$
$$\left. \left[j^{k+1} \left(f \circ g^{-1}\right)^{(k)} (g(a)) + (-1)^k (N-j)^{k+1} \left(f \circ g^{-1}\right)^{(k)} (g(b)) \right] \right| \leq$$
$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\alpha+2)}$$
$$\left(\frac{g(b) - g(a)}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{24}$$

(v) if $\left(f\circ g^{-1}\right)^{(k)}(g(a))=\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$, *for* $k=1,...,n-1$, *from (24) we obtain*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D^{\alpha}_{a+;g}f\right\|_{L_\infty([a,b])},\left\|D^{\alpha}_{b-;g}f\right\|_{L_\infty([a,b])}\right\}}{\Gamma(\alpha+2)}$$

$$\left(\frac{g(b)-g(a)}{N}\right)^{\alpha+1}\left[j^{\alpha+1}+(N-j)^{\alpha+1}\right], \tag{25}$$

$j=0,1,2,...,N$,

(vi) when $N=2$, $j=1$, *(25) turns to*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D^{\alpha}_{a+;g}f\right\|_{L_\infty([a,b])},\left\|D^{\alpha}_{b-;g}f\right\|_{L_\infty([a,b])}\right\}}{\Gamma(\alpha+2)}\frac{(g(b)-g(a))^{\alpha+1}}{2^{\alpha}}. \tag{26}$$

(vii) when $0<\alpha\leq 1$, *inequality (26) is again valid without any boundary conditions.*

Proof. We have by (7) that

$$f(x)-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(a))}{k!}(g(x)-g(a))^k=$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}g'(t)\left(D^{\alpha}_{a+;g}f\right)(t)\,dt, \tag{27}$$

$\forall\, x\in[a,b]$.

Also by (13) we obtain

$$f(x)-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(b))}{k!}(g(x)-g(b))^k=$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1}g'(t)\left(D^{\alpha}_{b-;g}f\right)(t)\,dt, \tag{28}$$

$\forall\, x\in[a,b]$.

By (27) we derive (by [4], p. 107)

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(a))}{k!}(g(x)-g(a))^k\right|\leq$$

$$\frac{\left\|D^{\alpha}_{a+;g}f\right\|_{L_\infty([a,b])}}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}, \tag{29}$$

and by (28) we obtain

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(b))}{k!}(g(x)-g(b))^k\right|\leq$$

$$\frac{\left\| D_{b-;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha}, \tag{30}$$

$\forall\, x \in [a,b]$.

Call

$$\varphi_1 := \frac{\left\| D_{a+;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}}{\Gamma(\alpha+1)}, \tag{31}$$

and

$$\varphi_2 := \frac{\left\| D_{b-;g}^{\alpha} f \right\|_{L_{\infty}([a,b])}}{\Gamma(\alpha+1)}. \tag{32}$$

Set

$$\varphi := \max\{\varphi_1, \varphi_2\}. \tag{33}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(a))}{k!} (g(x) - g(a))^k \right| \leq \varphi (g(x) - g(a))^{\alpha},$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(b))}{k!} (g(x) - g(b))^k \right| \leq \varphi (g(b) - g(x))^{\alpha}, \tag{34}$$

$\forall\, x \in [a,b]$.

Equivalently, we have

$$\sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(a))}{k!} (g(x) - g(a))^k - \varphi (g(x) - g(a))^{\alpha} \leq \tag{35}$$

$$f(x) \leq \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(a))}{k!} (g(x) - g(a))^k + \varphi (g(x) - g(a))^{\alpha},$$

and

$$\sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(b))}{k!} (g(x) - g(b))^k - \varphi (g(b) - g(x))^{\alpha} \leq \tag{36}$$

$$f(x) \leq \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(b))}{k!} (g(x) - g(b))^k + \varphi (g(b) - g(x))^{\alpha},$$

$\forall\, x \in [a,b]$.

Let any $t \in [a,b]$, then by integration against g over $[a,t]$ and $[t,b]$, respectively, we obtain

$$\sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(a))}{(k+1)!} (g(t) - g(a))^{k+1} - \frac{\varphi}{(\alpha+1)} (g(t) - g(a))^{\alpha+1}$$

$$\leq \int_a^t f(x)\, dg(x) \leq$$

$$\sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(a))}{(k+1)!} (g(t) - g(a))^{k+1} + \frac{\varphi}{(\alpha+1)} (g(t) - g(a))^{\alpha+1}, \tag{37}$$

and

$$-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(b))}{(k+1)!}(g(t)-g(b))^{k+1}-\frac{\varphi}{(\alpha+1)}(g(b)-g(t))^{\alpha+1}$$
$$\leq\int_t^b f(x)\,dg(x)\leq$$
$$-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(b))}{(k+1)!}(g(t)-g(b))^{k+1}+\frac{\varphi}{(\alpha+1)}(g(b)-g(t))^{\alpha+1}. \tag{38}$$

Adding (37) and (38), we obtain

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(f\circ g^{-1}\right)^{(k)}(g(a))(g(t)-g(a))^{k+1}-\right.\right.$$
$$\left.\left.\left(f\circ g^{-1}\right)^{(k)}(g(b))(g(t)-g(b))^{k+1}\right]\right\}-$$
$$\frac{\varphi}{(\alpha+1)}\left[(g(t)-g(a))^{\alpha+1}+(g(b)-g(t))^{\alpha+1}\right]$$
$$\leq\int_a^b f(x)\,dg(x)\leq$$
$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(f\circ g^{-1}\right)^{(k)}(g(a))(g(t)-g(a))^{k+1}-\right.\right.$$
$$\left.\left.\left(f\circ g^{-1}\right)^{(k)}(g(b))(g(t)-g(b))^{k+1}\right]\right\}+$$
$$\frac{\varphi}{(\alpha+1)}\left[(g(t)-g(a))^{\alpha+1}+(g(b)-g(t))^{\alpha+1}\right], \tag{39}$$

$\forall\, t\in[a,b]$.

Consequently we derive:

$$\left|\int_a^b f(x)\,dg(x)-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(f\circ g^{-1}\right)^{(k)}(g(a))(g(t)-g(a))^{k+1}\right.\right. \tag{40}$$
$$\left.\left.+(-1)^k\left(f\circ g^{-1}\right)^{(k)}(g(b))(g(b)-g(t))^{k+1}\right]\right|\leq$$
$$\frac{\varphi}{(\alpha+1)}\left[(g(t)-g(a))^{\alpha+1}+(g(b)-g(t))^{\alpha+1}\right],$$

$\forall\, t\in[a,b]$.

Let us consider

$$\theta(z):=(z-g(a))^{\alpha+1}+(g(b)-z)^{\alpha+1},\ \forall\, z\in[g(a),g(b)].$$

That is

$$\theta(g(t))=(g(t)-g(a))^{\alpha+1}+(g(b)-g(t))^{\alpha+1},\ \forall\, t\in[a,b].$$

We have that

$$\theta'(z)=(\alpha+1)\left[(z-g(a))^{\alpha}-(g(b)-z)^{\alpha}\right]=0,$$

giving $(z - g(a))^{\alpha} = (g(b) - z)^{\alpha}$ and $z - g(a) = g(b) - z$, that is $z = \frac{g(a)+g(b)}{2}$ the only critical number of θ. We have that $\theta(g(a)) = \theta(g(b)) = (g(b) - g(a))^{\alpha+1}$, and $\theta\left(\frac{g(a)+g(b)}{2}\right) = \frac{(g(b)-g(a))^{\alpha+1}}{2^{\alpha}}$, which is the minimum of θ over $[g(a), g(b)]$.

Consequently the right hand side of (40) is minimized when $g(t) = \frac{g(a)+g(b)}{2}$, with value $\frac{\varphi}{(\alpha+1)} \frac{(g(b)-g(a))^{\alpha+1}}{2^{\alpha}}$.

Assuming $\left(f \circ g^{-1}\right)^{(k)}(g(a)) = \left(f \circ g^{-1}\right)^{(k)}(g(b)) = 0$, for $k = 0, 1, ..., n-1$, then we obtain that

$$\left| \int_a^b f(x)\, dg(x) \right| \le \frac{\varphi}{(\alpha+1)} \frac{(g(b) - g(a))^{\alpha+1}}{2^{\alpha}}, \tag{41}$$

which is a sharp inequality.

When $g(t) = \frac{g(a)+g(b)}{2}$, then (40) becomes

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(g(b) - g(a))^{k+1}}{2^{k+1}} \right.$$
$$\left. \left[\left(f \circ g^{-1}\right)^{(k)}(g(a)) + (-1)^k \left(f \circ g^{-1}\right)^{(k)}(g(b)) \right] \right| \le$$
$$\frac{\varphi}{(\alpha+1)} \frac{(g(b) - g(a))^{\alpha+1}}{2^{\alpha}}. \tag{42}$$

Next let $N \in \mathbb{N}$, $j = 0, 1, 2, ..., N$ and $g(t_j) = g(a) + j\left(\frac{g(b)-g(a)}{N}\right)$, that is $g(t_0) = g(a)$, $g(t_1) = g(a) + \frac{(g(b)-g(a))}{N}, ..., g(t_N) = g(b)$.

Hence it holds

$$g(t_j) - g(a) = j\left(\frac{g(b) - g(a)}{N}\right), \quad g(b) - g(t_j) = (N - j)\left(\frac{g(b) - g(a)}{N}\right), \tag{43}$$

$j = 0, 1, 2, ..., N$.

We notice

$$\left(g(t_j) - g(a)\right)^{\alpha+1} + \left(g(b) - g(t_j)\right)^{\alpha+1} =$$
$$\left(\frac{g(b) - g(a)}{N}\right)^{\alpha+1} \left[j^{\alpha+1} + (N - j)^{\alpha+1}\right], \tag{44}$$

$j = 0, 1, 2, ..., N$,

and (for $k = 0, 1, ..., n-1$)

$$\left[\left(f \circ g^{-1}\right)^{(k)}(g(a)) \left(g(t_j) - g(a)\right)^{k+1} + \right.$$
$$\left. (-1)^k \left(f \circ g^{-1}\right)^{(k)}(g(b)) \left(g(b) - g(t_j)\right)^{k+1} \right] =$$
$$\left(\frac{g(b) - g(a)}{N}\right)^{k+1} \left[\left(f \circ g^{-1}\right)^{(k)}(g(a))\, j^{k+1} + \right.$$
$$\left. (-1)^k \left(f \circ g^{-1}\right)^{(k)}(g(b))\, (N - j)^{k+1} \right], \tag{45}$$

$j = 0, 1, 2, ..., N$.

By (40) we have

$$\left| \int_a^b f(x)\,dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{g(b)-g(a)}{N} \right)^{k+1} \right.$$

$$\left. \left[\left(f \circ g^{-1} \right)^{(k)} (g(a))\, j^{k+1} + (-1)^k \left(f \circ g^{-1} \right)^{(k)} (g(b))\, (N-j)^{k+1} \right] \right| \leq$$

$$\left(\frac{\varphi}{\alpha+1} \right) \left(\frac{g(b)-g(a)}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{46}$$

$j = 0, 1, 2, ..., N$.

If $\left(f \circ g^{-1} \right)^{(k)} (g(a)) = \left(f \circ g^{-1} \right)^{(k)} (g(b)) = 0$, $k = 1, ..., n-1$, then (46) becomes

$$\left| \int_a^b f(x)\,dg(x) - \left(\frac{g(b)-g(a)}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\left(\frac{\varphi}{\alpha+1} \right) \left(\frac{g(b)-g(a)}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{47}$$

$j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (47) becomes

$$\left| \int_a^b f(x)\,dg(x) - \left(\frac{g(b)-g(a)}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\left(\frac{\varphi}{\alpha+1} \right) 2 \frac{(g(b)-g(a))^{\alpha+1}}{2^{\alpha+1}} = \left(\frac{\varphi}{\alpha+1} \right) \frac{(g(b)-g(a))^{\alpha+1}}{2^{\alpha}}. \tag{48}$$

Let $0 < \alpha \leq 1$, then $n = \lceil \alpha \rceil = 1$.

In that case, without any boundary conditions, we derive from (48) again that

$$\left| \int_a^b f(x)\,dg(x) - \left(\frac{g(b)-g(a)}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\left(\frac{\varphi}{\alpha+1} \right) \frac{(g(b)-g(a))^{\alpha+1}}{2^{\alpha}}. \tag{49}$$

We have proved theorem in all possible cases. □

Next we give modified g-fractional Iyengar type inequalities:

Theorem 7. *Let g be a strictly increasing function and $g \in AC([a,b])$, and $f \in C([a,b])$. Let $0 < \alpha \leq 1$, and $F_k := D_{a+;g}^{k\alpha} f$, for $k = 0, 1, ..., n+1$; $n \in \mathbb{N}$. We assume that $F_k \circ g^{-1} \in AC([g(a), g(b)])$ and $\left(F_k \circ g^{-1} \right)' \circ g \in L_\infty([a,b])$. Also let $\overline{F_k} := D_{b-;g}^{k\alpha} f$, for $k = 0, 1, ..., n+1$, they fulfill $\overline{F_k} \circ g^{-1} \in AC([g(a), g(b)])$ and $\left(\overline{F_k} \circ g^{-1} \right)' \circ g \in L_\infty([a,b])$. Then*

(i)

$$\left| \int_a^b f(x)\,dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left[\left(D_{a+;g}^{i\alpha} f \right)(a) (g(t) - g(a))^{i\alpha+1} \right. \right. \right.$$

$$\left. \left. \left. + \left(D_{b-;g}^{i\alpha} f \right)(b) (g(b) - g(t))^{i\alpha+1} \right] \right\} \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]} \right\}}{\Gamma((n+1)\alpha+2)}$$

$$\left[(g(t) - g(a))^{(n+1)\alpha+1} + (g(b) - g(t))^{(n+1)\alpha+1} \right], \tag{50}$$

$\forall\, t \in [a, b]$,

(ii) at $g(t) = \frac{g(a)+g(b)}{2}$*, the right hand side of (50) is minimized, and we have:*

$$\left| \int_a^b f(x)\, dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \frac{(g(b) - g(a))^{i\alpha+1}}{2^{i\alpha+1}} \right. \right.$$

$$\left. \left. \left[\left(D_{a+;g}^{i\alpha} f \right)(a) + \left(D_{b-;g}^{i\alpha} f \right)(b) \right] \right\} \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]} \right\}}{\Gamma((n+1)\alpha+2)} \frac{(g(b) - g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}, \tag{51}$$

(iii) assuming $\left(D_{a+;g}^{i\alpha} f \right)(a) = \left(D_{b-;g}^{i\alpha} f \right)(b) = 0$*, for* $i = 0, 1, ..., n$*, we obtain*

$$\left| \int_a^b f(x)\, dg(x) \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]} \right\}}{\Gamma((n+1)\alpha+2)} \frac{(g(b) - g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}, \tag{52}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$*, it holds*

$$\left| \int_a^b f(x)\, dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left(\frac{g(b) - g(a)}{N} \right)^{i\alpha+1} \right. \right.$$

$$\left. \left. \left[\left(D_{a+;g}^{i\alpha} f \right)(a)\, j^{i\alpha+1} + \left(D_{b-;g}^{i\alpha} f \right)(b)\, (N-j)^{i\alpha+1} \right] \right\} \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]} \right\}}{\Gamma((n+1)\alpha+2)}$$

$$\left(\frac{g(b) - g(a)}{N} \right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1} \right], \tag{53}$$

(v) if $\left(D_{a+;g}^{i\alpha} f \right)(a) = \left(D_{b-;g}^{i\alpha} f \right)(b) = 0$*, for* $i = 1, ..., n$*, from (53) we obtain:*

$$\left| \int_a^b f(x)\, dg(x) - \left(\frac{g(b) - g(a)}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\max \left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,b]} \right\}}{\Gamma((n+1)\alpha+2)}$$

$$\left(\frac{g(b) - g(a)}{N} \right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1} \right], \tag{54}$$

for $j = 0, 1, 2, ..., N$,

(vi) when $N = 2$*,* $j = 1$*, (54) becomes*

$$\left| \int_a^b f(x)\, dg(x) - \left(\frac{g(b) - g(a)}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{\left\|D_{a+;g}^{(n+1)\alpha}f\right\|_{\infty,[a,b]},\left\|D_{b-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,b]}\right\}}{\Gamma\left((n+1)\alpha+2\right)}\frac{\left(g\left(b\right)-g\left(a\right)\right)^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}. \tag{55}$$

Proof. We have by (19) that

$$f\left(x\right)=\sum_{i=0}^{n}\frac{\left(g\left(x\right)-g\left(a\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{a+;g}^{i\alpha}f\right)\left(a\right)+$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_{a}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{(n+1)\alpha-1}g'\left(t\right)\left(D_{a+;g}^{(n+1)\alpha}f\right)\left(t\right)dt, \tag{56}$$

$\forall\, x\in[a,b]$.

Also by (16) we find

$$f\left(x\right)=\sum_{i=0}^{n}\frac{\left(g\left(b\right)-g\left(x\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{b-;g}^{i\alpha}f\right)\left(b\right)+$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_{x}^{b}\left(g\left(t\right)-g\left(x\right)\right)^{(n+1)\alpha-1}g'\left(t\right)\left(D_{b-;g}^{(n+1)\alpha}f\right)\left(t\right)dt, \tag{57}$$

$\forall\, x\in[a,b]$.

Clearly here it is $D_{a+;g}^{(n+1)\alpha}f, D_{b-;g}^{(n+1)\alpha}f\in C\left([a,b]\right)$.

By (56) we derive (by [4], p. 107)

$$\left|f\left(x\right)-\sum_{i=0}^{n}\frac{\left(g\left(x\right)-g\left(a\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{a+;g}^{i\alpha}f\right)\left(a\right)\right|\leq$$

$$\left\|D_{a+;g}^{(n+1)\alpha}f\right\|_{\infty,[a,b]}\frac{\left(g\left(x\right)-g\left(a\right)\right)^{(n+1)\alpha}}{\Gamma\left((n+1)\alpha+1\right)}, \tag{58}$$

and by (57) we obtain

$$\left|f\left(x\right)-\sum_{i=0}^{n}\frac{\left(g\left(b\right)-g\left(x\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{b-;g}^{i\alpha}f\right)\left(b\right)\right|\leq$$

$$\left\|D_{b-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,b]}\frac{\left(g\left(b\right)-g\left(x\right)\right)^{(n+1)\alpha}}{\Gamma\left((n+1)\alpha+1\right)}, \tag{59}$$

$\forall\, x\in[a,b]$.

Call

$$\gamma_1:=\frac{\left\|D_{a+;g}^{(n+1)\alpha}f\right\|_{\infty,[a,b]}}{\Gamma\left((n+1)\alpha+1\right)}, \tag{60}$$

and

$$\gamma_2:=\frac{\left\|D_{b-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,b]}}{\Gamma\left((n+1)\alpha+1\right)}. \tag{61}$$

Set

$$\gamma:=\max\left\{\gamma_1,\gamma_2\right\}. \tag{62}$$

That is

$$\left|f\left(x\right)-\sum_{i=0}^{n}\frac{\left(g\left(x\right)-g\left(a\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{a+;g}^{i\alpha}f\right)\left(a\right)\right|\leq\gamma\left(g\left(x\right)-g\left(a\right)\right)^{(n+1)\alpha}, \tag{63}$$

and

$$\left| f(x) - \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g}^{i\alpha} f \right)(b) \right| \leq \gamma \left(g(b) - g(x) \right)^{(n+1)\alpha}, \tag{64}$$

$\forall\, x \in [a, b]$.

Equivalently, we have

$$\sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{a+;g}^{i\alpha} f \right)(a) - \gamma \left(g(x) - g(a) \right)^{(n+1)\alpha} \leq f(x) \leq$$

$$\sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{a+;g}^{i\alpha} f \right)(a) + \gamma \left(g(x) - g(a) \right)^{(n+1)\alpha}, \tag{65}$$

and

$$\sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g}^{i\alpha} f \right)(b) - \gamma \left(g(b) - g(x) \right)^{(n+1)\alpha} \leq f(x) \leq$$

$$\sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g}^{i\alpha} f \right)(b) + \gamma \left(g(b) - g(x) \right)^{(n+1)\alpha}, \tag{66}$$

$\forall\, x \in [a, b]$.

Let any $t \in [a, b]$, then by integration against g over $[a, t]$ and $[t, b]$, respectively, we obtain

$$\sum_{i=0}^{n} \left(D_{a+;g}^{i\alpha} f \right)(a) \frac{(g(t) - g(a))^{i\alpha+1}}{\Gamma(i\alpha+2)} - \frac{\gamma}{((n+1)\alpha+1)} \left(g(t) - g(a) \right)^{(n+1)\alpha+1}$$

$$\leq \int_a^t f(x)\, dg(x) \leq$$

$$\sum_{i=0}^{n} \left(D_{a+;g}^{i\alpha} f \right)(a) \frac{(g(t) - g(a))^{i\alpha+1}}{\Gamma(i\alpha+2)} + \frac{\gamma}{((n+1)\alpha+1)} \left(g(t) - g(a) \right)^{(n+1)\alpha+1}, \tag{67}$$

and

$$\sum_{i=0}^{n} \frac{(g(b) - g(t))^{i\alpha+1}}{\Gamma(i\alpha+2)} \left(D_{b-;g}^{i\alpha} f \right)(b) - \frac{\gamma}{((n+1)\alpha+1)} \left(g(b) - g(t) \right)^{(n+1)\alpha+1}$$

$$\leq \int_t^b f(x)\, dg(x) \leq$$

$$\sum_{i=0}^{n} \frac{(g(b) - g(t))^{i\alpha+1}}{\Gamma(i\alpha+2)} \left(D_{b-;g}^{i\alpha} f \right)(b) + \frac{\gamma}{((n+1)\alpha+1)} \left(g(b) - g(t) \right)^{(n+1)\alpha+1}. \tag{68}$$

Adding (67) and (68), we obtain

$$\left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left[\left(D_{a+;g}^{i\alpha} f \right)(a) \left(g(t) - g(a) \right)^{i\alpha+1} + \left(D_{b-;g}^{i\alpha} f \right)(b) \left(g(b) - g(t) \right)^{i\alpha+1} \right] \right\}$$

$$- \frac{\gamma}{((n+1)\alpha+1)} \left[\left(g(t) - g(a) \right)^{(n+1)\alpha+1} + \left(g(b) - g(t) \right)^{(n+1)\alpha+1} \right]$$

$$\leq \int_a^b f(x)\, dg(x) \leq$$

$$\left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left[\left(D_{a+;g}^{i\alpha} f \right)(a) \left(g(t) - g(a) \right)^{i\alpha+1} + \left(D_{b-;g}^{i\alpha} f \right)(b) \left(g(b) - g(t) \right)^{i\alpha+1} \right] \right\}$$

$$+ \frac{\gamma}{((n+1)\alpha+1)} \left[\left(g(t) - g(a) \right)^{(n+1)\alpha+1} + \left(g(b) - g(t) \right)^{(n+1)\alpha+1} \right], \tag{69}$$

$\forall\, t \in [a,b]$.

Consequently, we derive:

$$\left| \int_a^b f(x)\,dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left[\left(D_{a+;g}^{i\alpha} f \right)(a) \left(g(t) - g(a) \right)^{i\alpha+1} \right. \right. \right.$$

$$\left. \left. \left. + \left(D_{b-;g}^{i\alpha} f \right)(b) \left(g(b) - g(t) \right)^{i\alpha+1} \right] \right\} \right| \leq$$

$$\frac{\gamma}{((n+1)\alpha+1)} \left[\left(g(t) - g(a) \right)^{(n+1)\alpha+1} + \left(g(b) - g(t) \right)^{(n+1)\alpha+1} \right], \tag{70}$$

$\forall\, t \in [a,b]$.

Let us consider

$$\phi(z) := (z - g(a))^{(n+1)\alpha+1} + (g(b) - z)^{(n+1)\alpha+1},$$

$\forall\, z \in [g(a), g(b)]$.

That is

$$\phi(g(t)) = (g(t) - g(a))^{(n+1)\alpha+1} + (g(b) - g(t))^{(n+1)\alpha+1},$$

$\forall\, t \in [a,b]$.

We have that

$$\phi'(z) = ((n+1)\alpha+1) \left[(z - g(a))^{(n+1)\alpha} - (g(b) - z)^{(n+1)\alpha} \right] = 0,$$

giving $(z - g(a))^{(n+1)\alpha} = (g(b) - z)^{(n+1)\alpha}$ and $z - g(a) = g(b) - z$, that is $z = \frac{g(a)+g(b)}{2}$ the only critical number of ϕ. We have that

$$\phi(g(a)) = \phi(g(b)) = (g(b) - g(a))^{(n+1)\alpha+1},$$

and

$$\phi\left(\frac{g(a)+g(b)}{2} \right) = \frac{(g(b) - g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}},$$

which is the minimum of ϕ over $[g(a), g(b)]$.

Consequently, the right hand side of (70) is minimized when $g(t) = \frac{g(a)+g(b)}{2}$, for some $t \in [a,b]$, with value $\frac{\gamma}{((n+1)\alpha+1)} \frac{(g(b)-g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}$.

Assuming $\left(D_{a+;g}^{i\alpha} f \right)(a) = \left(D_{b-;g}^{i\alpha} f \right)(b) = 0$, $i = 0, 1, ..., n$, then we obtain that

$$\left| \int_a^b f(x)\,dg(x) \right| \leq \frac{\gamma}{((n+1)\alpha+1)} \frac{(g(b) - g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}, \tag{71}$$

which is a sharp inequality.

When $g(t) = \frac{g(a)+g(b)}{2}$, then (70) becomes

$$\left| \int_a^b f(x)\,dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \frac{(g(b) - g(a))^{i\alpha+1}}{2^{i\alpha+1}} \right. \right.$$

$$\left. \left. \left[\left(D_{a+;g}^{i\alpha} f \right)(a) + \left(D_{b-;g}^{i\alpha} \right)(b) \right] \right\} \right| \leq$$

$$\frac{\gamma}{((n+1)\alpha+1)} \frac{(g(b) - g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}. \tag{72}$$

Next let $N \in \mathbb{N}$, $j = 0, 1, 2, ..., N$ and $g(t_j) = g(a) + j\left(\frac{g(b)-g(a)}{N}\right)$, that is $g(t_0) = g(a)$, $g(t_1) = g(a) + \frac{(g(b)-g(a))}{N}, ..., g(t_N) = g(b)$.

Hence it holds

$$g(t_j) - g(a) = j\left(\frac{g(b)-g(a)}{N}\right), \; g(b) - g(t_j) = (N-j)\left(\frac{g(b)-g(a)}{N}\right), \tag{73}$$

$j = 0, 1, 2, ..., N$.

We notice

$$(g(t_j) - g(a))^{(n+1)\alpha+1} + (g(b) - g(t_j))^{(n+1)\alpha+1} =$$

$$\left(\frac{g(b)-g(a)}{N}\right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1}\right], \tag{74}$$

$j = 0, 1, 2, ..., N$,

and (for $i = 0, 1, ..., n$)

$$\left[\left(D^{i\alpha}_{a+;g}f\right)(a)\,(g(t_j) - g(a))^{i\alpha+1} + \left(D^{i\alpha}_{b-;g}f\right)(b)\,(g(b) - g(t_j))^{i\alpha+1}\right] =$$

$$\left(\frac{g(b)-g(a)}{N}\right)^{i\alpha+1} \left[\left(D^{i\alpha}_{a+;g}f\right)(a)\, j^{i\alpha+1} + \left(D^{i\alpha}_{b-;g}\right) f(b)\,(N-j)^{i\alpha+1}\right], \tag{75}$$

for $j = 0, 1, 2, ..., N$.

By (70) we have

$$\left|\int_a^b f(x)\, dg(x) - \left\{\sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left(\frac{g(b)-g(a)}{N}\right)^{i\alpha+1}\right.\right.$$

$$\left.\left.\left[\left(D^{i\alpha}_{a+;g}f\right)(a)\, j^{i\alpha+1} + \left(D^{i\alpha}_{b-;g}f\right)(b)\,(N-j)^{i\alpha+1}\right]\right\}\right| \leq$$

$$\frac{\gamma}{((n+1)\alpha+1)} \left(\frac{g(b)-g(a)}{N}\right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1}\right], \tag{76}$$

$j = 0, 1, 2, ..., N$.

If $\left(D^{i\alpha}_{a+;g}f\right)(a) = \left(D^{i\alpha}_{b-;g}f\right)(b) = 0$, $i = 1, ..., n$, then (76) becomes

$$\left|\int_a^b f(x)\, dg(x) - \left(\frac{g(b)-g(a)}{N}\right)[jf(a) + (N-j)f(b)]\right| \leq$$

$$\frac{\gamma}{((n+1)\alpha+1)} \left(\frac{g(b)-g(a)}{N}\right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1}\right], \tag{77}$$

$j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (77) becomes

$$\left|\int_a^b f(x)\, dg(x) - \left(\frac{g(b)-g(a)}{2}\right)(f(a) + f(b))\right| \leq$$

$$\frac{\gamma}{((n+1)\alpha+1)} \frac{2\,(g(b)-g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha+1}} =$$

$$\frac{\gamma}{((n+1)\alpha+1)} \frac{(g(b)-g(a))^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}. \tag{78}$$

We have proved theorem in all possible cases. □

We give L_1 variants of last theorems:

Theorem 8. *All as in Theorem 6 with* $\alpha \geq 1$. *If* $\alpha = n \in \mathbb{N}$, *we assume that* $\left(f \circ g^{-1}\right)^{(n)} \circ g \in C([a,b])$. *Then*

(i)

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f \circ g^{-1}\right)^{(k)} (g(a)) (g(t) - g(a))^{k+1} \right.\right.$$

$$\left.\left. + (-1)^k \left(f \circ g^{-1}\right)^{(k)} (g(b)) (g(b) - g(t))^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_1([a,b],g)}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_1([a,b],g)} \right\}}{\Gamma(\alpha+1)}$$

$$\left[(g(t) - g(a))^{\alpha} + (g(b) - g(t))^{\alpha} \right], \tag{79}$$

$\forall\, t \in [a,b]$,

(ii) at $g(t) = \frac{g(a)+g(b)}{2}$, *the right hand side of (79) is minimized, and we find:*

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(g(b) - g(a))^{k+1}}{2^{k+1}} \right.$$

$$\left. \left[\left(f \circ g^{-1}\right)^{(k)} (g(a)) + (-1)^k \left(f \circ g^{-1}\right)^{(k)} (g(b)) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_1([a,b],g)}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_1([a,b],g)} \right\}}{\Gamma(\alpha+1)} \frac{(g(b) - g(a))^{\alpha}}{2^{\alpha-1}}, \tag{80}$$

(iii) if $\left(f \circ g^{-1}\right)^{(k)} (g(a)) = \left(f \circ g^{-1}\right)^{(k)} (g(b)) = 0$, *for* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\, dg(x) \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_1([a,b],g)}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_1([a,b],g)} \right\}}{\Gamma(\alpha+1)} \frac{(g(b) - g(a))^{\alpha}}{2^{\alpha-1}}, \tag{81}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds that*

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{g(b) - g(a)}{N} \right)^{k+1} \right.$$

$$\left. \left[j^{k+1} \left(f \circ g^{-1}\right)^{(k)} (g(a)) + (-1)^k (N - j)^{k+1} \left(f \circ g^{-1}\right)^{(k)} (g(b)) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_1([a,b],g)}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_1([a,b],g)} \right\}}{\Gamma(\alpha+1)}$$

$$\left(\frac{g(b) - g(a)}{N} \right)^{\alpha} \left[j^{\alpha} + (N - j)^{\alpha} \right], \tag{82}$$

(v) if $\left(f\circ g^{-1}\right)^{(k)}(g(a))=\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$, *for* $k=1,...,n-1$, *from* (82) *we obtain*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{N}\right)[jf(a)+(N-j)f(b)]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+;g}^{\alpha}f\right\|_{L_1([a,b],g)},\left\|D_{b-;g}^{\alpha}f\right\|_{L_1([a,b],g)}\right\}}{\Gamma(\alpha+1)}$$

$$\left(\frac{g(b)-g(a)}{N}\right)^{\alpha}\left[j^{\alpha}+(N-j)^{\alpha}\right], \tag{83}$$

$j=0,1,2,...,N$,

(vi) when $N=2$, $j=1$, (83) *turns to*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+;g}^{\alpha}f\right\|_{L_1([a,b],g)},\left\|D_{b-;g}^{\alpha}f\right\|_{L_1([a,b],g)}\right\}}{\Gamma(\alpha+1)}\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha-1}}. \tag{84}$$

Proof. From (27) we have

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(a))}{k!}(g(x)-g(a))^k\right|\leq$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}g'(t)\left|\left(D_{a+;g}^{\alpha}f\right)(t)\right|dt\leq$$

$$\frac{(g(x)-g(a))^{\alpha-1}}{\Gamma(\alpha)}\int_a^x g'(t)\left|\left(D_{a+;g}^{\alpha}f\right)(t)\right|dt\leq \tag{85}$$

$$\frac{(g(x)-g(a))^{\alpha-1}}{\Gamma(\alpha)}\int_a^b g'(t)\left|\left(D_{a+;g}^{\alpha}f\right)(t)\right|dt=$$

$$\frac{(g(x)-g(a))^{\alpha-1}}{\Gamma(\alpha)}\int_a^b \left|\left(D_{a+;g}^{\alpha}f\right)(t)\right|dg(t)=$$

$$\frac{\left\|D_{a+;g}^{\alpha}f\right\|_{L_1([a,b],g)}}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha-1},$$

$\forall\, x\in[a,b]$.

Similarly, from (28) we obtain

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(b))}{k!}(g(x)-g(b))^k\right|\leq$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1}g'(t)\left|\left(D_{b-;g}^{\alpha}f\right)(t)\right|dt\leq$$

$$\frac{(g(b)-g(x))^{\alpha-1}}{\Gamma(\alpha)}\int_x^b \left|\left(D_{b-;g}^{\alpha}f\right)(t)\right|dg(t)\leq \tag{86}$$

$$\frac{\left\| D_{b-;g}^{\alpha} f \right\|_{L_1([a,b],g)}}{\Gamma(\alpha)} (g(b) - g(x))^{\alpha - 1},$$

$\forall\, x \in [a,b]$.

Call

$$\delta := \max \left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_1([a,b],g)}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_1([a,b],g)} \right\}. \tag{87}$$

We have proved that

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(a))}{k!} (g(x) - g(a))^k \right| \leq$$

$$\frac{\delta}{\Gamma(\alpha)} (g(x) - g(a))^{\alpha - 1}, \tag{88}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(b))}{k!} (g(x) - g(b))^k \right| \leq$$

$$\frac{\delta}{\Gamma(\alpha)} (g(b) - g(x))^{\alpha - 1}, \tag{89}$$

$\forall\, x \in [a,b]$.

The rest of the proof is as in Theorem 6. □

It follows

Theorem 9. *All as in Theorem 7, with* $\frac{1}{n+1} \leq \alpha \leq 1$. *Call*

$$\rho := \max \left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{L_1([a,b],g)}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{L_1([a,b],g)} \right\}. \tag{90}$$

Then

(i)

$$\left| \int_a^b f(x)\, dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha + 2)} \left[\left(D_{a+;g}^{i\alpha} f \right)(a) (g(t) - g(a))^{i\alpha + 1} \right.\right.\right.$$

$$\left.\left.\left. + \left(D_{b-;g}^{i\alpha} f \right)(b) (g(b) - g(t))^{i\alpha + 1} \right] \right\} \right| \leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha + 1)} \left[(g(t) - g(a))^{(n+1)\alpha} + (g(b) - g(t))^{(n+1)\alpha} \right], \tag{91}$$

$\forall\, t \in [a,b]$,

(ii) at $g(t) = \frac{g(a)+g(b)}{2}$, *the right hand side of (91) is minimized, and we find:*

$$\left| \int_a^b f(x)\, dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha + 2)} \frac{(g(b) - g(a))^{i\alpha + 1}}{2^{i\alpha + 1}} \right.\right.$$

$$\left.\left. \left[\left(D_{a+;g}^{i\alpha} f \right)(a) + \left(D_{b-;g}^{i\alpha} f \right)(b) \right] \right\} \right| \leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha + 1)} \frac{(g(b) - g(a))^{(n+1)\alpha}}{2^{(n+1)\alpha - 1}}, \tag{92}$$

(iii) assuming $\left(D_{a+;g}^{i\alpha}f\right)(a)=\left(D_{b-;g}^{i\alpha}f\right)(b)=0$, $i=0,1,...,n$, *we obtain*

$$\left|\int_a^b f(x)\,dg(x)\right|\leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha+1)}\frac{(g(b)-g(a))^{(n+1)\alpha}}{2^{(n+1)\alpha-1}},\tag{93}$$

which is a sharp inequality,

(iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, *it holds that*

$$\left|\int_a^b f(x)\,dg(x)-\left\{\sum_{i=0}^{n}\frac{1}{\Gamma(i\alpha+2)}\left(\frac{g(b)-g(a)}{N}\right)^{i\alpha+1}\right.\right.$$

$$\left.\left.\left[\left(D_{a+;g}^{i\alpha}f\right)(a)\,j^{i\alpha+1}+\left(D_{b-;g}^{i\alpha}f\right)(b)\,(N-j)^{i\alpha+1}\right]\right\}\right|\leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha+1)}\left(\frac{g(b)-g(a)}{N}\right)^{(n+1)\alpha}\left[j^{(n+1)\alpha}+(N-j)^{(n+1)\alpha}\right],\tag{94}$$

(v) if $\left(D_{a+;g}^{i\alpha}f\right)(a)=\left(D_{b-;g}^{i\alpha}f\right)(b)=0$, $i=1,...,n$, *from (94) we find:*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{N}\right)[jf(a)+(N-j)f(b)]\right|\leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha+1)}\left(\frac{g(b)-g(a)}{N}\right)^{(n+1)\alpha}\left[j^{(n+1)\alpha}+(N-j)^{(n+1)\alpha}\right],\tag{95}$$

for $j=0,1,2,...,N$,

(vi) when $N=2$ *and* $j=1$, *(95) becomes*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\rho}{\Gamma((n+1)\alpha+1)}\frac{(g(b)-g(a))^{(n+1)\alpha}}{2^{(n+1)\alpha-1}}.\tag{96}$$

Proof. By (56) we obtain

$$\left|f(x)-\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a)\right|\leq$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_a^x(g(x)-g(t))^{(n+1)\alpha-1}g'(t)\left|\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\right|dt\leq$$

$$\frac{(g(x)-g(a))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}\int_a^x g'(t)\left|\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\right|dt\leq\tag{97}$$

$$\frac{(g(x)-g(a))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}\int_a^b g'(t)\left|\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\right|dt=$$

$$\frac{(g(x)-g(a))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}\int_a^b\left|\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\right|dg(t)=$$

$$\frac{\left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{L_1([a,b],g)}}{\Gamma((n+1)\alpha)} (g(x) - g(a))^{(n+1)\alpha - 1},$$

$\forall\, x \in [a,b]$.

Similarly, from (57) we derive

$$\left| f(x) - \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-;g}^{i\alpha} f \right)(b) \right| \leq$$
$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left| \left(D_{b-;g}^{(n+1)\alpha} f \right)(t) \right| dt \leq$$
$$\frac{(g(b) - g(x))^{(n+1)\alpha - 1}}{\Gamma((n+1)\alpha)} \int_x^b \left| \left(D_{b-;g}^{(n+1)\alpha} f \right)(t) \right| dg(t) \leq \tag{98}$$
$$\frac{\left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{L_1([a,b],g)}}{\Gamma((n+1)\alpha)} (g(b) - g(x))^{(n+1)\alpha - 1},$$

$\forall\, x \in [a,b]$.

We have proved that

$$\left| f(x) - \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{a+;g}^{i\alpha} f \right)(a) \right| \leq$$
$$\frac{\rho}{\Gamma((n+1)\alpha)} (g(x) - g(a))^{(n+1)\alpha - 1}, \tag{99}$$

and

$$\left| f(x) - \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-;g}^{i\alpha} f \right)(b) \right| \leq$$
$$\frac{\rho}{\Gamma((n+1)\alpha)} (g(b) - g(x))^{(n+1)\alpha - 1}, \tag{100}$$

$\forall\, x \in [a,b]$.

The rest of the proof is as in Theorem 7. □

Next follow L_p variants of Theorems 6 and 7.

Theorem 10. *All as in Theorem 6 with $\alpha \geq 1$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. If $\alpha = n \in \mathbb{N}$, we assume that $\left(f \circ g^{-1} \right)^{(n)} \circ g \in C([a,b])$. Set*

$$\mu := \max \left\{ \left\| D_{a+;g}^{\alpha} f \right\|_{L_q([a,b],g)}, \left\| D_{b-;g}^{\alpha} f \right\|_{L_q([a,b],g)} \right\}. \tag{101}$$

Then

(i)

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f \circ g^{-1} \right)^{(k)} (g(a)) (g(t) - g(a))^{k+1} \right. \right.$$
$$\left. \left. + (-1)^k \left(f \circ g^{-1} \right)^{(k)} (g(b)) (g(b) - g(t))^{k+1} \right] \right| \leq$$
$$\frac{\mu}{\Gamma(\alpha) \left(\alpha + \frac{1}{p} \right) (p(\alpha - 1) + 1)^{\frac{1}{p}}}$$

$$\left[(g(t) - g(a))^{\alpha + \frac{1}{p}} + (g(b) - g(t))^{\alpha + \frac{1}{p}} \right], \tag{102}$$

$\forall\, t \in [a, b]$,

(ii) at $g(t) = \frac{g(a)+g(b)}{2}$, *the right hand side of (102) is minimized, and we have:*

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(g(b) - g(a))^{k+1}}{2^{k+1}} \right.$$

$$\left. \left[\left(f \circ g^{-1}\right)^{(k)} (g(a)) + (-1)^k \left(f \circ g^{-1}\right)^{(k)} (g(b)) \right] \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha) \left(\alpha + \frac{1}{p}\right) (p(\alpha - 1) + 1)^{\frac{1}{p}}} \frac{(g(b) - g(a))^{\alpha + \frac{1}{p}}}{2^{\alpha - \frac{1}{q}}}, \tag{103}$$

(iii) if $\left(f \circ g^{-1}\right)^{(k)} (g(a)) = \left(f \circ g^{-1}\right)^{(k)} (g(b)) = 0$, *for* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\, dg(x) \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha) \left(\alpha + \frac{1}{p}\right) (p(\alpha - 1) + 1)^{\frac{1}{p}}} \frac{(g(b) - g(a))^{\alpha + \frac{1}{p}}}{2^{\alpha - \frac{1}{q}}}, \tag{104}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{g(b) - g(a)}{N} \right)^{k+1} \right.$$

$$\left. \left[j^{k+1} \left(f \circ g^{-1}\right)^{(k)} (g(a)) + (-1)^k (N - j)^{k+1} \left(f \circ g^{-1}\right)^{(k)} (g(b)) \right] \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha) \left(\alpha + \frac{1}{p}\right) (p(\alpha - 1) + 1)^{\frac{1}{p}}} \left(\frac{g(b) - g(a)}{N} \right)^{\alpha + \frac{1}{p}} \left[j^{\alpha + \frac{1}{p}} + (N - j)^{\alpha + \frac{1}{p}} \right], \tag{105}$$

(v) if $\left(f \circ g^{-1}\right)^{(k)} (g(a)) = \left(f \circ g^{-1}\right)^{(k)} (g(b)) = 0$, *for* $k = 1, ..., n-1$, *from (105) we obtain*

$$\left| \int_a^b f(x)\, dg(x) - \left(\frac{g(b) - g(a)}{N} \right) [j f(a) + (N - j) f(b)] \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha) \left(\alpha + \frac{1}{p}\right) (p(\alpha - 1) + 1)^{\frac{1}{p}}} \left(\frac{g(b) - g(a)}{N} \right)^{\alpha + \frac{1}{p}} \left[j^{\alpha + \frac{1}{p}} + (N - j)^{\alpha + \frac{1}{p}} \right], \tag{106}$$

$j = 0, 1, 2, ..., N$,

(vi) when $N = 2$, $j = 1$, *(106) turns to*

$$\left| \int_a^b f(x)\, dg(x) - \left(\frac{g(b) - g(a)}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha) \left(\alpha + \frac{1}{p}\right) (p(\alpha - 1) + 1)^{\frac{1}{p}}} \frac{(g(b) - g(a))^{\alpha + \frac{1}{p}}}{2^{\alpha - \frac{1}{q}}}. \tag{107}$$

Proof. From (27) we find

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(x) - g(a))^k \right| \leq$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left| \left(D_{a+;g}^{\alpha} f\right)(t) \right| dt =$$

(by [5], p. 439)

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} \left| \left(D_{a+;g}^{\alpha} f\right)(t) \right| dg(t) \leq \tag{108}$$

(by [6])

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x (g(x) - g(t))^{p(\alpha-1)} dg(t) \right)^{\frac{1}{p}} \left(\int_a^x \left| \left(D_{a+;g}^{\alpha} f\right)(t) \right|^q dg(t) \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\alpha)} \frac{(g(x) - g(a))^{\alpha - \frac{1}{q}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{a+;g}^{\alpha} f \right\|_{L_q([a,b],g)}.$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(x) - g(a))^k \right| \leq$$

$$\frac{\left\| D_{a+;g}^{\alpha} f \right\|_{L_q([a,b],g)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} (g(x) - g(a))^{\alpha - \frac{1}{q}}, \tag{109}$$

$\forall\, x \in [a,b]$.

Similarly, from (28) we obtain

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(b))}{k!} (g(x) - g(b))^k \right| \leq$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left| \left(D_{b-;g}^{\alpha} f\right)(t) \right| dt =$$

(by [5], p. 439)

$$\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} \left| \left(D_{b-;g}^{\alpha} f\right)(t) \right| dg(t) \leq$$

(by [6])

$$\frac{1}{\Gamma(\alpha)} \left(\int_x^b (g(t) - g(x))^{p(\alpha-1)} dg(t) \right)^{\frac{1}{p}} \left(\int_x^b \left| \left(D_{b-;g}^{\alpha} f\right)(t) \right|^q dg(t) \right)^{\frac{1}{q}} \leq \tag{110}$$

$$\frac{1}{\Gamma(\alpha)} \frac{(g(b) - g(x))^{\alpha - \frac{1}{q}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-;g}^{\alpha} f \right\|_{L_q([a,b],g)}.$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(b))}{k!} (g(x) - g(b))^k \right| \leq$$

$$\frac{\left\| D_{b-;g}^{\alpha} f \right\|_{L_q([a,b],g)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} (g(b) - g(x))^{\alpha - \frac{1}{q}}, \tag{111}$$

$\forall\, x \in [a,b]$.

We have proved that

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(a))}{k!} (g(x) - g(a))^k \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}} (g(x) - g(a))^{\alpha - \frac{1}{q}}, \tag{112}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(b))}{k!} (g(x) - g(b))^k \right| \leq$$

$$\frac{\mu}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}} (g(b) - g(x))^{\alpha - \frac{1}{q}}, \tag{113}$$

$\forall\, x \in [a,b]$.

The rest of the proof is as in Theorem 6. □

We continue with

Theorem 11. *All as in Theorem 7, with* $\frac{1}{n+1} \leq \alpha \leq 1$, *and* $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Set*

$$\theta := \max\left\{ \left\| D_{a+;g}^{(n+1)\alpha} f \right\|_{L_q([a,b],g)}, \left\| D_{b-;g}^{(n+1)\alpha} f \right\|_{L_q([a,b],g)} \right\}. \tag{114}$$

Then
(i)

$$\left| \int_a^b f(x)\, dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left[\left(D_{a+;g}^{i\alpha} f \right)(a) (g(t) - g(a))^{i\alpha+1} \right.\right.\right.$$

$$\left.\left.\left. + \left(D_{b-;g}^{i\alpha} f \right)(b) (g(b) - g(t))^{i\alpha+1} \right] \right\} \right| \leq$$

$$\frac{\theta}{\Gamma((n+1)\alpha)\left((n+1)\alpha + \frac{1}{p}\right)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}$$

$$\left[(g(t) - g(a))^{(n+1)\alpha + \frac{1}{p}} + (g(b) - g(t))^{(n+1)\alpha + \frac{1}{p}} \right], \tag{115}$$

$\forall\, t \in [a,b]$,

(ii) at $g(t) = \frac{g(a)+g(b)}{2}$, *the right hand side of (115) is minimized, and we have:*

$$\left| \int_a^b f(x)\, dg(x) - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \frac{(g(b)-g(a))^{i\alpha+1}}{2^{i\alpha+1}} \right.\right.$$

$$\left.\left. \left[\left(D_{a+;g}^{i\alpha} f \right)(a) + \left(D_{b-;g}^{i\alpha} f \right)(b) \right] \right\} \right| \leq$$

$$\frac{\theta}{\Gamma((n+1)\alpha)\left((n+1)\alpha + \frac{1}{p}\right)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}} \frac{(g(b)-g(a))^{(n+1)\alpha+\frac{1}{p}}}{2^{(n+1)\alpha-\frac{1}{q}}}, \tag{116}$$

(iii) assuming $\left(D^{i\alpha}_{a+;g}f\right)(a) = \left(D^{i\alpha}_{b-;g}f\right)(b) = 0$, $i = 0, 1, ..., n$, *we obtain*

$$\left|\int_a^b f(x)\,dg(x)\right| \leq$$

$$\frac{\theta}{\Gamma((n+1)\alpha)\left((n+1)\alpha + \frac{1}{p}\right)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \frac{(g(b) - g(a))^{(n+1)\alpha + \frac{1}{p}}}{2^{(n+1)\alpha - \frac{1}{q}}}, \tag{117}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds that*

$$\left|\int_a^b f(x)\,dg(x) - \left\{\sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha + 2)} \left(\frac{g(b) - g(a)}{N}\right)^{i\alpha+1}\right.\right.$$

$$\left.\left.\left[\left(D^{i\alpha}_{a+;g}f\right)(a)\, j^{i\alpha+1} + \left(D^{i\alpha}_{b-;g}f\right)(b)\,(N - j)^{i\alpha+1}\right]\right\}\right| \leq$$

$$\frac{\theta}{\Gamma((n+1)\alpha)\left((n+1)\alpha + \frac{1}{p}\right)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}}$$

$$\left(\frac{g(b) - g(a)}{N}\right)^{(n+1)\alpha + \frac{1}{p}} \left[j^{(n+1)\alpha + \frac{1}{p}} + (N - j)^{(n+1)\alpha + \frac{1}{p}}\right], \tag{118}$$

(v) if $\left(D^{i\alpha}_{a+;g}f\right)(a) = \left(D^{i\alpha}_{b-;g}f\right)(b) = 0$, $i = 1, ..., n$, *from (118) we obtain:*

$$\left|\int_a^b f(x)\,dg(x) - \left(\frac{g(b) - g(a)}{N}\right)[jf(a) + (N - j) f(b)]\right| \leq$$

$$\frac{\theta}{\Gamma((n+1)\alpha)\left((n+1)\alpha + \frac{1}{p}\right)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}}$$

$$\left(\frac{g(b) - g(a)}{N}\right)^{(n+1)\alpha + \frac{1}{p}} \left[j^{(n+1)\alpha + \frac{1}{p}} + (N - j)^{(n+1)\alpha + \frac{1}{p}}\right], \tag{119}$$

$j = 0, 1, 2, ..., N$,

(vi) when $N = 2$, $j = 1$, *(119) turns to*

$$\left|\int_a^b f(x)\,dg(x) - \left(\frac{g(b) - g(a)}{2}\right)(f(a) + f(b))\right| \leq$$

$$\frac{\theta}{\Gamma((n+1)\alpha)\left((n+1)\alpha + \frac{1}{p}\right)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \frac{(g(b) - g(a))^{(n+1)\alpha + \frac{1}{p}}}{2^{(n+1)\alpha - \frac{1}{q}}}. \tag{120}$$

Proof. By (56) we find

$$\left|f(x) - \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g}f\right)(a)\right| \leq \tag{121}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left|\left(D^{(n+1)\alpha}_{a+;g}f\right)(t)\right| dt =$$

(by [5])

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} \left|\left(D^{(n+1)\alpha}_{a+;g}f\right)(t)\right| dg(t) \leq$$

(by [6])

$$\frac{1}{\Gamma((n+1)\alpha)}\left(\int_a^x (g(x)-g(t))^{p((n+1)\alpha-1)}\,dg(t)\right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left|\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\right|^q dg(t)\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)}\frac{(g(x)-g(a))^{\frac{p((n+1)\alpha-1)+1}{p}}}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{a+;g}^{(n+1)\alpha}f\right\|_{L_q([a,b],g)}.$$

That is

$$\left|f(x)-\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a)\right| \leq$$

$$\frac{\left\|D_{a+;g}^{(n+1)\alpha}f\right\|_{L_q([a,b],g)}}{\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}(g(x)-g(a))^{(n+1)\alpha-\frac{1}{q}}, \tag{122}$$

$\forall\, x \in [a,b]$.

Similarly, from (57) we derive

$$\left|f(x)-\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b)\right| \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^b (g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left|\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\right|dt =$$

(by [5])

$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^b (g(t)-g(x))^{(n+1)\alpha-1}\left|\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\right|dg(t) \leq$$

(by [6])

$$\frac{1}{\Gamma((n+1)\alpha)}\left(\int_x^b (g(t)-g(x))^{p((n+1)\alpha-1)}\,dg(t)\right)^{\frac{1}{p}}$$

$$\left(\int_x^b \left|\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\right|^q dg(t)\right)^{\frac{1}{q}} \leq \tag{123}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\frac{(g(b)-g(x))^{\frac{p((n+1)\alpha-1)+1}{p}}}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{b-;g}^{(n+1)\alpha}f\right\|_{L_q([a,b],g)}.$$

That is

$$\left|f(x)-\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b)\right| \leq$$

$$\frac{\left\|D_{b-;g}^{(n+1)\alpha}f\right\|_{L_q([a,b],g)}}{\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}(g(b)-g(x))^{(n+1)\alpha-\frac{1}{q}}, \tag{124}$$

$\forall\, x \in [a,b]$.

We have proved that

$$\left|f(x)-\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a)\right| \leq$$

$$\frac{\theta}{\Gamma\left(\left(n+1\right)\alpha\right)\left(p\left(\left(n+1\right)\alpha-1\right)+1\right)^{\frac{1}{p}}}\left(g\left(x\right)-g\left(a\right)\right)^{\left(n+1\right)\alpha-\frac{1}{q}}, \tag{125}$$

and

$$\left|f\left(x\right)-\sum_{i=0}^{n}\frac{\left(g\left(b\right)-g\left(x\right)\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{b-;g}^{i\alpha}f\right)\left(b\right)\right|\leq$$

$$\frac{\theta}{\Gamma\left(\left(n+1\right)\alpha\right)\left(p\left(\left(n+1\right)\alpha-1\right)+1\right)^{\frac{1}{p}}}\left(g\left(b\right)-g\left(x\right)\right)^{\left(n+1\right)\alpha-\frac{1}{q}}, \tag{126}$$

$\forall\, x\in\left[a,b\right]$.

The rest of the proof is as in Theorem 7. □

Applications follow:

Proposition 1. *We assume that* $\left(f\circ\ln x\right)\in AC^{n}\left(\left[e^{a},e^{b}\right]\right)$, *where* $\mathbb{N}\ni n=\lceil\alpha\rceil$, $\alpha>0$. *We also assume that* $\left(f\circ\ln x\right)^{(n)}\circ e^{x}\in L_{\infty}\left(\left[a,b\right]\right)$, $f\in C\left(\left[a,b\right]\right)$. *Set*

$$T_{1}:=\max\left\{\left\|D_{a+;e^{x}}^{\alpha}f\right\|_{L_{\infty}\left(\left[a,b\right]\right)},\left\|D_{b-;e^{x}}^{\alpha}f\right\|_{L_{\infty}\left(\left[a,b\right]\right)}\right\}. \tag{127}$$

Then

(i)

$$\left|\int_{a}^{b}f\left(x\right)e^{x}dx-\sum_{k=0}^{n-1}\frac{1}{\left(k+1\right)!}\left[\left(f\circ\ln x\right)^{(k)}\left(e^{a}\right)\left(e^{t}-e^{a}\right)^{k+1}\right.\right.$$

$$\left.\left.\left(-1\right)^{k}\left(f\circ\ln x\right)^{(k)}\left(e^{b}\right)\left(e^{b}-e^{t}\right)^{k+1}\right]\right|\leq$$

$$\frac{T_{1}}{\Gamma\left(\alpha+2\right)}\left[\left(e^{t}-e^{a}\right)^{\alpha+1}+\left(e^{b}-e^{t}\right)^{\alpha+1}\right], \tag{128}$$

$\forall\, t\in\left[a,b\right]$,

(ii) at $t=\ln\left(\frac{e^{a}+e^{b}}{2}\right)$, *the right hand side of (128) is minimized, and we find:*

$$\left|\int_{a}^{b}f\left(x\right)e^{x}dx-\sum_{k=0}^{n-1}\frac{1}{\left(k+1\right)!}\frac{\left(e^{b}-e^{a}\right)^{k+1}}{2^{k+1}}\right.$$

$$\left.\left[\left(f\circ\ln x\right)^{(k)}\left(e^{a}\right)+\left(-1\right)^{k}\left(f\circ\ln x\right)^{(k)}\left(e^{b}\right)\right]\right|\leq$$

$$\frac{T_{1}}{\Gamma\left(\alpha+2\right)}\frac{\left(e^{b}-e^{a}\right)^{\alpha+1}}{2^{\alpha}}, \tag{129}$$

(iii) if $\left(f\circ\ln x\right)^{(k)}\left(e^{a}\right)=\left(f\circ\ln x\right)^{(k)}\left(e^{b}\right)=0$, *for* $k=0,1,...,n-1$, *we obtain*

$$\left|\int_{a}^{b}f\left(x\right)e^{x}dx\right|\leq T_{1}\frac{\left(e^{b}-e^{a}\right)^{\alpha+1}}{\Gamma\left(\alpha+2\right)2^{\alpha}}, \tag{130}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\, e^x dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{e^b - e^a}{N} \right)^{k+1} \right.$$

$$\left. \left[j^{k+1} (f \circ \ln x)^{(k)} (e^a) + (-1)^k (N-j)^{k+1} (f \circ \ln x)^{(k)} \left(e^b\right) \right] \right| \leq$$

$$\frac{T_1}{\Gamma(\alpha+2)} \left(\frac{e^b - e^a}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{131}$$

(v) if $(f \circ \ln x)^{(k)} (e^a) = (f \circ \ln x)^{(k)} \left(e^b\right) = 0$, for $k = 1, ..., n-1$, from (131) we obtain

$$\left| \int_a^b f(x)\, e^x dx - \left(\frac{e^b - e^a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{T_1}{\Gamma(\alpha+2)} \left(\frac{e^b - e^a}{N} \right)^{\alpha+1} \left[j^{\alpha+1} + (N-j)^{\alpha+1} \right], \tag{132}$$

$j = 0, 1, 2, ..., N$,

(vi) when $N = 2$, $j = 1$, (132) turns to

$$\left| \int_a^b f(x)\, e^x dx - \left(\frac{e^b - e^a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{T_1}{\Gamma(\alpha+2)} \frac{\left(e^b - e^a\right)^{\alpha+1}}{2^\alpha}, \tag{133}$$

(vii) when $0 < \alpha \leq 1$, inequality (133) is again valid without any boundary conditions.

Proof. By Theorem 6, for $g(x) = e^x$. □

We continue with

Proposition 2. *Here $f \in C([a,b])$, where $[a,b] \subset (0, +\infty)$. Let $0 < \alpha \leq 1$, and $G_k := D_{a+;\ln x}^{k\alpha} f$, for $k = 0, 1, ..., n+1$; $n \in \mathbb{N}$. We assume that $G_k \circ e^x \in AC([\ln a, \ln b])$ and $(G_k \circ e^x)' \circ \ln x \in L_\infty([a,b])$. Also let $\overline{G_k} := D_{b-;\ln x}^{k\alpha} f$, for $k = 0, 1, ..., n+1$, they fulfill $\overline{G_k} \circ e^x \in AC([\ln a, \ln b])$ and $\left(\overline{G_k} \circ e^x\right)' \circ \ln x \in L_\infty([a,b])$. Set*

$$T_2 := \max \left\{ \left\| D_{a+;\ln x}^{(n+1)\alpha} f \right\|_{\infty,[a,b]}, \left\| D_{b-;\ln x}^{(n+1)\alpha} f \right\|_{\infty,[a,b]} \right\}. \tag{134}$$

Then

(i)

$$\left| \int_a^b \frac{f(x)}{x} dx - \left\{ \sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left[\left(D_{a+;\ln x}^{i\alpha} f \right)(a) \left(\ln \frac{t}{a} \right)^{i\alpha+1} \right. \right. \right.$$

$$\left. \left. \left. + \left(D_{b-;\ln x}^{i\alpha} f \right)(b) \left(\ln \frac{b}{t} \right)^{i\alpha+1} \right] \right\} \right| \leq$$

$$\frac{T_2}{\Gamma((n+1)\alpha+2)} \left[\left(\ln \frac{t}{a} \right)^{(n+1)\alpha+1} + \left(\ln \frac{b}{t} \right)^{(n+1)\alpha+1} \right], \tag{135}$$

$\forall\, t \in [a,b]$,

(ii) at $t = e^{\left(\frac{\ln ab}{2}\right)}$, *the right hand side of (135) is minimized, and we have:*

$$\left|\int_a^b \frac{f(x)}{x} dx - \left\{\sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \frac{\left(\ln \frac{b}{a}\right)^{i\alpha+1}}{2^{i\alpha+1}}\right.\right.$$

$$\left.\left.\left[\left(D^{i\alpha}_{a+;\ln x} f\right)(a) + \left(D^{i\alpha}_{b-;\ln x} f\right)(b)\right]\right\}\right| \leq$$

$$\frac{T_2}{\Gamma((n+1)\alpha+2)} \frac{\left(\ln \frac{b}{a}\right)^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}, \tag{136}$$

(iii) assuming $\left(D^{i\alpha}_{a+;\ln x} f\right)(a) = \left(D^{i\alpha}_{b-;\ln x} f\right)(b) = 0$, $i = 0, 1, ..., n$, *we obtain*

$$\left|\int_a^b \frac{f(x)}{x} dx\right| \leq \frac{T_2}{\Gamma((n+1)\alpha+2)} \frac{\left(\ln \frac{b}{a}\right)^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}, \tag{137}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left|\int_a^b \frac{f(x)}{x} dx - \left\{\sum_{i=0}^{n} \frac{1}{\Gamma(i\alpha+2)} \left(\frac{\ln \frac{b}{a}}{N}\right)^{i\alpha+1}\right.\right.$$

$$\left.\left.\left[\left(D^{i\alpha}_{a+;\ln x} f\right)(a)\, j^{i\alpha+1} + \left(D^{i\alpha}_{b-;\ln x} f\right)(b)\,(N-j)^{i\alpha+1}\right]\right\}\right| \leq$$

$$\frac{T_2}{\Gamma((n+1)\alpha+2)} \left(\frac{\ln \frac{b}{a}}{N}\right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1}\right], \tag{138}$$

(v) if $\left(D^{i\alpha}_{a+;\ln x} f\right)(a) = \left(D^{i\alpha}_{b-;\ln x} f\right)(b) = 0$, $i = 1, ..., n$, *from (138) we find:*

$$\left|\int_a^b \frac{f(x)}{x} dx - \left(\frac{\ln \frac{b}{a}}{N}\right)(jf(a) + (N-j) f(b))\right| \leq$$

$$\frac{T_2}{\Gamma((n+1)\alpha+2)} \left(\frac{\ln \frac{b}{a}}{N}\right)^{(n+1)\alpha+1} \left[j^{(n+1)\alpha+1} + (N-j)^{(n+1)\alpha+1}\right], \tag{139}$$

for $j = 0, 1, 2, ..., N$,

(vi) if $N = 2$ *and* $j = 1$, *(139) becomes*

$$\left|\int_a^b \frac{f(x)}{x} dx - \left(\frac{\ln \frac{b}{a}}{2}\right)(f(a) + f(b))\right| \leq$$

$$\frac{T_2}{\Gamma((n+1)\alpha+2)} \frac{\left(\ln \frac{b}{a}\right)^{(n+1)\alpha+1}}{2^{(n+1)\alpha}}. \tag{140}$$

Proof. By Theorem 7, for $g(x) = \ln x$. □

We could give many other interesting applications that are based in our other theorems, due to lack of space we skip this task.

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References

1. Iyengar, K.S.K. Note on an inequality. *Math. Stud.* **1938**, *6*, 75–76.
2. Anastassiou, G. Advanced Fractional Taylor's formulae. *J. Comput. Anal. Appl.* **2016**, *21*, 1185–1204.
3. Anastassiou, G.; Argyros, I. *Intelligent Numerical Methods: Applications to Fractional Calculus*; Springer: Heidelberg, Germany; New York, NY, USA, 2016.
4. Royden, H.L. *Real Analysis*, 2nd ed.; MacMillan Publishing Co., Inc.: New York, NY, USA, 1968.
5. Royden, H.L.; Fitzpatrick, P.M. *Real Analysis*, 4th ed.; Pearson: New York, NY, USA, 2010.
6. Dragomir, S.S. Inequalities for the Riemann-Stieljes integral of $(p, q) - H-$Dominated integrators with applications. *Appl. Math. E-Notes* **2015**, *15*, 243–260.