Article

An Arc-Sine Law for Last Hitting Points in the Two-Parameter Wiener Space

Jeong-Gyoo KIM

School of Games, Hongik University, Sejong 30016, Korea; jgkim@hongik.ac.kr or jgkimath@yonsei.ac.kr

Abstract: We develop the two-parameter version of an arc-sine law for a last hitting time. The existing arc-sine laws are about a stochastic process X_t with one parameter t. If there is another varying key factor of an event described by a process, then we need to consider another parameter besides t. That is, we need a system of random variables with two parameters, say $X_{s,t}$, which is far more complex than one-parameter processes. In this paper we challenge to develop such an idea, and provide the two-parameter version of an arc-sine law for a last hitting time. An arc-sine law for a two-parameter process is hardly found in literature. We use the properties of the two-parameter Wiener process for our development. Our result shows that the probability of last hitting points in the two-parameter Wiener space turns out to be arcsine-distributed. One can use our results to predict an event happened in a system of random variables with two parameters, which is not available among existing arc-sine laws for one parameter processes.

Keywords: arc-sine law; last hitting points; two-parameter Wiener space

MSC: 28C20; 46G12; 60A10; 60G15

1. Introduction

We develop a two-parameter version of an arc-sine law for a last hitting time, and this is achieved in the two-parameter Wiener space. If a random variable X has its cumulative distribution $P(X \leq x) = (2/\pi) \arcsin(\sqrt{x})$, then X is called arcsine-distributed. It is well known that Lévy proved an arc-sine theorem for Brownian motion [1] in 1939 and Erdös and Kac extended Levy's result to general random variables [2] in 1947. Since then a number of similar laws for various processes have been introduced by researchers, most of which are studied in the Wiener space, for example, Reference [3]. One recent work [4] studies the arc-sine law in the analogue of the Wiener space; the analogue of the Wiener space is introduced as 'a kind of the generalisation of the Wiener space' in Reference [5].

Some phenomena in many fields of science have been explained by arc-sine distributions; Barato et al. applied to thermodynamics [6]. Arc-sine laws play crucial roles in problems arising in dynamic systems using random walks. Due to the aspect of the U-shaped density $\frac{1}{\pi\sqrt{x(1-x)}}$ of an arc-sine distribution, arc-sine laws notably have been applied in the fields of mathematical finance. Dale and Workman discuss the law in relation to the treasury bill future market [7].

Among many arc-sine laws, we are interested in the process called a last hitting time in this paper, which is also called the second arc-sine law (In some literature, such as Reference [8], the name may indicate a slightly different event). In order to state it we assume that $\{X_t, t > 0\}$ is the Wiener process defined on the Wiener space $C_0[0,1]$. Let $L_X = \sup\{t \in [0,1] : X_t = 0\}$ be the last time the process hits zero. Then the probability $P(L_X \leq s) = \frac{2}{\pi} \arcsin(\sqrt{s})$ [9].

The existing arc-sine laws are about a stochastic process X_t with one parameter t, which is a time parameter in most applications. If there is another varying key factor of an event described

by the process, then we need to consider another parameter besides t. That is, we need a system of random variables with two parameters, say $X_{s,t}$. In this paper we challenge to develop such an idea. A two-parameter process in relation to our interest is very rare in literature. To the best of the author's knowledge, arc-sine laws have not been studied yet in two-parameter processes. We devise a two-parameter process analogous to a last hitting time and develop the two parameter version of the arc-sine law using the properties of the two-parameter Wiener process. Though the probability of an event described by $X_{s,t}$ is much more complicated than that of X_t, a new arc-sine law has been accomplished based on our development.

In the next section we briefly introduce the two-parameter Wiener space for background knowledge of our development. Section 3 provides the procedure of how to construct a probability density function (pdf) for the stochastic process of our interest. Using the pdf, the arc-sine law for last hitting points in the the two-parameter Wiener space is calculated in Section 4. Finally conclusions are given in Section 5.

2. Background: The Two-Parameter Wiener Process

In this section we briefly introduce the two-parameter Wiener process for background knowledge of our development as it is far less accessible than the Wiener process. This section is mostly based on Reference [10]. For readers who are not familiar with the two-parameter Wiener process, a more detailed digest of it can be found in the author's recent work [11].

Given a set $R = [0, S] \times [0, T]$ in $\mathbb{R}^2$, $C(R)$ will denote the space of real-valued continuous functions on R. We are interested in the subset of $C(R)$ defined below.

Definition 1. *Let $R = [0, S] \times [0, T]$ in $\mathbb{R}^2$. We denote the set of continuous functions by*

$$C_2(R) := \{x \mid x(\cdot, \cdot) : R \to \mathbb{R}, x \text{ is continuous on } R, \text{ and } x(0, \cdot) = x(\cdot, 0) = 0\}$$

Definition 2 (Yeh). *Let $R = [0, S] \times [0, T]$ in $\mathbb{R}^2$, and $0 = s_0 < s_1 < \cdots < s_m = S$, $0 = t_0 < t_1 < \cdots < t_n = T$,*

$$I := \{x \in C_2(R) \mid (x(s_1, t_1), \ldots, x(s_m, t_n)) \in D\}, \tag{1}$$

where $D = (a_{11}, b_{11}] \times \cdots \times (a_{mn}, b_{mn}] \subset \mathbb{R}^{mn}$, $-\infty \le a_{jk} \le b_{jk} \le \infty$, and $j = 1, \ldots, m, k = 1, \ldots, n$.

(1) I is called a strict interval in $C_2(R)$.
(2) If D is a measurable set in $\mathbb{R}^{mn}$, then I is called a cylinder.
(3) $(s_1, t_1), \ldots, (s_m, t_n)$ are called restriction points of I.
(4) $\mathcal{I} =$ the collection of all strict intervals I. (Then $\mathcal{I}$ is a semi-algebra.)

Throughout the paper, we denote an exponential function by $\exp\{\cdot\}$, usually for a random variable; we use it and $e^{\{\cdot\}}$ interchangeably.

Definition 3 (Yeh). *For $I \in \mathcal{I}$, the two-parameter Wiener measure m_2 of I is defined by*

$$m_2(I) \;=\; \int_E \ldots (mn) \cdots \int W(\vec{u}; \vec{s}, \vec{t}) \, d\vec{u},$$

where

$$W(\vec{u}; \vec{s}, \vec{t}) \;=\; W(u_{11}, u_{12}, \ldots, u_{mn}; s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n)$$

$$= \prod_{k=1}^{n} \prod_{j=1}^{m} \left(\frac{1}{\sqrt{\pi(s_j - s_{j-1})(t_k - t_{k-1})}} \cdot \exp\{-\frac{(u_{j,k} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})}\} \right)$$

$$= \frac{1}{\sqrt{\pi^{mn}[(s_1 - 0)(s_2 - s_1)\dots(S - s_{m-1})][(t_1 - 0)(t_2 - t_1)\dots(T - t_{n-1})]}}$$

$$\cdot \exp\left\{ -\sum_{k=1}^{n} \sum_{j=1}^{m} \left(\frac{(u_{j,k} - u_{j-1,k} - u_{j,k-1} + u_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \right) \right\} \tag{2}$$

Here, $u_{j,0} = u_{0,k} = 0$ for $0 \leq j \leq m, 0 \leq k \leq n$.

The completed measure space of the two-parameter Wiener measure m_2 is denoted by $(C_2(R), \mathcal{M}_2, m_2)$ and called the two-parameter Wiener space. The space is also called the Yeh-Wiener space in some literature because the space was first introduced by Yeh in Reference [10] and further work [12]. One can see the measure space is a probability space since $m_2(C_2(R)) = 1$.

Proposition 1 (Yeh). *Let $F : C_2(R) \longrightarrow \mathbb{R}$. Let $f(u_{11}, u_{12}, \dots, u_{mn})$ be a Lebesgue measurable function on $\mathbb{R}^{mn}$ and $F(x) = f(x(s_1, t_1), \dots, x(s_m, t_n))$, then F is two-parameter Wiener measurable and*

$$\int_{C_2(R)} F(x) \, dm_2(x) = \int_{\mathbb{R}^{mn}} f(\vec{u}) W(\vec{u}; \vec{s}, \vec{t}) \, d\vec{u}.$$

The equality means that if one side exists then the other side also exists and both sides coincide. It tells us that the integral over the infinite dimensional function space $C_2(R)$ on the left is reduced to an ordinary Lebesgue integral over the finite dimensional space $\mathbb{R}^{mn}$ on the right [10].

Example 1.

(a) $E(x(s,t)^{2k-1}) = \int_{C_2(R)} [x(s,t)]^{2k-1} \, dm_2(x) = 0, \ k = 1, 2, 3 \dots$

(b) $E(x(s,t)^2) = \int_{C_2(R)} x(s,t)^2 \, dm_2(x) = \frac{st}{2}$

Here, $E(\cdot)$ denotes the expectation of a random variable.

Now we introduce the two-parameter Wiener process defined on $(C_2(R), \mathcal{M}_2, m_2)$ to be used in the next sections.

Definition 4 (Yeh). *Let $(s, t) \in R$ and $X_{s,t}$ is defined on the two-parameter Wiener space by*

$$\begin{array}{rcl} X_{s,t} : C_2(R) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & X_{s,t}(x) := x(s,t) \end{array} \tag{3}$$

Then $\{X_{s,t} : s, t \geq 0\}$ is called the two-parameter Wiener process.

From Example 1, we can see that $X_{s,t} = x(s,t)$ follows a normal distribution $N(0, \frac{st}{2})$. Obviously, for all $x \in C_2(R)$, $X_{0,t}(x) = 0$ for all t and $X_{s,0}(x) = 0$ for all s as $x(0, t) = x(s, 0) = 0$.

The function $X_{s,t}$ is a random variable and the collection $\{X_{s,t} : s, t \geq 0\}$ was originally called a two-time parameter Wiener process by Yeh [10]; later, it is called the Yeh-Wiener process in the literature, such as References [13–15]. We keep the name, the two-parameter Wiener process, throughout this paper.

By the definition of m_2 in Definition 3

$$m_2\{x \in C_2(R) \mid \alpha < X_{s,t}(x) \leq \beta\} = \frac{1}{\sqrt{\pi st}} \int_{\alpha}^{\beta} \exp\{-\frac{u^2}{st}\} du. \tag{4}$$

Since m_2 is a probability measure, we can express it by P:

$$P\{\alpha \ < X_{s,t} \le \beta \ \} = \frac{1}{\sqrt{\pi st}} \int_{\alpha}^{\beta} \exp\{-\frac{u^2}{st}\}du.$$

In the later sections we will use the following well-known integral in our calculations.

Proposition 2. *For a positive real number a,* $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$

3. Construction of a PDF for T_c, a Two-Parameter Version of a First Hitting Time

In this section, we aim to build a probability density function (pdf) of an event $\{X_{s,t}(x) > c\} = \{x \in C_2(R)|\ x(s,t) > c\}$, where c is a real number.

Let x be in $C_2(R)$, that is, x is continuous on $R = [0, S] \times [0, T]$ and $x(s, \cdot) = x(\cdot, t) = 0$ for all $0 \le s \le S$ and $0 \le t \le T$. A one-parameter process $\tau(x) = \inf\{t \in [0, T] \mid x(t) = c,\ x \in C_0[0, T]\}$ is called the first point that $x(t)$ hits c or a first hitting time to c, and its distribution is well-known.

We define the two-parameter version of $\tau(x)$ denoted by $T_c(x)$ for $x \in C_2(R)$ below. For this we adapt the inequality $<$ for one-dimensional $t \in [0, T]$ to two-dimensional $(s, t) \in R$.

Definition 5. *Let (u, v) and (s, t) are two points in the rectangle R.*

(1) *We define $(u, v) < (s, t)$ if $u < s$ and $v < t$.*
    We also define $(u, v) \le (s, t)$ if either $(u, v) < (s, t)$ or $[u = s$ and $v = t]$.
(2) *Similarly, we define $(u, v) <^c (s, t)$ if either $u < s$ or $v < t$.*
    We also define $(u, v) \le^c (s, t)$ if either $(u, v) <^c (s, t)$ or $[u = s$ and $v = t]$.

In Definition 5, if R is divided into four subrectangles with common vertex (s, t), then the symbol $<$ is for the description of the bottom left subrectangle and the symbol $<^c$ is used for the description of the complement of the top right subrectangle.

Definition 6. *Let c be a real number. $T_c(x)$ is the set of points $(s, t) \in R$ satisfying:*

(i) *$x(s, t) = c$ and*
(ii) *$x(u, v) < c$ for any (u, v) with $0 < u \le s, 0 < v \le t$ but $(u, v) \ne (s, t)$.*

For one-variable functions $x(t)$, the first hitting point $\tau(x)$ is a single point in the domain $[0, T]$ as introduced at the beginning of this section. On the other hand, for two-variable functions $x(s, t)$, the set $T_c(x) \subset R$ is a curve in general. If $(s, t) \in T_c(x)$ and $(u, v) < (s, t)$, then $x(u, v)$ never hits c. Hence, (s, t) can be called a least point hitting c in some sense or the first point in a radial direction from $(0, 0)$.

As introduced in Section 2, m_2 is the two-parameter Wiener measure defined on $C_2(R)$, where R is a rectangle $[0, S] \times [0, T]$. When a specific point (s, t) in R is chosen, our interest is the region beyond the point (s, t); that is, we are interested in the subrectangle $[s, S] \times [t, T]$ of R. If we regard the set of x's (surfaces) restricted on this subrectangle as a new space, m_2 is insufficient to configure the set of x's defined on the subrectangle; we cannot apply the two-parameter Wiener measure m_2 directly. The reason is that $x(s, \cdot)$ or $x(\cdot, t)$ on the boundary of the subrectangle may not be zero as required for m_2. This is one of the greatest difficulties in the development of a process in the two-parameter Wiener space compared to the Wiener space (For the Wiener space, if a specific point $t \in [0, T]$ is chosen, then curves x restricted on its subinterval $[t, T]$ can be regarded as a new Wiener space by translating x by $x(t)$ so that we can make the translated x has value zero at t. Then the Wiener measure can be applied directly to a set of those x's defined on the subinterval).

Remark 1. *Let $s < s' \leq S$ and $t < t' \leq T$. Then we have the following relationships of the two-parameter Wiener measure m_2 (see Definition 3).*

$$(1) \quad m_2(\{x \mid x(s,t') < \alpha\,\}) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{\pi st'}} \exp\{-\frac{u_{01}^2}{st'}\} du_{01}$$

$$(2) \quad m_2(\{x \mid x(s',t) < \alpha\,\}) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{\pi s't}} \exp\{-\frac{u_{10}^2}{s't}\} du_{10}$$

$$(3) \quad m_2(\{x \mid x(s',t') < \alpha\,\}) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{\pi s't'}} \exp\{-\frac{u_{11}^2}{s't'}\} du_{11}$$

Being motivated by the relationships in the remark above, we introduce a restriction of m_2 on a subrectangle of R as follows.

Definition 7. *Let $(s,t) \in R$, then $[s,S] \times [t,T]$ is a subrectangle of R. We denote the subrectangle by $\tilde{R}$; $\tilde{R} = [s,S] \times [t,T]$. Let $\tilde{x} := x|_{\tilde{R}}$ for $x \in C_2(R)$.*
Define $\tilde{C}(\tilde{R}) := \{\tilde{x} \mid x \in C_2(R)\}$. Let b be a real number and define $\tilde{C}(\tilde{R})_b = \{\tilde{x} \in \tilde{C}(\tilde{R}) \mid \tilde{x}(s,t) = b\}$.

Definition 8. *Let us define a measure w on $\tilde{C}(\tilde{R})$, satisfying the followings. For a real number α and a point $(s',t') \in \tilde{R}$,*

$$
\begin{aligned}
w(\{\tilde{x} \mid \tilde{x}(s',t') < \alpha\,\}) \;=\; & \int_{-\infty}^{\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi(s'-s)(t'-t)}} \exp\{-\frac{(u_{11}-u_{10}-u_{01}+u_{00})^2}{(s'-s)(t'-t)}\} \\
\times \; & \frac{1}{\sqrt{\pi s(t'-t)}} \exp\{-\frac{(u_{10}-u_{00})^2}{s(t'-t)}\} \times \frac{1}{\sqrt{\pi(s'-s)t}} \exp\{-\frac{(u_{01}-u_{00})^2}{(s'-s)t}\} \\
\times \; & \frac{1}{\sqrt{\pi st}} \exp\{-\frac{u_{00}^2}{st}\} \, du_{00} du_{10} du_{01} du_{11} \\
= \; & \int_{-\infty}^{\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi^2 st(s'-s)(t'-t)} \exp\{-\frac{(u_{11}-u_{10}-u_{01}+u_{00})^2}{(s'-s)(t'-t)}\} \\
\times \; & \exp\{-\frac{(u_{10}-u_{00})^2}{(s)(t'-t)}\} \exp\{-\frac{(u_{01}-u_{00})^2}{(s'-s)(t)}\} \exp\{-\frac{u_{00}^2}{st}\} \, du_{00} du_{10} du_{01} du_{11}.
\end{aligned}
$$

We will calculate the measure w when $u_{00} = b$, a specific value of $\tilde{x}$ at (s,t), that is, $\tilde{x}(s,t) = b$ and denote it by w_b in the next theorem.

Theorem 1. *If w_b is the measure when $u_{00} = b$ in Definition 8 (here, b is the value of $\tilde{x}$ at (s,t), that is, $\tilde{x}(s,t) = b$), then*

$$w_b(\{\tilde{x} \mid \tilde{x}(s',t') < \alpha\,\}) \;=\; \int_{-\infty}^{\alpha} \frac{1}{\sqrt{\pi(s't'-st)}} \exp\{-\frac{(u_{11}-b)^2}{s't'-st}\} du_{11}. \tag{5}$$

Proof. By Definition 8

$$
\begin{aligned}
w_b(\{\tilde{x} \mid \tilde{x}(s',t') < \alpha\,\}) \;=\; & \int_{-\infty}^{\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi(s'-s)(t'-t)}} \exp\{-\frac{(u_{11}-u_{10}-u_{01}+b)^2}{(s'-s)(t'-t)}\} \\
\times \; & \frac{1}{\sqrt{\pi s(t'-t)}} \exp\{-\frac{(u_{10}-b))^2}{s(t'-t)}\} \frac{1}{\sqrt{\pi(s'-s)t}} \\
\times \; & \exp\{-\frac{(u_{01}-b)^2}{(s'-s)t}\} \, du_{10} du_{01} du_{11}
\end{aligned}
$$

Each exponent of exponential functions $\exp\{\cdot\}$ in the integrand of w_b is a quadratic function of an integral variable. Using completing the square on a quadratic function with respect to u_{10}, the exponents of $\exp\{-\frac{(u_{11}-u_{10}-u_{01}+b)^2}{(s'-s)(t'-t)}\}$ and $\exp\{-\frac{(u_{10}-b))^2}{s(t'-t)}\}$ can be expressed by

$$-\frac{\{u_{10}-\frac{At+(t'-t)b}{t'}\}^2}{\frac{t}{t'}(s'-s)(t'-t)}-\frac{\{-At+(t'-t)b\}^2+t'tA^2+t'(t'-t)b^2}{t't(s'-s)(t'-t)},$$

where $A=u_{11}-u_{01}+b$. Then the inner integral has the form of $(constant)\times\int_{-\infty}^{\infty}e^{-(u_{10}-d)^2/a}\,du_{10}$ for some constants a and d. Since $\int_{-\infty}^{\infty}e^{-(u_{10}-d)^2/a}\,du_{10}=\sqrt{\pi a}$ as in Proposition 2, the inner integral with respect to u_{10} is calculated; then a, d and the front $(constant)$ are functions of two variables u_{01} and u_{00}. Following the same technique for u_{01} as done for u_{10}, the middle integral is calculated too and becomes a function of u_{11}, which is the variable of the outer integral.

Though the process of these calculations are tremendously lengthy, the resulting integral has been compactly expressed as a single exponential function (In the author's calculations, five sheets of A4 paper are filled with equations written in small-sized letters; the width of a page was not long enough for a line even in landscape orientation). Therefore, the triple integral can be reduced to a single integral with respect to u_{11}:

$$w_b(\{\tilde{x}\mid\tilde{x}(s',t')<\alpha\,\})\quad=\quad\int_{-\infty}^{\alpha}\frac{1}{\sqrt{\pi(s't'-st)}}\exp\{-\frac{(u_{11}-b)^2}{s't'-st}\}du_{11}.$$

This is the Equation (5).　□

Lemma 1. *For a real number $c<b$ and $(s',t')\in\tilde{R}=[s,S]\times[t,T]$,*

$$w_b(\{\tilde{x}\in\tilde{C}(\tilde{R})\mid T_c(\tilde{x})<(s',t')\})=2\,w_b(\{\tilde{x}\in\tilde{C}(\tilde{R})\mid\tilde{x}(s',t')<c\}).$$

Proof. We use a trivial set relationship of an event, $\{\tilde{x}\mid T_c(\tilde{x})<(s',t')\}=\{\tilde{x}\mid T_c(\tilde{x})<(s',t'),\,\tilde{x}(s',t')<c\}\cup\{\tilde{x}\mid T_c(\tilde{x})<(s',t'),\,\tilde{x}(s',t')>c\}\cup\{\tilde{x}\mid T_c(\tilde{x})<(s',t'),\,\tilde{x}(s',t')=c\}$, Since the last set among the three sets on the right side has two-parameter Wiener measure (or probability) zero, we do not consider it. The rest of the two sets have an equal probability. Thus we examine only one of them.

Obviously, $\{\tilde{x}\mid T_c(\tilde{x})<(s',t'),\,\tilde{x}(s',t')>c\}\subseteq\{\tilde{x}\mid\tilde{x}(s',t')>c\}$ and we need to show the converse inclusion. If $\tilde{x}$ is contained in the set on the right side, that is, $\tilde{x}(s',t')>c$, then (s',t') cannot be $T_c(\tilde{x})$ and $T_c(\tilde{x})<\tilde{x}(s',t')$. This means that $\tilde{x}$ is contained in the set on the left side. Therefore, the sets on the both sides are equivalent and they have the same measure w_b as desired.　□

For $b<c$, we have a similar result to Lemma 1 and state in the next corollary.

Corollary 1. *For $b<c$ and $(s',t')\in\tilde{R}$,*

$$w_b(\{\tilde{x}\mid T_c(\tilde{x})<(s',t')\})=2w_b(\{\tilde{x}\mid\tilde{x}(s',t')<c\}).$$

Using Lemma 1, we can express the measure w_b of an event of T_c as a single integral.

Theorem 2. *Let $\tilde{x}\in\tilde{C}(\tilde{R})$. For $c<b$ and $(s',t')\in\tilde{R}$,*

$$w_b(\{\tilde{x}\mid T_c(\tilde{x})<(s',t')\})=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{c-b}{\sqrt{s't'-st}}}e^{-v^2}dv.$$

For $c\geq b$ and $(s',t')\in\tilde{R}$, $w_b(\{\tilde{x}\mid T_c(\tilde{x})<(s',t')\})=\frac{2}{\sqrt{\pi}}\int_{\frac{c-b}{\sqrt{s't'-st}}}^{\infty}e^{-v^2}dv.$

Proof. We know that $w_b(\{\tilde{x}| \ T_c(\tilde{x}) \ < \ (s',t')\}) \ = \ 2\,w_b(\{\tilde{x}| \ \tilde{x}(s',t') \ < \ c\})$ by Lemma 1. Using Equation (5) for the set on the right, we have

$$
\begin{aligned}
w_b(\{\tilde{x}| \ T_c(\tilde{x}) < (s',t')\}) \ &= \ 2\,w_b(\{\tilde{x}| \ \tilde{x}(s',t') < c\}) \\
&= \ 2 \int_{-\infty}^{c} \frac{1}{\sqrt{\pi(s't'-st)}} \exp\{-\frac{(u_{11}-b)^2}{s't'-st}\}du_{11} \\
&= \ \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\frac{c-b}{\sqrt{s't'-st}}} \exp\{-v^2\}dv.
\end{aligned}
$$

Here, the last equation is derived by the change of variables $v = \frac{u_{11}-b}{\sqrt{(s't'-st)}}$. Therefore, $w_b(\{\tilde{x}| \ T_c(\tilde{x}) < (s',t')\}) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\frac{c-b}{\sqrt{s't'-st}}} e^{-v^2} dv$. The second statement for $c \geq b$ follows from the exactly same manner as the case $c < b$ and Corollary 1. $\quad\square$

Remark 2. *Let $\tilde{R} = [s,S] \times [t,T]$ and $(s',t') \in \tilde{R}$.*

(1) *Suppose that $b > 0$.*

  Let $A(s',t') := \{\tilde{x}| \ T_0(\tilde{x}) < (s',t')\}$, $B(s',t') := \{\tilde{x}| \ T_0(\tilde{x}) \geq (s',t')\}$, $F(s',t') := w_b\{\tilde{x}| \ T_0(\tilde{x}) < (s',t')\}$ and $G(s',t') := w_b\{\tilde{x}| \ T_0(\tilde{x}) \geq^c (s',t')\}$. (See Definition 5 for the symbol $\geq^c$)

  Then $A(s',t') \cap B(s',t') = \varnothing$ and

$$
\begin{aligned}
A(s',t') \cup B(s',t') \ &= \ \{\tilde{x}\,|\,\exists(s_0,t_0) \in \tilde{R} \text{ with } (s_0,t_0) < (s',t'), \text{ such that } \tilde{x}(s_0,t_0) = 0\} \\
&\quad \cup \ \{\tilde{x}\,|\,\exists(s_0,t_0) \in \tilde{R} \text{ with } (s_0,t_0) \geq (s',t'), \text{ such that } \tilde{x}(s_0,t_0) = 0\} \\
&= \ \{\tilde{x}\,|\,\exists(s_0,t_0) \in [s,S] \times [t,T], \text{ such that } \tilde{x}(s_0,t_0) = 0\}
\end{aligned}
$$

(2) *By (1), $w_b(A(s',t') \cup B(s',t')) = F(s',t') + G(s',t')$ is a constant function of (s',t') associated with b; so $\frac{\partial}{\partial s'}(F(s',t') + G(s',t')) = 0$ and $\frac{\partial^2}{\partial s'\partial t'}(F(s',t') + G(s',t')) = 0$. Therefore,*

$$
\frac{\partial^2}{\partial s'\partial t'}G(s',t') = -\frac{\partial^2}{\partial s'\partial t'}F(s',t'). \tag{6}
$$

(3) *In the case of $b < 0$ or $b = 0$, we can obtain the equality (6) using the same way as (1) and (2) above.*

Theorem 3. *For $(s',t') \in \tilde{R}$,*

$$
w_b\{\tilde{x}| \ T_0(\tilde{x}) < (s',t')\} = \int_{t'}^{\infty}\int_{s'}^{\infty}\left(\frac{|b|}{\sqrt{(s't'-st)^3\pi}} + \frac{(-\frac{3}{2})s't'|b|}{\sqrt{(s't'-st)^5\pi}} + \frac{s't'|b|^3}{\sqrt{(s't'-st)^7\pi}}\right)\exp\left\{-\frac{b^2}{s't'-st}\right\}ds'dt'.
$$

Proof. Let $F(s',t') := w_b(\{\tilde{x}| \ T_0(\tilde{x}) < (s',t')\})$ as in Remark 2. First we will show that

$$
\frac{\partial^2 F}{\partial t'\partial s'} = \left(\frac{b}{\sqrt{(s't'-st)^3\pi}} + \frac{(-\frac{3}{2})s't'b}{\sqrt{(s't'-st)^5\pi}} + \frac{s't'b^3}{\sqrt{(s't'-st)^7\pi}}\right)\exp\left\{-\frac{b^2}{s't'-st}\right\}. \tag{7}
$$

From Theorem 2, $w_b(\{\tilde{x}| \ T_0(\tilde{x}) < (s',t)\}) = \frac{2}{\sqrt{\pi}} \int_{\infty}^{-b/\sqrt{s't'-st}} \exp\{-v^2\}dv$. In order to get $\frac{\partial F}{\partial s'}$, we use the Fundamental Theorem of Calculus; the integral is a function of s' which appears only on an integral boundary. Then the same technique is applied to the second partial derivative $\frac{\partial^2}{\partial t'\partial s'}$.

$$
\frac{\partial F}{\partial s'} \ = \ \frac{2}{\sqrt{\pi}}\exp\left\{-\frac{b^2}{s't'-st}\right\}\cdot\frac{\partial}{\partial s'}\left(\frac{-b}{\sqrt{s't'-st}}\right) = \frac{t'b}{\sqrt{(s't'-st)^3\pi}}\exp\left\{-\frac{b^2}{s't'-st}\right\}
$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial t' \partial s'} &= \frac{b}{\sqrt{(s't'-st)^3\pi}}\exp\left\{-\frac{b^2}{s't'-st}\right\} + \frac{t'b}{\sqrt{(s't'-st)^5\pi}}(-\frac{3}{2})s'\exp\left\{-\frac{b^2}{s't'-st}\right\} \\
&\quad + \frac{t'b}{\sqrt{(s't'-st)^3\pi}}\exp\left\{-\frac{b^2}{s't'-st}\right\}\cdot(-1)\frac{-s'b^2}{(s't'-st)^2} \\
&= \left(\frac{b}{\sqrt{(s't'-st)^3\pi}} + \frac{(-\frac{3}{2})s't'b}{\sqrt{(s't'-st)^5\pi}} + \frac{s't'b^3}{\sqrt{(s't'-st)^7\pi}}\right)\exp\left\{-\frac{b^2}{s't'-st}\right\}
\end{aligned}
$$

Next we will use Equation (7) with $|b|$ and $|b|^3$ for both cases $\int_T^\infty \cdots dt'$ and $\int_{-\infty}^T \cdots dt'$. At the beginning of the proof we regarded $F(s',t')$ as a cumulative distribution function of the event $\{\tilde{x}|\ T_0(\tilde{x}) < (s',t')\}$; that is, $\frac{\partial^2 F}{\partial t' \partial s'}$ is known to be the pdf of F. We can compute $w_b\{\tilde{x}|\ T_0(\tilde{x}) > (s',t')\}$ using the pdf as follows:

$$
w_b\{\tilde{x}|\ T_0(\tilde{x}) < (s',t')\} = \int_{t'}^\infty\int_{s'}^\infty\left(\frac{|b|}{\sqrt{(s't'-st)^3\pi}} + \frac{(-\frac{3}{2})s't'|b|}{\sqrt{(s't'-st)^5\pi}} + \frac{s't'|b|^3}{\sqrt{(s't'-st)^7\pi}}\right)\exp\left\{-\frac{b^2}{s't'-st}\right\}ds'dt'
$$

and we obtain the desired result.　□

The pdf $\frac{\partial^2 F}{\partial t' \partial s'}$ in the proof of Theorem 3 will play a key role in our main theorem.

4. The Two-Parameter Version of an Arc-Sine Law for a Hitting Time

Introducing the one-parameter case for the Wiener process $\{X_t, t > 0\}$, a last hitting time $L_X = \sup\{t \in [0,T] : X_t = 0\}$ has the probability $P\{x \in C_0[0,T] \mid L_X(x) \le s\} = \frac{2}{\pi}\arcsin(\sqrt{s})$ ([9]), which is known to be an arc-sine law.

We now define a process L in the two-parameter Wiener space $C_2(R)$ that is the two parameter version of a last hitting time L_X and aim to develop for our process L a law similar to the existing arc-sine law for L_X. We keep the notations $R = [0,S] \times [0,T]$ and $x \in C_2(R)$. Differently from the one-dimensional feature of the parameter t, the word "last" for a two-dimensional parameter (s,t) will have a weakened meaning. We define an analogous concept to the last hitting time in the one-dimension as follows.

Definition 9. *Let $L(x)$ be the set of points (s,t) in a rectangle R with the following properties:*

(i)　*$x(s,t) = 0$ and for any $(s',t') > (s,t)$, $x(s',t') \neq 0$;*

(ii)　*on the boundary $s = S$ or $t = T$ of R, if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists a point hitting zero in the region $(S - \varepsilon_1, S] \times (T - \varepsilon_2, T] \subset R$, then we define $L(x) = (S,T)$.*

Here, the inequality $>$ used in (i) has been defined in Definition 5.

Remark 3. *(1) If $(s,t) \in L(x)$, then x never hits zero beyond the point (s,t). Hence, in some sense, the point (s,t) can be called the last point in a radial direction from the origin $(0,0)$. Therefore, we can call L the set of last hitting points of radial directions.*

(2) Considering a degenerate case of the definition of L, if $x(\cdot,t)$ is an increasing function of t and $x(s,\cdot)$ is an increasing function of s, then we define $L(x) = (0,0)$.

Lemma 2. *For $(s,t) \in R$, the two-parameter Wiener measure of L can be expressed by*

$$
m_2\{x|\ L(x) < (s,t)\} = \int_{-\infty}^\infty \frac{1}{\sqrt{\pi st}}\exp\{\frac{-u_1^2}{st}\}\,w_{u_1}(\ T_0(\tilde{x}) > (S,T))\,du_1.
$$

Proof. Let $L(x) < (s,t)$. Then the surface of x over R does not hit zero at (s,t), that is, $x(s,t) \neq 0$, say $x(s,t) = u_1$. Also for any (s',t') with $(s',t') > (s,t)$, $x(s',t') \neq 0$. For all $x \in \{L(x) < (s,t)\}$, x does not hit zero on $[s,S] \times [t,T]$.

The subrectangle has been denoted by $\tilde{R} = [s, S] \times [t, T]$ (Definition 7) and a new measure denoted by w_b on $\tilde{R}$ (Definition 8) has been defined and used in Theorems 2 and 3. In the event discussed here, b in the theorems is replaced by u_1. This means that x may hit zero after S for s and T for t; that is, $T_0(\tilde{x}) > (S, T)$. The event can be written as $\{x|\ L(x) < (s, t)\} = \{x|\ x(s, t) = u_1, T_0(\tilde{x}) > (S, T)\}$. Therefore, $m_2\{x|\ L(x) < (s, t)\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi st}} \exp\{\frac{-u_1^2}{st}\} w_{u_1}(\ T_0(\tilde{x}) > (S, T))\, du_1$. $\square$

We provide the main theorem of our work.

Theorem 4. *For* $(s, t) \in R$,

$$m_2\{x|\ L(x) < (s, t)\} = \frac{2}{\pi} \arcsin \sqrt{\frac{st}{ST}}.$$

Proof. From Lemma 2, $m_2(\{x|\ L(x) < (s, t)\}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi st}} \exp\{\frac{-u_1^2}{st}\} w_{u_1}(\ T_0(\tilde{x}) > (S, T))\, du_1$.
Using Equation (6) and Theorem 3 for w_{u_1},

$$w_{u_1}\{x|\ T_0(x) > (S, T)\} = -\int_T^{\infty}\int_S^{\infty} \left(\frac{|u_1|}{\sqrt{(s't'-st)^3}\,\pi} + \frac{(-\frac{3}{2})s't'|u_1|}{\sqrt{(s't'-st)^5}\,\pi} + \frac{s't'|u_1|^3}{\sqrt{(s't'-st)^7}\,\pi} \right)$$
$$\times \exp\left\{-\frac{u_1^2}{s't'-st}\right\} ds'\, dt'$$

and so $m_2(\{x|\ L(x) < (s, t)\})$ is a triple integral of the form of $\int_{-\infty}^{\infty}\int_T^{\infty}\int_S^{\infty} \cdots ds'\, dt'\, du_1$. The integrand of the triple integral consists of three terms (fractions are involved each); we calculate each term of the triple integral one by one.

$$\text{(i)} \quad -\int_{-\infty}^{\infty}\int_T^{\infty}\int_S^{\infty} \frac{1}{\sqrt{\pi st}} \exp\{\frac{-u_1^2}{st}\} \frac{|u_1|}{\sqrt{(s't'-st)^3}\,\pi} \exp\left\{-\frac{u_1^2}{s't'-st}\right\} ds'\, dt'\, du_1$$
$$= -\int_T^{\infty}\int_S^{\infty} \frac{2}{\pi\sqrt{st(s't'-st)^3}} \int_0^{\infty} u_1 \exp\{\frac{-u_1^2}{st} - \frac{u_1^2}{s't'-st}\} du_1\, ds'\, dt'$$
$$= -\int_T^{\infty}\int_S^{\infty} \frac{2}{\pi\sqrt{st(s't'-st)^3}} \{\frac{st(s't'-st)}{2s't'}\} ds'\, dt'$$
$$= -\int_T^{\infty}\int_S^{\infty} \frac{\sqrt{st}}{\pi\sqrt{(s't'-st)}\,s't'} ds'\, dt',$$

here, the Fubini theorem is used in the first equality. The second equality comes from $\int_0^{\infty} u e^{-au^2}\, du = \frac{1}{2a}$ for a positive real number a.

$$\text{(ii)} \quad \frac{3}{2}\int_{-\infty}^{\infty}\int_T^{\infty}\int_S^{\infty} \frac{1}{\sqrt{\pi st}} \exp\{\frac{-u_1^2}{st}\} \frac{s't'|u_1|}{\sqrt{(s't'-st)^5}\,\pi} \exp\left\{-\frac{u_1^2}{s't'-st}\right\} ds'\, dt'\, du_1$$
$$= 3\int_T^{\infty}\int_S^{\infty} \frac{s't'}{\pi\sqrt{st(s't'-st)^5}} \int_0^{\infty} u_1 \exp\{-\frac{s't'u_1^2}{st(s't'-st)}\} du_1\, ds'\, dt'$$
$$= \frac{3}{2}\int_T^{\infty}\int_S^{\infty} \frac{\sqrt{st}}{\pi\sqrt{(s't'-st)^3}} ds'\, dt'.$$

$$(iii) \qquad -\int_{-\infty}^{\infty}\int_{T}^{\infty}\int_{S}^{\infty}\frac{1}{\sqrt{\pi st}}\exp\{\frac{-u_1^2}{st}\}\frac{s't'|u_1|^3}{\sqrt{(s't'-st)^7\pi}}\exp\left\{-\frac{u_1^2}{s't'-st}\right\}ds'dt'du_1$$

$$= -2\int_{T}^{\infty}\int_{S}^{\infty}\frac{s't'}{\pi\sqrt{st(s't'-st)^7}}\int_{0}^{\infty}u_1^3\exp\{-\frac{s't'}{st(s't'-st)}u_1^2\}\,du_1ds'dt'$$

$$= -\int_{T}^{\infty}\int_{S}^{\infty}\frac{\sqrt{(st)^3}}{\pi\sqrt{(s't'-st)^3}\,s't'}\,ds'dt'.$$

Here, the second equality comes from $\int_{0}^{\infty}u^3 e^{-a\,u^2}\,du = \frac{1}{2a^2}$ for a positive real number a. Adding the three terms (i), (ii) and (iii) yields

$$m_2(\{x|\,L(x) < (s,t)\}) = \frac{1}{\pi}\int_{T}^{\infty}\int_{S}^{\infty}-\frac{\sqrt{st}}{\sqrt{(s't'-st)}s't'}+\frac{3}{2}\frac{\sqrt{st}}{\pi\sqrt{(s't'-st)^3}}-\frac{\sqrt{(st)^3}}{\sqrt{(s't'-st)^3}\,s't'}\,ds'dt'.$$

Substituting $u = s'/s$ and $v = t'/t$, so $ds' = sdu$ and $dt' = tdv$, the integrand is replaced by

$$-\frac{1}{\frac{s'}{s}\frac{t'}{t}st\sqrt{\frac{s't'}{st}-1}}+\frac{3}{2}\frac{1}{st\sqrt{(\frac{s'}{s}\frac{t'}{t}-1)^3}}-\frac{1}{\frac{s'}{s}\frac{t'}{t}st\sqrt{(\frac{s'}{s}\frac{t'}{t}-1)^3}} = \frac{1}{st}\frac{1}{2}\frac{uv}{uv\sqrt{(uv-1)^3}}$$

and the integral is simplified to

$$m_2(\{x|\,L(x) < (s,t)\}) = \frac{1}{\pi}\int_{T/t}^{\infty}\int_{S/s}^{\infty}\frac{1}{2}\frac{1}{\sqrt{(uv-1)^3}}\,dudv. \tag{8}$$

The inner integral of Equation (8) ($\frac{1}{2\pi}$ will be multiplied later) with respect to u becomes

$$\int_{S/s}^{\infty}\frac{1}{\sqrt{(uv-1)^3}}\,du = \frac{-2}{v}(uv-1)^{-1/2}\Big]_{S/s}^{\infty} = \frac{2}{v\sqrt{\frac{S}{s}v-1}}.$$

Set $w = \frac{s}{S}\frac{1}{v}$ for the outer integral of (8), then $v = \frac{s}{S}\frac{1}{w}$, $dv = \frac{-s}{Sw^2}\,dw$ and

$$\int_{T/t}^{\infty}\frac{2}{v\sqrt{\frac{S}{s}v-1}}\,dv = \int_{st/ST}^{0}\frac{2}{\sqrt{(\frac{1}{w}-1)}}\frac{-1}{w}\,dw = \int_{0}^{st/ST}\frac{2}{\sqrt{w(1-w)}}\,dw.$$

In all,

$$m_2(\{x|\,L(x) < (s,t)\}) \;=\; \frac{1}{2\pi}\int_{0}^{st/ST}\frac{2}{\sqrt{w(1-w)}}\,dw = \frac{2}{\pi}\arcsin\sqrt{\frac{st}{ST}}.$$

Finally, we have obtained that the probability of $\{x|\,L(x) < (s,t)\}$ follows an arc-sine distribution as desired. □

5. Conclusions

We have developed a two-parameter process analogous to a last hitting time for the Wiener process and provided the probability distribution of the process. This can be called the two-parameter version of the arc-sine law for a last hitting time. Differently from the one-dimensional feature of the parameter t, we were cautious to say the word "the last time" for a two-dimensional parameter (s,t). We devised an analogous concept for the two-dimensional parameter, which can be called last points hitting b in radial directions, so our word "last" has a weakened meaning.

We have calculated the probability of the event of last points associated with the process and confirmed that it follows an arc-sine distribution. The resulting probability calculated in Theorem 4 shows a similar form to the arc-sine law for the last hitting time of the Wiener process.

However, the derivation of our result is not as simple as the derivation for a one-parameter process. The probability of our two-parameter process is naturally led to a multiple integral and required some device on measures as well as very lengthy calculations and a series of substitutions. Nevertheless, we have obtained that it follows an arc-sine distribution.

Our result can be added to many existing arc-sine laws, in particular, uniquely for two-dimensional parameters. One can use our results to predict an event happened in a system of random variables with two parameters, which is not available among existing laws for one parameter processes. The result of our work can be adapted to other acrsine laws, such as the first or third laws, which is our ongoing work.

Conflicts of Interest: The author declares no conflict of interest. The funder had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Lévy, P. Sur certains processus stochastiques homogénes. *Compos. Math.* **1939**, *7*, 283–339.
2. Erdös, P.; Kac, M. On the number of positive sums of independent random variables. *Bull. Am. Math. Soc.* **1947**, *53*, 1011–1020. [CrossRef]
3. Abundo, M. The arcsine law for the first instant at which a diffusion process equals the ultimate value of a functional. *Int. J. Pure Appl. Math.* **2006**, *30*, 13–22.
4. Ryu, K.S. The arcsine law in the analogue of Wiener space. *J. Chungcheong Math. Soc.* **2015**, *28*, 615–622. [CrossRef]
5. Ryu, K.S.; Im, K. A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula. *Trans. Am. Math. Soc.* **2002**, *354*, 4921–4951. [CrossRef]
6. Barato, A.C.; Roldán, É.; Martínez, I.A.; Pigolotti, S. Arcsine laws in stochastic thermodynamics. *Phys. Rev. Lett.* **2018**, *121*, 1–5. [CrossRef] [PubMed]
7. Dale, C.; Workman, R. The arc sine law and the treasury bill futures market. *Financ. Anal. J.* **1980**, *36*, 71–74. [CrossRef]
8. Mörters, P.; Peres, Y. *Brownian Motion*, 1st ed.; Cambridge University Press: Cambridge, UK, 2010.
9. Durrett, R. *Probability: Theory and Examples*, 4th ed.; Cambridge University Press: Cambridge, UK, 2010.
10. Yeh, J. Wiener measure in a space of functions of two variables. *Trans. Am. Math Soc.* **1960**, *95*, 433–450. [CrossRef]
11. Kim, J.-G. An average of surfaces as functions in the two-parameter Wiener space for a probabilistic 3D shape model. *Bull. Korean Math. Soc.* **2019**. [CrossRef]
12. Yeh, J. Cameron-Martin translation theorems in the Wiener space of functions of two variables. *Trans. Am. Math Soc.* **1963**, *107*, 409–420. [CrossRef]
13. Chan, A.H.C. Some lower bounds for the distribution of the supremum of the Yeh-Wiener process over a rectangular region. *J. Appl. Probab.* **1975**, *12*, 824–830. [CrossRef]
14. Paranjape, S.; Park, C. Distribution of the Supremum of the Two-Parameter Yeh-Wiener process on the boundary. *J. Appl. Probab.* **1973**, *10*, 875–880. [CrossRef]
15. Park, C.; Skoug, D. Distribution estimates of barrier-crossing probabilities of the Yeh-Wiener process. *Pac. J. Math.* **1978**, *78*, 455–466. [CrossRef]