mathematics

Article

Space of Quasi-Periodic Limit Functions and Its Applications

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Received: 23 October 2019; Accepted: 15 November 2019; Published: 19 November 2019

Abstract: We introduce a class consisting of what we call quasi-periodic limit functions and then establish the relation between quasi-periodic limit functions and asymptotically quasi-periodic functions. At last, these quasi-periodic limit functions are applied to study the existence of asymptotically quasi-periodic solutions of abstract Cauchy problems.

Keywords: quasi-periodic limit functions; asymptotically quasi-periodic functions; abstract Cauchy problems

MSC: 34C25, 34C27, 34A12

1. Introduction

The study of quasi-periodic functions may go back to P. Bohl in 1893 and E. Esclangon in 1904; see [1,2]. These quasi-periodic functions, which appeared in astronomical perturbation problems, are those that can be approximated uniformly on $\mathbb{R}$ by $\sum_{k=1}^{n} c_k A_k(t)$ with $c_k \in \mathbb{C}$ and $A_k(t) = e^{i(m_1 a_1 + m_2 a_2 + \cdots + m_n a_n)t}$, where $m_j \in \mathbb{N}$ and $a_j \in \mathbb{R}$. It turns out that any quasi-periodic function can be obtained from a periodic function depending on several variables [3]. Let f be a periodic function from $\mathbb{R}^n$ into $\mathbb{C}$. Regarding the variable t_i, if the periodic function $f(t_1, t_2, \cdots, t_n)$ has a period 2π, then $F(t) = f(\omega_1 t, \omega_2 t, \cdots, \omega_n t)$ with $\omega_j \in \mathbb{R}/\{0\}$ is quasi-periodic.

The application of quasi-periodic functions has been taken in several directions. One direction is the study of quasi-periodic solutions to nonlinear PDE. Up until now, the existence of quasi-periodic solutions of different kinds of nonlinear equations has been shown by the KAM (Kolmogorov–Arnold–Moser) theory or the C-W-B (Craig–Wayne–Bourgain) method. We briefly mention some recent work in this direction and refer the reader to [4–11] for some pioneering work. In [12], F. Giuliani focused on the generalized KdV equations and discussed the existence of Cantor families of quasi-periodic solutions, which extended the results in [13]. In [14], based on a modified KAM theorem, the existence of quasi-periodic response solutions of reversible systems that have Liouvillian frequencies was studied. Regarding the beam equations defined on compact Lie groups, in [15], the authors showed the existence of quasi-periodic solutions by the Nash–Moser iteration; meanwhile, based on the KAM theorem, in [16], Y. Wang proved the existence of quasi-periodic solutions to beam equations when the nonlinear term has the time and space variables. In [17], the authors discussed the Whitney smooth family of quasi-periodic solutions to beam equations. For other applications of quasi-periodic functions, we mention the work by Küpper and Yuan [18] on quasi-periodic solutions for differential equations with piecewise constant argument (see also, e.g., [19,20]).

On the other hand, consider the inhomogeneous abstract Cauchy problem:

$$\begin{cases} x'(t) = Ax(t) + F(t), \ t \in \mathbb{R}^+; \\ x(0) = x_0 \in X, \end{cases} \tag{1}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$, that is there exist $M > 0$ and $r > 0$ such that $\|T(t)\| \leq Me^{-rt}$ for all $t \in \mathbb{R}^+$. In order to study the existence of asymptotically periodic solutions of (1), the authors in [21] proposed the concept of the ω-periodic limit function, which is a generalization of the asymptotically ω-periodic function. Later, in [22], the concept of the squared mean periodic limit process was proposed and the application to stochastic differential equations was studied. The theory of ω-periodic limit functions has the advantage of considering the asymptotical periodicity as a whole. For other applications of the ω-periodic limit function, we refer the reader to [23].

It follows from the method in Theorem 4.4 in [21] that the solution of (1) is asymptotically ω-periodic when the coefficient F is the ω-periodic limit. Moreover, if we assume $F = F_1 + F_2$, F_1 is the ω_1-periodic limit and F_2 is the ω_2-periodic limit, then the solution of (1) is asymptotically quasi-periodic. Here, a question arises: To make sure the solution of (1) is asymptotically quasi-periodic, what kind of function can F be?

Motivated by the above discussions, in the present paper, we propose a class of functions, and we call them quasi-periodic limit functions, which could be regarded as a generalization of (asymptotically) quasi-periodic functions. We will show that the quasi-periodic limit functions contribute to studying the existence of asymptotically quasi-periodic solutions of (1). The main results are Theorem 2 and Theorem 4. To show this, we develop a very general method. We believe the method in this paper could contribute to studying the existence of asymptotically quasi-periodic solutions of some kinds of equations, such as fractional differential equations.

This paper is arranged in three main sections.

In Section 2, we define the notion of quasi-periodic limit functions and study their properties. Especially, we discuss the relation between quasi-periodic limit functions and asymptotically quasi-periodic functions. In Section 3, these quasi-periodic limit functions are applied to study the existence of asymptotically quasi-periodic solutions of abstract Cauchy problems. In Section 4, we propose some related questions. We believe that the questions found here are of interest in the theory of asymptotical quasi-periodicity and that their answers would certainly help to develop this field.

2. Space of Quasi-Periodic Limit Functions

In this paper, we denote the interval $[0, \infty)$ by $\mathbb{R}^+$. Let $(X, \|\cdot\|)$ be a Banach space, and let $C_b(\mathbb{R}^+ \times \mathbb{R}^+, X)$ be the space of bounded and continuous functions from $\mathbb{R}^+ \times \mathbb{R}^+$ into X, endowed with the uniform convergence norm $\|\cdot\|_\infty$. Assume that $\omega_1, \omega_2, \cdots, \omega_n \in \mathbb{R}$. $\omega_1, \omega_2, \cdots, \omega_n$ are called rationally independent if $k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n \neq 0$ for all $k_1, k_2, \cdots, k_n \in Q\backslash\{0\}$, where Q is the set of all rational numbers. In this paper, we always assume $\omega_1, \omega_2, \cdots, \omega_n$ are rationally independent and $\omega_1, \omega_2, \cdots, \omega_n > 0$. Let f be a periodic function from $\mathbb{R}^n$ into X. If the periodic function $f(t_1, t_2, \cdots, t_n)$ in $t_1, t_2, \cdots, t_n$ with the same periodic 2π, then $F(t) = f(\frac{2\pi}{\omega_1}t, \frac{2\pi}{\omega_2}t, \cdots, \frac{2\pi}{\omega_n}t)$ is said to be quasi-periodic. If we define $g(t_1, t_2, \cdots, t_n) = f(\frac{2\pi}{\omega_1}t_1, \frac{2\pi}{\omega_2}t_2, \cdots, \frac{2\pi}{\omega_n}t_n)$, then:

$$
\begin{aligned}
& g(t_1, t_2, \cdots, t_k + \omega_k, \cdots, t_n) \\
=& f\Big(\frac{2\pi}{\omega_1} t_1, \frac{2\pi}{\omega_2} t_2, \cdots, \frac{2\pi}{\omega_k}(t_k + \omega_k), \cdots, \frac{2\pi}{\omega_n} t_n\Big) \\
=& f\Big(\frac{2\pi}{\omega_1} t_1, \frac{2\pi}{\omega_2} t_2, \cdots, \frac{2\pi}{\omega_k} t_k + 2\pi, \cdots, \frac{2\pi}{\omega_n} t_n\Big) \\
=& f\Big(\frac{2\pi}{\omega_1} t_1, \frac{2\pi}{\omega_2} t_2, \cdots, \frac{2\pi}{\omega_k} t_k, \cdots, \frac{2\pi}{\omega_n} t_n\Big) \\
=& g(t_1, t_2, \cdots, t_k, \cdots, t_n)
\end{aligned}
$$

and we can define the same quasi-periodic function by g, that is $F(t) = g(t, t, \cdots, t)$. A continuous function $f : \mathbb{R}^+ \to X$ is called asymptotically quasi-periodic if it admits a decomposition $f(t) = g(t) + h(t)$, where $g : \mathbb{R} \to X$ is a quasi-periodic function and $h : \mathbb{R}^+ \to X$ is a continuous function with $\lim_{t\to+\infty} \|h(t)\| = 0$. Let $\omega > 0$. A bounded continuous function $f : \mathbb{R}^+ \to X$ is called the ω-periodic limit if there is a function $g : \mathbb{R}^+ \to X$ such that $\lim_{n\to\infty} f(t + n\omega) = g(t)$. The set of ω-periodic limit functions will be denoted by $P_\omega L(\mathbb{R}^+, X)$.

For the sake of simplicity, we establish the concept by a function with two variables.

Definition 1. *Let $f \in C_b(\mathbb{R}^+ \times \mathbb{R}^+, X)$. If $g(t,s) = \lim_{n\to\infty} f(t + n\omega_1, s)$ and $h(t,s) = \lim_{n\to\infty} f(t, s + n\omega_2)$ are well defined for each pair $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $n \in \mathbb{N}$, then f is said to be the (ω_1, ω_2)-periodic limit.*

Hypothesis 1 (H1). *For each $t \in \mathbb{R}^+$, $\lim_{n\to\infty} f(t + n\omega_1, s) = g(t,s)$ uniformly for $s \in \mathbb{R}^+$.*

Hypothesis 2 (H2). *For each $s \in \mathbb{R}^+$, $\lim_{n\to\infty} f(t, s + n\omega_2) = h(t,s)$ uniformly for $t \in \mathbb{R}^+$.*

Definition 2. *Let f be a (ω_1, ω_2)-periodic limit function. Assume H1 and H2 hold, then $F(t) = f(t,t)$ is said to be the (ω_1, ω_2)-quasi-periodic limit. The collection of all (ω_1, ω_2)-periodic limit functions that satisfies H1 and H2 (respectively, (ω_1, ω_2)-quasi-periodic limit functions) will be denoted by $PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$ $(QPL_{(\omega_1,\omega_2)}(\mathbb{R}^+, X))$.*

Remark 1. *The functions g and h in Definition 2 is measurable, but not necessarily continuous.*

Now, we present some examples of (ω_1, ω_2)-quasi-periodic limit functions.

Example 1. *Let $f(t,s) = f_1(t) + f_2(s)$, where $f_1 \in P_{\omega_1} L(\mathbb{R}^+, X)$ and $f_2 \in P_{\omega_2} L(\mathbb{R}^+, X)$. By the definition of the ω-periodic limit function, we have $\lim_{n\to\infty} f_1(t + n\omega_1) = g_1(t)$ and $\lim_{n\to\infty} f_2(t + n\omega_2) = g_2(t)$. Let $g(t,s) = g_1(t) + f_2(s)$ and $h(t,s) = f_1(t) + g_2(s)$. Thus, for each $t \in \mathbb{R}^+$, $\lim_{n\to\infty} f(t + n\omega_1, s) = g(t,s)$ uniformly for $s \in \mathbb{R}^+$, and for each $s \in \mathbb{R}^+$, $\lim_{n\to\infty} f(t, s + n\omega_2) = h(t,s)$ uniformly for $t \in \mathbb{R}^+$. Then, $F(t) = f(t,t) = f_1(t) + f_2(t)$ is the (ω_1, ω_2)-quasi-periodic limit.*

Example 2. *Let $f(t,s) = f_1(t) f_2(s)$, where $f_1 \in P_{\omega_1} L(\mathbb{R}^+, \mathbb{C})$ and $f_2 \in P_{\omega_2} L(\mathbb{R}^+, X)$. Then, $F(t) = f(t,t) = f_1(t) f_2(t)$ is the (ω_1, ω_2)-quasi-periodic limit.*

Example 3. *For $n \geq 0$, define a function f on $[n\omega_1, (n+1)\omega_1] \times [m\omega_2 \times (m+1)\omega_2]$ as follows:*

$$f(t,s)=\begin{cases}1, & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ f^1_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3},n\omega_1+\frac{\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ f^2_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{2\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ f^3_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3},m\omega_2+\frac{\omega_2}{3}],\\ f^4_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{2\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3}],\\ f^5_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3},n\omega_1+\frac{\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3},m\omega_2+\frac{\omega_2}{3}],\\ f^6_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{2\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3}]\times[m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3},m\omega_2+\frac{\omega_2}{3}],\\ f^7_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3},n\omega_1+\frac{\omega_1}{3}]\times[m\omega_2+\frac{2\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3}],\\ f^8_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{2\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3}]\times[m\omega_2+\frac{2\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3}],\\ 0, & otherwise,\end{cases}$$

where $f^1_{n,m}(t,s)=\frac{n+3}{\omega_1}[t-(n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3})]$, $f^2_{n,m}(t,s)=-\frac{n+3}{\omega_1}[t-(n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3})]$, $f^3_{n,m}(t,s)=\frac{m+3}{\omega_2}[s-(m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3})]$, $f^4_{n,m}(t,s)=-\frac{m+3}{\omega_2}[s-(m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3})]$, $f^5_{n,m}(t,s)=\frac{\frac{n+3}{\omega_1}[t-(n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3})]}{\frac{\omega_2}{m+3}}[s-(m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3})]$, $f^6_{n,m}(t,s)=-\frac{\frac{n+3}{\omega_1}[t-(n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3})]}{\frac{\omega_2}{m+3}}[s-(m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3})]$, $f^7_{n,m}(t,s)=-\frac{\frac{n+3}{\omega_1}[t-(n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3})]}{\frac{\omega_2}{m+3}}[s-(m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3})]$, $f^8_{n,m}(t,s)=\frac{\frac{n+3}{\omega_1}[t-(n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3})]}{\frac{\omega_2}{m+3}}[s-(m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3})]$. *The graph of the function f in each rectangle $[n\omega_1,(n+1)\omega_1]\times[m\omega_1\times(m+1)\omega_2]$ $(n\geq 1)$ consists of ten parts, and $f(t,s):\mathbb{R}^+\times\mathbb{R}^+\to[0,1]$ is continuous. If we define the function g and h on $[n\omega_1,(n+1)\omega_1]\times[m\omega_2\times(m+1)\omega_2]$ by:*

$$g(t,s)=\begin{cases}1, & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ f^3_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3}-\frac{\omega_2}{m+3},m\omega_2+\frac{\omega_2}{3}],\\ f^4_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{2\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}+\frac{\omega_2}{m+3}],\\ 0, & otherwise\end{cases}$$

and:

$$h(t,s)=\begin{cases}1, & (t,s)\in[n\omega_1+\frac{\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ f^1_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{\omega_1}{3}-\frac{\omega_1}{n+3},n\omega_1+\frac{\omega_1}{3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ f^2_{n,m}(t,s), & (t,s)\in[n\omega_1+\frac{2\omega_1}{3},n\omega_1+\frac{2\omega_1}{3}+\frac{\omega_1}{n+3}]\times[m\omega_2+\frac{\omega_2}{3},m\omega_2+\frac{2\omega_2}{3}],\\ 0, & otherwise.\end{cases}$$

Then, for each $t\in\mathbb{R}^+$, $\lim_{n\to\infty}f(t+n\omega_1,s)=g(t,s)$ uniformly for $s\in\mathbb{R}^+$, and for each $s\in\mathbb{R}^+$, $\lim_{n\to\infty}f(t,s+n\omega_2)=h(t,s)$ uniformly for $t\in\mathbb{R}^+$. Thus, $F(t)=f(t,t)$ is the (ω_1,ω_2)-quasi-periodic limit.

Next, we present the following properties of (ω_1,ω_2)-quasi-periodic limit functions.

Proposition 1. *Let F, F_1 and F_2 be the (ω_1,ω_2)-quasi-periodic limit. Assume $g(t,s)=\lim_{n\to\infty}f(t+n\omega_1,s)$, $h(t,s)=\lim_{n\to\infty}f(t,s+n\omega_2)$ are well defined for each pair $(t,s)\in\mathbb{R}^+\times\mathbb{R}^+$ and $F(t)=f(t,t)$. Then, the following statements are true:*

(1) *F_1+F_2 is the (ω_1,ω_2)-quasi-periodic limit;*
(2) *cF is the (ω_1,ω_2)-quasi-periodic limit for any $c\in\mathbb{C}$;*
(3) *$g(t+\omega_1,s)=g(t,s)$, $h(t,s+\omega_2)=h(t,s)$ for each pair $(t,s)\in\mathbb{R}^+\times\mathbb{R}^+$;*
(4) *g and h are bounded on $\mathbb{R}^+\times\mathbb{R}^+$; moreover, $\|g\|_\infty\leq\|f\|_\infty$ and $\|h\|_\infty\leq\|f\|_\infty$;*

(5) *F is bounded on $\mathbb{R}^+$; moreover, $\|F\|_\infty \leq \|f\|_\infty$.*

Proposition 2. *Let f be a (ω_1, ω_2)-periodic limit function.*

(1) *Assume H1 holds, then $k(t,s) = \lim_{n\to\infty} g(t, s+n\omega_2)$ is well defined for each pair $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$.*
(2) *Assume H2 holds, then $\bar{k}(t,s) = \lim_{n\to\infty} h(t+n\omega_1, s)$ is well defined for each pair $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$.*
(3) *Assume H1 and H2 hold, then $k(t,s) = \bar{k}(t,s)$.*
(4) *Assume H1 and H2 hold, then $k(t+\omega_1, s) = k(t,s) = k(t, s+\omega_2)$.*

Proof. (1) We only need to show that $\{g(t, s+n\omega_2)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for each pair $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$. Let $\varepsilon > 0$. By Hypothesis of H1, for fixed $t \in \mathbb{R}^+$, there exists $N_1 \in \mathbb{N}$ such that $\|f(t + p\omega_1, s) - g(t,s)\| < \frac{\varepsilon}{4}$ uniformly for $s \in \mathbb{R}^+$ when $p \geq N_1$. For the above t, choose $p \geq N_1$, and fix $s \in \mathbb{R}^+$. By Definition 1, there exists $N_2 \in \mathbb{N}$ such that $\|f(t+p\omega_1, s+n\omega_2) - h(t+p\omega_1, s)\| < \frac{\varepsilon}{4}$ when $n \geq N_2$. Therefore,

$$\begin{aligned}
&\|g(t,s+n\omega_2) - g(t,s+m\omega_2)\| \\
\leq &\|g(t,s+n\omega_2) - f(t+p\omega_1, s+n\omega_2)\| + \|f(t+p\omega_1, s+n\omega_2) - h(t+p\omega_1, s)\| \\
&+ \|h(t+p\omega_1, s) - f(t+p\omega_1, s+m\omega_2)\| + \|f(t+p\omega_1, s+m\omega_2) - g(t, s+m\omega_2)\| \\
< &\varepsilon
\end{aligned}$$

when $m, n \geq N_2$.

(2) In a similar way as (1), one can show (2).

(3) Let $\varepsilon > 0$, and fix $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$. By Hypothesis H1, there exists $N_1 \in \mathbb{N}$ such that:

$$\|f(t+n\omega_1, s') - g(t,s')\| < \frac{\varepsilon}{4} \tag{2}$$

uniformly for $s' \in \mathbb{R}^+$ when $n \geq N_1$. By Hypothesis H2, there exists $N_2 \in \mathbb{N}$ such that:

$$\|f(t', s+n\omega_2) - h(t', s)\| < \frac{\varepsilon}{4} \tag{3}$$

uniformly for $t' \in \mathbb{R}^+$ when $n \geq N_2$. By the conclusion of (1), there exists $N_3 \in \mathbb{N}$ such that:

$$\|g(t, s+n\omega_2) - k(t,s)\| < \frac{\varepsilon}{4} \tag{4}$$

when $n \geq N_3$. By the conclusion of (2), there exists $N_4 \in \mathbb{N}$ such that:

$$\|h(t+n\omega_1, s) - \bar{k}(t,s)\| < \frac{\varepsilon}{4} \tag{5}$$

when $n \geq N_4$. Select $N_5 = \max\{N_1, N_2, N_3, N_4\}$. (2), (3), (4), and (5) imply:

$$\begin{aligned}
&\|k(t,s) - \bar{k}(t,s)\| \\
\leq &\|k(t,s) - g(t, s+N_5\omega_2)\| + \|g(t, s+N_5\omega_2) - f(t+N_5\omega_1, s+N_5\omega_2)\| \\
&+ \|f(t+N_5\omega_1, s+N_5\omega_2) - h(t+N_5\omega_1, s)\| + \|h(t+N_5\omega_1, s) - \bar{k}(t,s)\| \\
< &\varepsilon,
\end{aligned}$$

which shows $k(t,s) = \bar{k}(t,s)$.

(4) $k(t,s) = \lim_{n\to\infty} g(t, s+n\omega_2) = \lim_{n\to\infty} g(t+\omega_1, s+n\omega_2) = k(t+\omega_1, s)$. Similarly, $k(t,s) = k(t, s+\omega_2)$. □

Hypothesis 3 (H3)**.** *f is uniformly continuous.*

Proposition 3. *Let f be a (ω_1, ω_2)-periodic limit function. Assume H3 holds, then g and h are uniformly continuous.*

Proof. For $\varepsilon > 0$ given, there exists $\delta > 0$ such that $\|f(t,s) - f(t',s')\| < \frac{\varepsilon}{3}$ when $|t - t'| < \delta$, $|s - s'| < \delta$ for $t, t', s, s' \in \mathbb{R}^+$. Choose $t, t', s, s' \in \mathbb{R}^+$ such that $|t - t'| < \delta$, $|s - s'| < \delta$, then there exists $N_1 \in \mathbb{N}$ such that $\|g(t,s) - f(t + n\omega_1, s)\| < \frac{\varepsilon}{3}$ when $n \geq N_1$, and there exists $N_2 \in \mathbb{N}$ such that $\|g(t',s') - f(t' + n\omega_1, s')\| < \frac{\varepsilon}{3}$ when $n \geq N_2$. Choose $N_3 = \max\{N_1, N_2\}$. Then, one has:

$$\begin{aligned} &\|g(t,s) - g(t',s')\| \\ \leq &\|g(t,s) - f(t + N_3\omega_1, s)\| + \|f(t + N_3\omega_1, s) - f(t' + N_3\omega_1, s')\| + \|f(t' + N_3\omega_1, s') - g(t',s')\| \\ < &\varepsilon. \end{aligned}$$

Therefore, g is uniformly continuous. Similarly, h is uniformly continuous. □

In the following propositions, if F is a (ω_1, ω_2)-quasi-periodic limit function, then f, g, h are defined in Definition 2, and k is defined in Proposition 2.

Proposition 4. *Let F be a (ω_1, ω_2)-quasi-periodic limit function. Assume H3 holds, then k is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous, g is uniformly continuous by Proposition 3. Therefore, there exists $\delta > 0$ such that $\|g(t,s) - g(t',s')\| < \frac{\varepsilon}{3}$ when $|t - t'| < \delta$, $|s - s'| < \delta$ for $t, t', s, s' \in \mathbb{R}^+$. Choose $t, t', s, s' \in \mathbb{R}^+$ such that $|t - t'| < \delta$, $|s - s'| < \delta$, then by Proposition 2 (1), there exists $N_1 \in \mathbb{N}$ such that $\|k(t,s) - g(t, s + n\omega_2)\| < \frac{\varepsilon}{3}$ when $n \geq N_1$, and there exists $N_2 \in \mathbb{N}$ such that $\|k(t',s') - g(t', s' + n\omega_2)\| < \frac{\varepsilon}{3}$ when $n \geq N_2$. Choose $N_3 = \max\{N_1, N_2\}$. Then, one has:

$$\begin{aligned} &\|k(t,s) - k(t',s')\| \\ \leq &\|k(t,s) - g(t, s + N_3\omega_2)\| + \|g(t, s + N_3\omega_2) - g(t', s' + N_3\omega_2)\| + \|g(t', s' + N_3\omega_2) - k(t',s')\| \\ < &\varepsilon. \end{aligned}$$

Thus, k is uniformly continuous. □

Proposition 5. *Let F be a (ω_1, ω_2)-quasi-periodic limit function. Assume H3 holds, and denote $r(t,s) = f(t,s) - k(t,s)$, then $r \in C_0(\mathbb{R}^+ \times \mathbb{R}^+, X)$, that is for any $\varepsilon > 0$, there exists $M > 0$ such that $\|r(t,s)\| < \varepsilon$ when $t > M$, $s > M$.*

Proof. It is equivalent to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f(t + n\omega_1, s + m\omega_2) - k(t + n\omega_1, s + m\omega_2)\| < \varepsilon$ uniformly for $t \in [0, \omega_1]$, $s \in [0, \omega_2]$ when $n, m \geq N$. By Proposition 2 (4), one has $k(t,s) = k(t + n\omega_1, s + m\omega_2)$ for any $m, n \in \mathbb{N}$. Note that:

$$\begin{aligned} &f(t + n\omega_1, s + m\omega_2) - k(t + n\omega_1, s + m\omega_2) \\ = &f(t + n\omega_1, s + m\omega_2) - g(t, s + m\omega_2) + g(t, s + m\omega_2) - k(t,s). \end{aligned}$$

Let $\varepsilon > 0$. By Proposition 3 and Proposition 4, g and k are uniformly continuous. Thus, there exists $\delta > 0$ such that $\|f(t,s) - f(t',s')\| < \varepsilon$, $\|g(t,s) - g(t',s')\| < \varepsilon$, $\|k(t,s) - k(t',s')\| < \varepsilon$ when $|t - t'| < \delta$, $|s - s'| < \delta$ for $t, t', s, s' \in \mathbb{R}^+$.

Divide $[0, \omega_1]$ into p equal intervals such that $\frac{\omega_1}{p} < \delta$, $p \in \mathbb{N}$. Define $t_0 = 0, t_1 = \frac{\omega_1}{p}, t_2 = \frac{2\omega_1}{p}, \cdots, t_{p-1} = \frac{(p-1)\omega_1}{p}, t_p = \omega_1$. Then, there exists $N_1 \in \mathbb{N}$ such that $\|f(t_i + n\omega_1, s + m\omega_2) - g(t_i, s + m\omega_2)\| < \varepsilon$ uniformly for $s \in [0, \omega_2]$ and $m \in \mathbb{N}$, $i = 0, 1, 2, \cdots, p$ when $n \geq N_1$. Choose any $t \in [0, \omega_1]$, one can pick $t_{i_0} \in \{t_i\}$ such that $|t - t_{i_0}| < \delta$. Then, one has:

$$
\begin{aligned}
&\|f(t+n\omega_1, s+m\omega_2) - g(t, s+m\omega_2)\| \\
\leq &\|f(t+n\omega_1, s+m\omega_2) - f(t_{i_0}+n\omega_1, s+m\omega_2)\| + \|f(t_{i_0}+n\omega_1, s+m\omega_2) - g(t_{i_0}, s+m\omega_2)\| \\
&+ \|g(t_{i_0}, s+m\omega_2) - g(t, s+m\omega_2)\| \\
< &3\varepsilon
\end{aligned}
$$

when $n \geq N_1$ uniformly for $s \in [0, \omega_2]$ and $m \in \mathbb{N}$.

Fix $t \in \mathbb{R}^+$. In a similar way, by dividing the interval $[0, \omega_2]$ and using the uniform continuousness of g, k, one can show there exists $N_2 \in \mathbb{N}$ such that:

$$
\|g(t, s+m\omega_2) - k(t, s)\| < 3\varepsilon \tag{6}
$$

when $m \geq N_2$ uniformly for $s \in [0, \omega_2]$.

In a similar way again, by dividing the interval $[0, \omega_1]$ and using (6), one can show there exists $N_3 \in \mathbb{N}$ such that $\|g(t, s+m\omega_2) - k(t, s)\| < 5\varepsilon$ when $m \geq N_3$ uniformly for $s \in [0, \omega_2]$ and $t \in [0, \omega_1]$. Note that $g(t+n\omega_1, s) = g(t, s)$, $k(t+n\omega_1, s) = k(t, s)$. Therefore, for any $t \in \mathbb{R}^+$, one has $\|g(t, s+m\omega_2) - k(t, s)\| < 5\varepsilon$ when $m \geq N_3$ uniformly for $s \in [0, \omega_2]$.

Choose $N_4 = \max\{N_1, N_3\}$; one has:

$$
\begin{aligned}
&\|f(t+n\omega_1, s+m\omega_2) - k(t+n\omega_1, s+m\omega_2)\| \\
\leq &\|f(t+n\omega_1, s+m\omega_2) - g(t, s+m\omega_2)\| + \|g(t, s+m\omega_2) - k(t, s)\| \\
< &8\varepsilon
\end{aligned}
$$

when $n, m \geq N_4$ uniformly for $t \in [0, \omega_1]$ and $s \in [0, \omega_2]$. □

Proposition 6. *Let F be a (ω_1, ω_2)-quasi-periodic limit function. Assume H3 holds, then F is asymptotically quasi-periodic.*

Proof. Denote $K(t) = k(t, t)$, $R(t) = r(t, t)$. By Proposition 5, $F(t) = K(t) + R(t)$, $R \in C_0(\mathbb{R}^+, X)$. By Proposition 2 (4) and Proposition 4, K is quasi-periodic. Therefore, F is asymptotically quasi-periodic. □

Let us introduce the following conditions.

Hypothesis 4 (H4). $\lim_{n\to\infty} f(t+n\omega_1, s) = g(t, s)$ *uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.*

Hypothesis 5 (H5). $\lim_{n\to\infty} f(t, s+n\omega_2) = h(t, s)$ *uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.*

Proposition 7. *Let f be a (ω_1, ω_2)-periodic limit function. Assume H4 and H5 hold, then:*

(1) $k(t, s) = \lim_{n\to\infty} g(t, s+n\omega_2)$ *uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$;*
(2) $\bar{k}(t, s) = \lim_{n\to\infty} h(t+n\omega_1, s)$ *uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$;*
(3) $k(t, s) = \bar{k}(t, s)$;
(4) $k(t+\omega_1, s) = k(t, s) = k(t, s+\omega_2)$.

Proof. (1) We only need to show that $\{g(t, s+n\omega_2)\}_{n\in\mathbb{N}}$ is a Cauchy sequence uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$. Let $\varepsilon > 0$. By Hypothesis H4, there exists $N_1 \in \mathbb{N}$ such that $\|f(t+p\omega_1, s) - g(t, s)\| < \frac{\varepsilon}{4}$ uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$ when $p \geq N_1$. By Hypothesis H5, there exists $N_2 \in \mathbb{N}$ such that $\|f(t, s+n\omega_2) - h(t, s)\| < \frac{\varepsilon}{4}$ uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$ when $n \geq N_2$.

Therefore,

$$\begin{aligned}&\|g(t,s+n\omega_2)-g(t,s+m\omega_2)\|\\ \le&\|g(t,s+n\omega_2)-f(t+p\omega_1,s+n\omega_2)\|+\|f(t+p\omega_1,s+n\omega_2)-h(t+p\omega_1,s)\|\\ &+\|h(t+p\omega_1,s)-f(t+p\omega_1,s+m\omega_2)\|+\|f(t+p\omega_1,s+m\omega_2)-g(t,s+m\omega_2)\|\\ <&\varepsilon\end{aligned}$$

uniformly for $t\in\mathbb{R}^+$ and $s\in\mathbb{R}^+$ when $m,n\ge N_2$.

(2) In a similar way as (1), one can show (2).

(3) and (4) are the conclusions of Proposition 2 (3) (4). □

Proposition 8. *Let F be a (ω_1,ω_2)-quasi-periodic limit function. Assume H4 and H5 hold, then F is asymptotically quasi-periodic.*

Proof. Denote $r(t,s)=f(t,s)-k(t,s)$; we first show that $r\in C_0(\mathbb{R}^+\times\mathbb{R}^+,X)$.

We only need to show that for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $\|f(t+n\omega_1,s+m\omega_2)-k(t+n\omega_1,s+m\omega_2)\|<\varepsilon$ uniformly for $t\in[0,\omega_1]$, $s\in[0,\omega_2]$ when $n,m\ge N$. By Proposition 7 (4), one has $k(t,s)=k(t+n\omega_1,s+m\omega_2)$ for any $m,n\in\mathbb{N}$.

By Hypothesis H4, there exists $N_1\in\mathbb{N}$ such that $\|f(t+n\omega_1,s)-g(t,s)\|<\frac{\varepsilon}{2}$ uniformly for $t\in\mathbb{R}^+$ and $s\in\mathbb{R}^+$ when $n\ge N_1$. By Proposition 7, there exists $N_2\in\mathbb{N}$ such that $\|g(t,s+m\omega_2)-k(t,s)\|<\frac{\varepsilon}{2}$ uniformly for $t\in\mathbb{R}^+$ and $s\in\mathbb{R}^+$ when $m\ge N_2$. Choose $N=\max\{N_1,N_2\}$. Therefore,

$$\begin{aligned}&\|f(t+n\omega_1,s+m\omega_2)-k(t+n\omega_1,s+m\omega_2)\|\\ \le&\|f(t+n\omega_1,s+m\omega_2)-g(t,s+m\omega_2)\|+\|g(t,s+m\omega_2)-k(t,s)\|\\ <&\varepsilon\end{aligned}$$

uniformly for $t\in[0,\omega_1]$, $s\in[0,\omega_2]$ when $n,m\ge N$.

Next, we show that k is continuous. Take any $t_0,s_0\in\mathbb{R}^+$. Since f is continuous, there exists $\delta>0$ such that:

$$\|f(t_0+N_1\omega_1,s_0+N_2\omega_2)-f(t'+N_1\omega_1,s'+N_2\omega_2)\|<\varepsilon$$

when $|t_0-t'|<\delta$, $|s_0-s'|<\delta$. Therefore,

$$\begin{aligned}&\|k(t_0,s_0)-k(t',s')\|\\ =&\|k(t_0,s_0)-g(t_0,s_0+N_2\omega_2)\|+\|g(t_0,s_0+N_2\omega_2)-f(t_0+N_1\omega_1,s_0+N_2\omega_2)\|\\ &+\|f(t_0+N_1\omega_1,s_0+N_2\omega_2)-f(t'+N_1\omega_1,s'+N_2\omega_2)\|\\ &+\|f(t'+N_1\omega_1,s'+N_2\omega_2)-g(t',s'+N_2\omega_2)\|+\|g(t',s'+N_2\omega_2)-k(t',s')\|\\ <&3\varepsilon\end{aligned}$$

when $|t_0-t'|<\delta$, $|s_0-s'|<\delta$.

Denote $K(t)=k(t,t)$, $R(t)=r(t,t)$. Then, $F(t)=K(t)+R(t)$, $R\in C_0(\mathbb{R}^+,X)$, and K is quasi-periodic. Therefore, F is asymptotically quasi-periodic. □

The collection of all (ω_1,ω_2)-periodic limit functions that satisfies H4 and H5 will be denoted by $AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+\times\mathbb{R}^+,X)$.

Theorem 1. *$AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+\times\mathbb{R}^+,X)$ is a Banach space.*

Proof. Let $\{f_k\} \subset AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$ such that:

$$\lim_{k\to\infty} f_k(t,s) = f(t,s)$$

uniformly in $t, s \in \mathbb{R}^+$.

Since $\{f_k\} \subset AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$, for each $k \in \mathbb{N}$, one has:

$$\lim_{n\to\infty} f_k(t+n\omega_1, s) = g_k(t,s)$$

uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.

At the same time, for each $k \in \mathbb{N}$, one has:

$$\lim_{n\to\infty} f_k(t, s+n\omega_2) = h_k(t,s)$$

uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.

Then, the sequence of $\{g_k(t,s)\}$ is a Cauchy sequence in X uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$ because the following inequality:

$$\|g_k(t,s) - g_h(t,s)\| \le \|g_k(t,s) - f_k(t+n\omega_1,s)\| + \|f_k(t+n\omega_1,s) - f_h(t+n\omega_1,s)\| + \|f_h(t+n\omega_1,s) - g_h(t,s)\|.$$

Thus, the sequence $\{g_k(t,s)\}$ converges to a function $g(t,s)$ uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.

Now, we only need to show:

$$\lim_{n\to\infty} f(t+n\omega_1, s) = g(t,s)$$

uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$. However, we can get it from the following inequality.

$$\|f(t+n\omega_1,s) - g(t,s)\| \le \|f(t+n\omega_1,s) - f_k(t+n\omega_1,s)\| + \|f_k(t+n\omega_1,s) - g_k(t,s)\| + \|g_k(t,s) - g(t,s)\|.$$

In a similar way, we can show that there exists a function h such that:

$$\lim_{n\to\infty} f(t, s+n\omega_2) = h(t,s)$$

uniformly for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$. Therefore, $f \in AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$. □

3. Existence of Asymptotically Quasi-Periodic Solutions of Abstract Cauchy Problems

In this section, we first introduce the following definition:

Definition 3. *A function $x \in C_b(\mathbb{R}^+, X)$ is said to be a mild solution of Problem (1) if:*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F(s)ds,\ t \in \mathbb{R}^+,$$

where $(T(t))_{t\ge 0}$ is an exponentially stable C_0-semigroup.

Lemma 1. *Let F be a (ω_1,ω_2)-quasi-periodic limit function and $(T(t))_{t\ge 0}$ be an exponentially stable C_0-semigroup, then $U(t) = \int_0^t T(t-s)F(s)ds$ is asymptotically quasi-periodic.*

Proof. By the definition of the (ω_1,ω_2)-quasi-periodic limit function, there exists a $f \in C_b(\mathbb{R}^+ \times \mathbb{R}^+, X)$ such that for each $t \in \mathbb{R}^+$, $\lim_{n\to\infty} f(t+n\omega_1, s) = g(t,s)$ uniformly for $s \in \mathbb{R}^+$, for each

$s \in \mathbb{R}^+$, $\lim_{n\to\infty} f(t, s + n\omega_2) = h(t, s)$ uniformly for $t \in \mathbb{R}^+$ and $F(t) = f(t, t)$. Assume $\|f(t, s)\| \le K$ for $t, s \in \mathbb{R}^+$. Then, $\|g(t, s)\| \le K$ and $\|h(t, s)\| \le K$ for $t, s \in \mathbb{R}^+$.

Define:

$$u(x, y) = \begin{cases} \int_0^x T(x - s)f(s, s + y - x)ds, \ y \ge x, \ x, y \in \mathbb{R}^+, \\ \int_0^y T(y - s)f(s + x - y, s)ds, \ x \ge y, \ x, y \in \mathbb{R}^+. \end{cases} \tag{7}$$

Clearly, u is continuous and bounded.

Note that $U(t) = u(t, t) = \int_0^t T(t - s)f(s, s)ds = \int_0^t T(t - s)F(s)ds$. By Proposition 8, we only need to show that H4 and H5 hold for u. Next, we only show that H4 holds for u because the case for H5 is similar.

Define $f(t, s) = f(t, 0)$ and $g(t, s) = g(t, 0)$ when $s < 0$. Then, we obtain $\lim_{n\to\infty} f(t + n\omega_1, s) = \lim_{n\to\infty} f(t + n\omega_1, 0) = g(t, 0) = g(t, s)$ when $s < 0$. Moreover, $\lim_{n\to\infty} f(t + n\omega_1, s) = g(t, s)$ uniformly for $s \in \mathbb{R}$. Note that $\int_0^y T(y - s)f(s + x - y, s)ds = \int_{x-y}^x T(x - s)f(s, s + y - x)ds$ when $x \ge y$. Then, one has:

$$u(x, y) = \begin{cases} \int_0^x T(x - s)f(s, s + y - x)ds, \ y \ge x, \ x, y \in \mathbb{R}^+, \\ \int_0^x T(x - s)f(s, s + y - x)ds - \int_0^{x-y} T(x - s)f(s, s + y - x)ds, \ x \ge y, \ x, y \in \mathbb{R}^+. \end{cases}$$

Denote $u_1(x, y) = \int_0^x T(x - s)f(s, s + y - x)ds$ and:

$$u_2(x, y) = \begin{cases} 0, \ y \ge x, \\ \int_0^{x-y} T(x - s)f(s, s + y - x)ds, \ x \ge y, \end{cases} = \begin{cases} 0, \ y \ge x, \\ \int_0^{x-y} T(x - s)f(s, 0)ds, \ x \ge y. \end{cases}$$

To show $\lim_{n\to\infty} u(x + n\omega_1, y) = v(x, y)$ uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$, we only need to show $\lim_{n\to\infty} u_1(x + n\omega_1, y) = v_1(x, y)$ uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$ and $\lim_{n\to\infty} u_2(x + n\omega_1, y) = v_2(x, y)$ uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$.

Step 1. Let $x \in \mathbb{R}^+$, $y \in \mathbb{R}^+$.

$$\begin{aligned} u_1(x + n\omega_1, y) &= \int_0^{x+n\omega_1} T(x + n\omega_1 - s)f(s, s + y - x - n\omega_1)ds \\ &= \int_{-n\omega_1}^x T(x - s)f(s + n\omega_1, s + y - x)ds \\ &= \int_{-n\omega_1}^0 T(x - s)f(s + n\omega_1, s + y - x)ds + \int_0^x T(x - s)f(s + n\omega_1, s + y - x)ds \\ &= \int_0^{n\omega_1} T(x + s)f(n\omega_1 - s, y - x - s)ds + \int_0^x T(x - s)f(s + n\omega_1, s + y - x)ds \\ &= I_1(x, y, n) + I_2(x, y, n). \end{aligned}$$

We discuss the terms $I_i(x, y, n)$ $(i = 1, 2)$ separately. First, we show that $I_1(x, y, n)$ is a Cauchy sequence in X for each $x \in \mathbb{R}^+$ and each $y \in \mathbb{R}^+$.

For any $p \in \mathbb{N}$, $n \in \mathbb{N}$, one has:

$$\begin{aligned}
& I_1(x,y,n+p) - I_1(x,y,n) \\
=& \int_0^{(n+p)\omega_1} T(x+s) f((n+p)\omega_1 - s, y-x-s) ds \\
& - \int_0^{n\omega_1} T(x+s) f(n\omega_1 - s, y-x-s) ds \\
=& \int_{n\omega_1}^{(n+p)\omega_1} T(x+s) f((n+p)\omega_1 - s, y-x-s) ds \\
& + \int_0^{n\omega_1} T(x+s)[f((n+p)\omega_1 - s, y-x-s) - f(n\omega_1 - s, y-x-s)] ds \\
=& I_3(x,y,n,p) + I_4(x,y,n,p).
\end{aligned}$$

We see that:

$$\begin{aligned}
\|I_3(x,y,n,p)\| \leq & \int_{n\omega_1}^{(n+p)\omega_1} \|T(x+s)\| \|f((n+p)\omega_1 - s, y-x-s)\| ds \\
\leq & KM \int_{n\omega_1}^{(n+p)\omega_1} e^{-r(x+s)} ds \\
\leq & KM \int_{n\omega_1}^{\infty} e^{-r(x+s)} ds \\
\leq & \frac{KM}{r} e^{-rn\omega_1}.
\end{aligned}$$

Let $\varepsilon > 0$. We can choose $N_1 \in \mathbb{N}$ such that $\frac{KM}{r} e^{-rn\omega_1} < \varepsilon$ when $n \geq N_1$. Then, one gets $\|I_3(x,y,n,p)\| < \varepsilon$ when $n \geq N_1$ uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$.

For $n \geq N_1$, we consider:

$$\begin{aligned}
I_4(x,y,n,p) =& \int_0^{N_1\omega_1} T(x+s)[f((n+p)\omega_1 - s, y-x-s) - f(n\omega_1 - s, y-x-s)] ds \\
& + \int_{N_1\omega_1}^{n\omega_1} T(x+s)[f((n+p)\omega_1 - s, y-x-s) - f(n\omega_1 - s, y-x-s)] ds \\
=& I_5(x,y,n,p) + I_6(x,y,n,p).
\end{aligned}$$

Now, we estimate the term $I_5(x,y,n,p)$.

$$\begin{aligned}
& \|I_5(x,y,n,p)\| \\
\leq & \int_0^{N_1\omega_1} \|T(x+s)\| \|f((n+p)\omega_1 - s, y-x-s) - f(n\omega_1 - s, y-x-s)\| ds \\
= & \int_0^{N_1\omega_1} \|T(x+N_1\omega_1 - s)\| \|f((n-N_1+p)\omega_1 + s, s - N_1\omega_1 + y - x) \\
& - f((n-N_1)\omega_1 + s, s - N_1\omega_1 + y - x)\| ds \\
\leq & M \int_0^{N_1\omega_1} e^{-r(N_1\omega_1 - s)} \|f((n-N_1+p)\omega_1 + s, s - N_1\omega_1 + y - x) - g(s, s - N_1\omega_1 + y - x)\| ds \\
+ & M \int_0^{N_1\omega_1} e^{-r(N_1\omega_1 - s)} \|f((n-N_1)\omega_1 + s, s - N_1\omega_1 + y - x) - g(s, s - N_1\omega_1 + y - x)\| ds.
\end{aligned}$$

For each $s \in [0, N_1\omega_1]$, we have:

$$e^{-r(N_1\omega_1 - s)} \|f((n-N_1)\omega_1 + s, s - N_1\omega_1 + y - x) - g(s, s - N_1\omega_1 + y - x)\| \leq 2K e^{-r(N_1\omega_1 - s)}$$

and:

$$\int_0^{N_1\omega_1} 2Ke^{-r(N_1\omega_1-s)}ds = \frac{2K}{r}(1-e^{-rN_1\omega_1}).$$

Since $\lim_{n\to\infty} f(t+n\omega_1,s) = g(t,s)$ uniformly for $s\in\mathbb{R}$, for each $s\in[0,N_1\omega_1]$, one has:

$$e^{-r(N_1\omega_1-s)}\|f((n-N_1)\omega_1+s,s-N_1\omega_1+y-x)-g(s,s-N_1\omega_1+y-x)\|\to 0$$

as $n\to\infty$. By Lebesgue's dominated convergence theorem, we obtain:

$$\lim_{n\to\infty}\int_0^{N_1\omega_1} e^{-r(N_1\omega_1-s)}\|f((n-N_1)\omega_1+s,s-N_1\omega_1+y-x)-g(s,s-N_1\omega_1+y-x)\|ds = 0. \quad (8)$$

Since $\lim_{n\to\infty} f(t+n\omega_1,s) = g(t,s)$ uniformly for $s\in\mathbb{R}$, (8) holds uniformly for $x\in\mathbb{R}^+$ and $y\in\mathbb{R}^+$. Moreover,

$$\lim_{n\to\infty}\int_0^{N_1\omega_1} e^{-r(N_1\omega_1-s)}\|f((n-N_1+p)\omega_1+s,s-N_1\omega_1+y-x)-g(s,s-N_1\omega_1+y-x)\|ds = 0$$

uniformly for $x\in\mathbb{R}^+$ and $y\in\mathbb{R}^+$. Thus, we can select $N_2\in\mathbb{N}$ $(N_2>N_1)$ such that $\|I_5(x,y,n,p)\|<\varepsilon$ when $n\geq N_2$ uniformly for $x\in\mathbb{R}^+$ and $y\in\mathbb{R}^+$.

Next, we estimate the term $I_6(x,y,n,p)$.

$$\begin{aligned}\|I_6(x,y,n,p)\| \leq& \int_{N_1\omega_1}^{n\omega_1}\|T(x+s)\|\|f((n+p)\omega_1-s,y-x-s)-f(n\omega_1-s,y-x-s)\|ds\\ \leq& 2KM\int_{N_1\omega_1}^{n\omega_1} e^{-r(x+s)}ds\\ \leq& 2KM\int_{N_1\omega_1}^{\infty} e^{-r(x+s)}ds\\ \leq& \frac{2KM}{r}e^{-rN_1\omega_1}\\ <& 2\varepsilon\end{aligned}$$

uniformly for $x\in\mathbb{R}^+$ and $y\in\mathbb{R}^+$.

Thus, $\|I_1(x,y,n+p)-I_1(x,y,n)\|\leq\|I_3(x,y,n,p)\|+\|I_5(x,y,n,p)\|+\|I_6(x,y,n,p)\|<4\varepsilon$ when $n\geq N_2$. This shows that $I_1(x,y,n)$ is a Cauchy sequence, and we denote $l_1(x,y)=\lim_{n\to\infty} I_1(x,y,n)$. Besides, from the proof, we also know that $\lim_{n\to\infty} I_1(x,y,n) = l_1(x,y)$ uniformly for $x\in\mathbb{R}^+$ and $y\in\mathbb{R}^+$.

Finally, we consider the term $I_2(x,y,n)$. Note that $\int_0^x T(x-s)g(s,s+y-x)ds$ is well defined for each $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$. For $m\omega_1 \le x < (m+1)\omega_1$, then one has:

$$
\begin{aligned}
&\left\| I_2(x,y,n) - \int_0^x T(x-s)g(s,s+y-x)ds \right\| \\
\le &\int_0^x \|T(x-s)\| \|f(s+n\omega_1, s+y-x) - g(s,s+y-x)\| ds \\
\le &M \int_0^x e^{-r(x-s)} \|f(s+n\omega_1, s+y-x) - g(s,s+y-x)\| ds \\
\le &M \int_0^{m\omega_1} e^{-r(x-s)} \|f(s+n\omega_1, s+y-x) - g(s,s+y-x)\| ds \\
&+ M \int_{m\omega_1}^{x} \|f(s+n\omega_1, s+y-x) - g(s,s+y-x)\| ds \\
\le &M \sum_{k=0}^{m-1} \int_{k\omega_1}^{(k+1)\omega_1} e^{-r(x-s)} \|f(s+n\omega_1, s+y-x) - g(s,s+y-x)\| ds \\
&+ M \int_{m\omega_1}^{(m+1)\omega_1} \|f(s+n\omega_1, s+y-x) - g(s,s+y-x)\| ds
\end{aligned}
$$

For each $t \in [0,\omega_1]$, we have $\|f(t+n\omega_1,s) - g(t,s)\| \to 0$ as $n \to \infty$ uniformly for $s \in \mathbb{R}$ and $\|f(t+n\omega_1,s) - g(t,s)\| \le 2K$. By Lebesgue's dominated convergence theorem, we obtain:

$$
\lim_{n\to\infty} \int_0^{\omega_1} \|f(t+n\omega_1,s) - g(t,s)\| dt = 0
$$

uniformly for $s \in \mathbb{R}$. For $\varepsilon > 0$ given, we select $N_3 \in \mathbb{N}$ such that:

$$
\int_0^{\omega_1} \|f(t+n\omega_1,s) - g(t,s)\| dt < \varepsilon
$$

when $n \ge N_3$ uniformly for $s \in \mathbb{R}$. For any $i \in \mathbb{N}$, one has:

$$
\begin{aligned}
&\int_{i\omega_1}^{(i+1)\omega_1} \|f(t+n\omega_1,s) - g(t,s)\| dt \\
= &\int_0^{\omega_1} \|f(t+i\omega_1+n\omega_1,s) - g(t+i\omega_1,s)\| dt \\
= &\int_0^{\omega_1} \|f(t+i\omega_1+n\omega_1,s) - g(t,s)\| dt < \varepsilon
\end{aligned}
$$

when $n \ge N_3$ uniformly for $s \in \mathbb{R}$.

Therefore,

$$
\begin{aligned}
&\left\| I_2(x,y,n) - \int_0^x T(x-s)g(s,s+y-x)ds \right\| \\
\le &M \sum_{k=0}^{m-1} e^{-r(x-(k+1)\omega_1)} \varepsilon + M\varepsilon \\
\le &\Big(\frac{1}{1-e^{-r\omega_1}} + 1 \Big) M\varepsilon
\end{aligned}
$$

Hence, $\lim_{n\to\infty} I_2(x,y,n) = \int_0^x T(x-s)g(s,s+y-x)ds$ uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$.

Therefore,

$$
\lim_{n\to\infty} u_1(x+n\omega,y) = \lim_{n\to\infty} I_1(x,y,n) + \lim_{n\to\infty} I_2(x,y,n) = l_1(x,y) + \int_0^x T(x-s)g(s,s+y-x)ds
$$

uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$.

Step 2. For any $x \in \mathbb{R}^+$. Firstly, let $y \in [0, x]$.

$$\begin{aligned}
u_2(x + n\omega_1, y) =& \int_0^{x+n\omega_1-y} T(x + n\omega_1 - s) f(s, 0) ds \\
=& \int_{-n\omega_1}^{x-y} T(x - s) f(s + n\omega_1, 0) ds \\
=& \int_{-n\omega_1}^{0} T(x - s) f(s + n\omega_1, 0) ds + \int_0^{x-y} T(x - s) f(s + n\omega_1, 0) ds \\
=& \int_0^{n\omega_1} T(x + s) f(n\omega_1 - s, 0) ds + \int_0^{x-y} T(x - s) f(s + n\omega_1, 0) ds. \\
=& I_7(x, n) + I_8(x, y, n).
\end{aligned}$$

In a similar way as the case $I_1(x, y, n)$, we can show that $I_7(x, n)$ is a Cauchy sequence, and we denote $l_2(x) = \lim_{n\to\infty} I_7(x, n)$. Besides, we know $l_2(x) = \lim_{n\to\infty} I_7(x, n)$ uniformly for $x \in \mathbb{R}^+$.

Note that:

$$\begin{aligned}
&\left\| I_8(x, y, n) - \int_0^{x-y} T(x - s) g(s, 0) ds \right\| \\
\leq& \int_0^{x-y} \|T(x - s)\| \|f(s + n\omega_1, 0) - g(s, 0)\| ds \\
\leq& \int_0^{x} \|T(x - s)\| \|f(s + n\omega_1, 0) - g(s, 0)\| ds \\
\leq& M \int_0^{x} e^{-r(x-s)} \|f(s + n\omega_1, 0) - g(s, 0)\| ds.
\end{aligned}$$

Then, in a similar way as the case $I_2(x, y, n)$, we can show $\lim_{n\to\infty} I_8(x, y, n) = \int_0^{x-y} T(x - s) g(s, 0) ds$ uniformly for $x \in \mathbb{R}^+$ and $y \in [0, x]$.

Therefore,

$$\lim_{n\to\infty} u_2(x + n\omega, y) = \lim_{n\to\infty} I_7(x, n) + \lim_{n\to\infty} I_8(x, y, n) = l_2(x) + \int_0^{x-y} T(x - s) g(s, 0) ds$$

uniformly for $x \in \mathbb{R}^+$ and $y \in [0, x]$.

Secondly, let $y > x$. Let $\varepsilon > 0$, and choose $N_1 \in \mathbb{N}$ such that $\frac{KM}{r} e^{-rn\omega_1} < \varepsilon$ when $n \geq N_1$. We now prove that $u_2(x + n\omega_1, y)$ is a Cauchy sequence in X for each $y \in (x, +\infty)$.

Case 1: $y - x \in (0, N_1\omega_1]$. Let $n > N_1$, $p \in \mathbb{N}$.

$$\begin{aligned}
&u_2(x + (n+p)\omega_1, y) - u_2(x + n\omega_1, y)\\
=&\int_0^{x+(n+p)\omega_1 - y} T(x + (n+p)\omega_1 - s) f(s,0)ds - \int_0^{x+n\omega_1-y} T(x + n\omega_1 - s) f(s,0)ds\\
=&\int_{-(n+p)\omega_1}^{x-y} T(x-s) f(s + (n+p)\omega_1, 0)ds - \int_{-n\omega_1}^{x-y} T(x-s) f(s + n\omega_1, 0)ds\\
=&\int_{y-x}^{(n+p)\omega_1} T(x+s) f((n+p)\omega_1 - s, 0)ds - \int_{y-x}^{n\omega_1} T(x+s) f(n\omega_1 - s, 0)ds\\
=&\int_{y-x}^{n\omega_1} T(x+s)[f((n+p)\omega_1 - s, 0) - f(n\omega_1 - s, 0)]ds + \int_{n\omega_1}^{(n+p)\omega_1} T(x+s) f((n+p)\omega_1 - s, 0)ds\\
=&\int_{y-x}^{N_1\omega_1} T(x+s)[f((n+p)\omega_1 - s, 0) - f(n\omega_1 - s, 0)]ds\\
&+ \int_{N_1\omega_1}^{n\omega_1} T(x+s)[f((n+p)\omega_1 - s, 0) - f(n\omega_1 - s, 0)]ds\\
&+ \int_{n\omega_1}^{(n+p)\omega_1} T(x+s) f((n+p)\omega_1 - s, 0)ds\\
=&I_9(x,y,n,p) + I_{10}(x,y,n,p) + I_{11}(x,y,n,p).
\end{aligned}$$

Now, we estimate the term $I_9(x,y,n,p)$.

$$\begin{aligned}
\|I_9(x,y,n,p)\| \leq& \int_0^{N_1\omega_1} T(x+s)[f((n+p)\omega_1 - s, 0) - f(n\omega_1 - s, 0)]ds\\
=& \int_0^{N_1\omega_1} T(x + N_1\omega_1 - s)[f((n - N_1 + p)\omega_1 + s, 0) - f((n - N_1)\omega_1 + s, 0)]ds\\
\leq& M \int_0^{N_1\omega_1} e^{-r(N_1\omega_1 - s)} \|f((n - N_1 + p)\omega_1 + s, 0) - g(s,0)\|ds\\
&+ M \int_0^{N_1\omega_1} e^{-r(N_1\omega_1 - s)} \|f((n - N_1)\omega_1 + s, 0) - g(s,0)\|ds.
\end{aligned}$$

By Lebesgue's dominated convergence theorem, there exists $N_4 \in \mathbb{N}$ $(N_4 > N_1)$ such that $\|I_9(x,y,n,p)\| < \varepsilon$ when $n \geq N_4$ uniformly for $y - x \in (0, N_1\omega_1]$ and $x \in \mathbb{R}^+$. Since $\frac{KM}{r} e^{-rn\omega_1} < \varepsilon$ $(n \geq N_1)$, then we have:

$$\begin{aligned}
\|I_{10}(x,y,n,p)\| \leq& 2KM \int_{N_1\omega_1}^{n\omega_1} e^{-r(x+s)} ds\\
\leq& 2KM \int_{N_1\omega_1}^{\infty} e^{-r(x+s)} ds\\
\leq& \frac{2KM}{r} e^{-rN_1\omega_1}\\
<& 2\varepsilon
\end{aligned}$$

and:

$$\begin{aligned}
\|I_{11}(x,y,n,p)\| \leq& KM \int_{n\omega_1}^{(n+p)\omega_1} e^{-r(x+s)} ds\\
\leq& KM \int_{n\omega_1}^{\infty} e^{-r(x+s)} ds\\
\leq& \frac{KM}{r} e^{-rn\omega_1}\\
<& \varepsilon
\end{aligned}$$

uniformly for $y - x \in (0, N_1\omega_1]$ and $x \in \mathbb{R}^+$.

Therefore, $\|u_2(x + (n + p)\omega_1, y) - u_2(x + n\omega_1, y)\| < 4\varepsilon$ when $n \geq N_4$ uniformly for $y - x \in (0, N_1\omega_1]$ and $x \in \mathbb{R}^+$.

Case 2: $y - x \in (N_1\omega_1, N_4\omega_1]$. Note that $\frac{KM}{r}e^{-rn\omega_1} < \varepsilon$ $(n \geq N_1)$.

$$\begin{aligned}
&\|u_2(x + (n + p)\omega_1, y) - u_2(x + n\omega_1, y)\| \\
\leq&\Big\| \int_0^{x+(n+p)\omega_1-y} T(x + (n + p)\omega_1 - s)f(s, 0)ds\Big\| + \Big\| \int_0^{x+n\omega_1-y} T(x + n\omega_1 - s)f(s, 0)ds\Big\| \\
=&\Big\| \int_{-(n+p)\omega_1}^{x-y} T(x - s)f(s + (n + p)\omega_1, 0)ds\Big\| + \Big\| \int_{-n\omega_1}^{x-y} T(x - s)f(s + n\omega_1, 0)ds\Big\| \\
=&\Big\| \int_{y-x}^{(n+p)\omega_1} T(x + s)f((n + p)\omega_1 - s, 0)ds\Big\| + \Big\| \int_{y-x}^{n\omega_1} T(x + s)f(n\omega_1 - s, 0)ds\Big\| \\
\leq&KM \int_{y-x}^{(n+p)\omega_1} e^{-r(x+s)}ds + KM \int_{y-x}^{n\omega_1} e^{-r(x+s)}ds \\
\leq&KM \int_{y-x}^{\infty} e^{-r(x+s)}ds + KM \int_{y-x}^{\infty} e^{-r(x+s)}ds \\
\leq&\frac{KM}{r}e^{-r(y-x)} + \frac{KM}{r}e^{-r(y-x)} \\
<&2\varepsilon
\end{aligned}$$

when $n \geq N_4$ uniformly for $y - x \in (N_1\omega_1, N_4\omega_1]$ and $x \in \mathbb{R}^+$.

Case 3: $y - x \in (N_4\omega_1, +\infty)$. Consider $n \geq N_4$. Note that:

$$u_2(x + n\omega_1, y) = \begin{cases} \int_{y-x}^{n\omega_1} T(x + s)f(n\omega_1 - s, 0)ds, & y \leq x + n\omega_1, \\ 0, & y \geq x + n\omega_1 \end{cases}$$

and:

$$u_2(x + (n + p)\omega_1, y) = \begin{cases} \int_{y-x}^{(n+p)\omega_1} T(x + s)f((n + p)\omega_1 - s, 0)ds, & y \leq x + (n + p)\omega_1, \\ 0, & y \geq x + (n + p)\omega_1. \end{cases}$$

Then, one has:

$$\begin{aligned}
&\|u_2(x + n\omega_1, y)\| \\
=&\Big\| \int_{y-x}^{n\omega_1} T(x + s)f(n\omega_1 - s, 0)ds\Big\| \\
\leq&KM \int_{y-x}^{n\omega_1} e^{-r(x+s)}ds \\
\leq&KM \int_{y-x}^{\infty} e^{-r(x+s)}ds \\
\leq&\frac{KM}{r}e^{-r(y-x)} \\
<&\varepsilon
\end{aligned}$$

when $y - x \in (N_4\omega_1, n\omega_1]$.

Besides, $\|u_2(x + n\omega_1, y)\| = 0$ when $y - x \in (n\omega_1, +\infty)$. Therefore, $\|u_2(x + n\omega_1, y)\| < \varepsilon$ when $y - x \in (N_4\omega_1, +\infty)$. Similarly, $\|u_2(x + (n + p)\omega_1, y)\| < \varepsilon$ when $y - x \in (N_4\omega_1, +\infty)$. Thus,

$\|u_2(x+(n+p)\omega_1,y)-u_2(x+n\omega_1,y)\| < 2\varepsilon$ when $n \geq N_4$ uniformly for $y-x \in (N_4\omega_1,+\infty)$ and $x \in \mathbb{R}^+$.

Therefore, $\|u_2(x+(n+p)\omega_1,y)-u_2(x+n\omega_1,y)\| < 4\varepsilon$ when $n \geq N_4$ uniformly for $y-x \in (0,+\infty)$ and $x \in \mathbb{R}^+$. This shows that $u_2(x+n\omega_1,y)$ is a Cauchy sequence, and we denote $l_3(x,y) = \lim_{n\to\infty} u_2(x+n\omega_1,y)$. Besides, from the proof, we also know that $\lim_{n\to\infty} u_2(x+n\omega_1,y) = l_3(x,y)$ uniformly for $y \in (x,+\infty)$ and $x \in \mathbb{R}^+$.

Therefore,

$$\lim_{n\to\infty} u_2(x+n\omega_1,y) = \begin{cases} l_2(x) + \int_0^{x-y} T(x-s)g(s,0)ds, & y \in [0,x], \\ l_3(x,y), & y \in (x,+\infty) \end{cases}$$

uniformly for $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$.

This completes the proof. □

Theorem 2. *Let F be a (ω_1,ω_2)-quasi-periodic limit function. Then, the mild solution of Problem (1) is asymptotically quasi-periodic.*

Remark 2. *Theorem 4.4 in [21] implies that the mild solution of Problem (1) is asymptotically quasi-periodic when $F = F_1 + F_2$, where $F_1 \in P_{\omega_1}L(\mathbb{R}^+, X)$, $F_2 \in P_{\omega_2}L(\mathbb{R}^+, X)$. However, the method in [21] is not available when $F = F_1F_2$, where $F_1 \in P_{\omega_1}L(\mathbb{R}^+, \mathbb{C})$, $F_2 \in P_{\omega_2}L(\mathbb{R}^+, X)$. Therefore, we propose the concept of the quasi-periodic limit function and develop the method to get a more general result.*

Next, consider the following abstract Cauchy problem:

$$\begin{cases} x'(t) = Ax(t) + F(t,x(t)), \ t \in \mathbb{R}^+; \\ x(0) = x_0 \in X, \end{cases} \tag{9}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$.

Let us introduce the following definition.

Definition 4. *A joint continuous function $f : \mathbb{R}^+ \times \mathbb{R}^+ \times X \to X$ is said to be the (ω_1,ω_2)-periodic limit uniformly for x in bounded subsets of X if for every bounded subset K of X, $\{f(t,s,x) : t,s \in \mathbb{R}^+, x \in K\}$ is bounded, for each $t \in \mathbb{R}^+$ $\lim_{n\to\infty} f(t+n\omega_1,s,x) = g(t,s,x)$ uniformly for $s \in \mathbb{R}^+$ and $x \in K$, for each $s \in \mathbb{R}^+$ $\lim_{n\to\infty} f(t,s+n\omega_2,x) = h(t,s,x)$ uniformly for $t \in \mathbb{R}^+$ and $x \in K$. Denote by $PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+ \times X, X)$ the set of all such functions. If we define $F(t,x) = f(t,t,x)$, then $F(t,x)$ is said to be the (ω_1,ω_2)-quasi-periodic limit uniformly for x in bounded subsets of X.*

The following is a composition theorem.

Theorem 3. *Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \times X \to X$ be the (ω_1,ω_2)-periodic limit uniformly for x in bounded subsets of X, and assume that f satisfies a Lipschitz condition in x uniformly in $t,s \in \mathbb{R}^+$:*

$$\|f(t,s,x) - f(t,s,y)\| \leq L\|x-y\|$$

for all $x,y \in X$ and $t,s \in \mathbb{R}^+$, where $L > 0$. Let $\varphi \in PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$. The function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to X$ is defined by $F(t,s) = f(t,s,\varphi(t,s))$. Then, $F \in PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$.

Proof. Since $\varphi \in PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$, we have for each $t \in \mathbb{R}^+$:

$$\lim_{n\to\infty} \varphi(t+n\omega_1,s) = \varphi_g(t,s) \tag{10}$$

uniformly for $s \in \mathbb{R}^+$.

Select a bounded subset K of X such that $\varphi(t,s), \varphi_g(t,s) \in K$ for $t,s \in \mathbb{R}^+$. Thus, $F(t,s)$ is bounded.

On the other side, one has for each $t \in \mathbb{R}^+$:

$$\lim_{n\to\infty} f(t+n\omega_1, s, x) = g(t,s,x) \tag{11}$$

uniformly for $s \in \mathbb{R}^+$ and $x \in K$.

Let us consider the function $G : \mathbb{R}^+ \times \mathbb{R}^+ \to X$ defined by $G(t,s) = g(t,s,\varphi_g(t,s))$. Note that for each $t \in \mathbb{R}^+$:

$$\begin{aligned}\|F(t+n\omega_1,s) - G(t,s)\| \leq& \|f(t+n\omega_1, s, \varphi(t+n\omega_1,s)) - f(t+n\omega_1, s, \varphi_g(t,s))\| \\ &+ \|f(t+n\omega_1, s, \varphi_g(t,s)) - g(t,s,\varphi_g(t,s))\| \\ \leq& L\|\varphi(t+n\omega_1,s) - \varphi_g(t,s)\| + \|f(t+n\omega_1, s, \varphi_g(t,s)) - g(t,s,\varphi_g(t,s))\|\end{aligned}$$

uniformly for $s \in \mathbb{R}^+$.

We deduce from (10) and (11) that for each $t \in \mathbb{R}^+$:

$$\lim_{n\to\infty} F(t+n\omega_1, s) = G(t,s)$$

uniformly for $s \in \mathbb{R}^+$.

In a similar way, we can show there exists a function H such that for each $s \in \mathbb{R}^+$:

$$\lim_{n\to\infty} F(t, s+n\omega_2) = H(t,s)$$

uniformly for $t \in \mathbb{R}^+$. □

Definition 5. *A function $x \in C_b(\mathbb{R}^+, X)$ is said to be a mild solution of Problem (9) if:*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F(s,x(s))ds,\ t \in \mathbb{R}^+,$$

where $(T(t))_{t\geq 0}$ is an exponentially stable C_0-semigroup.

Now, we can establish the following theorem.

Theorem 4. *Let $F : \mathbb{R}^+ \times X \to X$ be the (ω_1, ω_2)-quasi-periodic limit uniformly for x in bounded subsets of X, and assume $F(t,x) = f(t,t,x)$, where $f \in PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+ \times X, X)$. Assume that f satisfies a Lipschitz condition in x uniformly in $t,s \in \mathbb{R}^+$:*

$$\|f(t,s,x) - f(t,s,y)\| \leq L\|x-y\|$$

for all $x,y \in X$ and $t,s \in \mathbb{R}^+$, where $L > 0$. If $ML < r$, then there exists an asymptotically quasi-periodic mild solution of Problem (9).

Proof. Define the function $T : \mathbb{R}^+ \times \mathbb{R}^+ \to X$ by:

$$T(x,y) = \begin{cases} T(y)x_0,\ y \geq x, \\ T(x)x_0,\ x \geq y. \end{cases}$$

Then, we can define the operator Γ on the space $AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$ by:

$$\Gamma\varphi(x,y) = T(x,y) + \begin{cases} \int_0^x T(x-s)f(s, s+y-x, \varphi(s, s+y-x))ds, \ y \geq x, \\ \int_0^y T(y-s)f(s+x-y, s, \varphi(s+x-y, s))ds, \ x \geq y, \end{cases}$$

where $\varphi \in AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X) \subset PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$.

Clearly, $T(x,y) \in AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$. By Lemma 1 and Theorem 3, one has $\Gamma\varphi \in AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$. Note that the space $AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$ is a Banach space by Theorem 1. Finally, for $\varphi_1, \varphi_2 \in AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$, one has:

$$\|\Gamma\varphi_1 - \Gamma\varphi_2\|_\infty \leq \frac{ML}{r}\|\varphi_1 - \varphi_2\|_\infty,$$

which shows that Γ is a contraction. By the contraction mapping principle, there exists a unique $\overline{\varphi}(x,y) \in AQP_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, X)$ such that $\Gamma\overline{\varphi}(x,y) = \overline{\varphi}(x,y)$, that is:

$$\overline{\varphi}(x,y) = T(x,y) + \begin{cases} \int_0^x T(x-s)f(s, s+y-x, \overline{\varphi}(s, s+y-x))ds, \ y \geq x, \\ \int_0^y T(y-s)f(s+x-y, s, \overline{\varphi}(s+x-y, s))ds, \ x \geq y. \end{cases}$$

If we denote $\overline{\Phi}(t) = \overline{\varphi}(t,t)$, then we have $\Gamma\overline{\Phi}(t) = \overline{\Phi}(t)$, that is:

$$\overline{\Phi}(t) = T(t)x_0 + \int_0^t T(t-s)F(s, \overline{\Phi}(s))ds.$$

Therefore, $\overline{\Phi}$ is a solution of Problem (9). Moreover, $\overline{\Phi}$ is asymptotically quasi-periodic by Proposition 8. □

Remark 3. *The operator Γ in Theorem 4 may be constructed in a different way, so we cannot show the uniqueness of the solution of Problem (9).*

Example 4. *Consider the following problem:*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + A(t)f(u(t,x)), \ t \in \mathbb{R}^+, x \in [0,\pi]; \\ u(t,0) = u(t,\pi) = 0, \ t \in \mathbb{R}^+; \\ u(0,x) = g(x), \ x \in [0,\pi]. \end{cases} \tag{12}$$

where the functions $g : [0,\pi] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are appropriate bounded continuous functions and A defined as $A(t) = a(t,t)$, where $a \in PL_{(\omega_1,\omega_2)}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$. Besides, f satisfies:

$$|f(x) - f(y)| \leq L|x-y|, \ x, y \in \mathbb{R},$$

where $L > 0$. Let $X = L^2([0,\pi])$, and let A be the operator given by $Au = u''$ with domain $D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}$. Clearly, A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, A has a discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = (\frac{2}{\pi})^{\frac{1}{2}}\sin(n\xi)$. Furthermore, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$ for $x \in X$. Therefore, one has $\|T(t)\| \leq e^{-t}, t \in \mathbb{R}^+$. Therefore, if $\|a\|_\infty L < 1$, (12) has an asymptotically quasi-periodic mild solution by Theorem 4.

4. Some Remarks and Questions

Proposition 6 and Proposition 8 make sure that these quasi-periodic limit functions can be regarded as a generalization of asymptotically quasi-periodic functions. Here, a question arises: Could we generalize quasi-periodic limit functions again?

It seems more general if we define $F(t) = f(t,t)$ without H1 and H2 in Definition 2. However, we have the following example.

Example 5. *Define:*

$$f(x,y) = \begin{cases} f_1(x,y), & y \geq k_1 x, \\ f_2(x,y), & k_2 x < y < k_1 x, \\ f_3(x,y), & y \leq k_2 x, \end{cases}$$

where $x, y \in \mathbb{R}^+$, $k_1 > 1, 0 < k_2 < 1$. f_1 *is a continuous function such that* $\lim_{n\to\infty} f_1(x, y + n\omega_2) = g_1(x,y)$. f_3 *is a continuous function such that* $\lim_{n\to\infty} f_3(x + n\omega_1, y) = h_3(x,y)$. $f_1(0,0) = f_3(0,0)$. f_2 *is an arbitrary continuous function such that* f *is continuous on* $\mathbb{R}^+ \times \mathbb{R}^+$. *Then,* f *is a* (ω_1, ω_2)*-periodic limit function. However,* $F(t) = f(t,t) = f_2(t,t)$ $(t \geq t_0, t_0 > 0)$ *can be an arbitrary continuous function.*

It is interesting to ask whether there is another way to generalize asymptotically quasi-periodic functions without using a function with several variables.

Let $x \in C_b(\mathbb{R}^+, X)$. If for every $\varepsilon > 0$, there exist $f_i \in P_{\omega_1}L(\mathbb{R}^+, \mathbb{C})$, $g_i \in P_{\omega_2}L(\mathbb{R}^+, \mathbb{C})$, $F_i \in P_{\omega_2}L(\mathbb{R}^+, X)$ and $G_i \in P_{\omega_1}L(\mathbb{R}^+, X)(i = 1, 2, \cdots, N)$ such that:

$$\left\| x - \sum_{i=1}^{N} (f_i F_i + g_i G_i) \right\| < \varepsilon,$$

then denote the set of x by $\overline{QPL}_{(\omega_1,\omega_2)}(\mathbb{R}^+, X)$.

Note that $\sum_{i=1}^{N} (f_i F_i + g_i G_i) \in QPL_{(\omega_1,\omega_2)}(\mathbb{R}^+, X)$ for each $N \in \mathbb{N}$. Therefore, there is a natural question as follows.

Question 4.2: Does $\overline{QPL}_{(\omega_1,\omega_2)}(\mathbb{R}^+, X)$ equal $QPL_{(\omega_1,\omega_2)}(\mathbb{R}^+, X)$?

Author Contributions: Investigation, R.X., Z.X. and J.L. All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Funding: The work of the first author is supported by the research project of Tianjin Municipal Education Commission (Grant No. 160022), the Doctoral Scientific Research Foundation of Tianjin University of Commerce (Grant No. R160101), and the National Nurture Fund of Tianjin University of Commerce (Grant No. 170113). The research of the second author is supported by the National Natural Science Foundation of China (Grant No. 11501507) and the Natural Science Foundation of Zhejiang Province (Grant No. LY19A010013). The work of the third author is supported by the Natural Science Foundation of Hebei Province (Grant No. A2016203341).

Conflicts of Interest: The authors declare no conflict of interest.

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