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Article

Extended Rectangular $M_{r\xi}$-Metric Spaces and Fixed Point Results

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Abstract: In this paper, we enlarge the classes of rectangular M_b-metric spaces and extended rectangular b-metric spaces by considering the class of extended rectangular $M_{r\xi}$-metric spaces and utilize the same to prove an analogue of Banach contraction principle in such spaces. We adopt an example to highlight the utility of our main result. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

Keywords: extended rectangular $M_{r\xi}$-metric space; fixed point

MSC: 47H10; 54H25

1. Introduction

Fixed point theory is a very wide domain of mathematical research. It has extensive applications in various fields within and beyond mathematics which also includes various type of real word problems. Indeed, the fundamental result of metric fixed point theory is the classical Banach contraction principle which continues to inspire researchers to prove new results enriching the principle in several ways. One possible way is to improve this principle by enlarging the class of spaces. In 1993, S. Czerwik [1] extensively used the concept of b-metric space by replacing triangular inequality with a relatively more general condition which is also utilized to extend Banach contraction theorem. By now there already exists considerable literature in b-metric spaces and for the work of this kind one can consult to [2–8] and similar others. In 2017, Kamran et al. [9] introduced a new type of generalized b-metric space and termed it as extended b-metric space.

In 2000, Branciari [10] generalized the class of metric spaces by replacing the triangular inequality with a relatively more general inequality namely: quadrilateral inequality which involves four points instead of three points and utilized this to prove an analogue of Banach contraction theorem. In 2008, George et al. [11] further enlarged the class of rectangular metric spaces by introducing the the class of rectangular b-metric spaces and proved an analogue of Banach contraction principle in such spaces. Recently, Asim et al. [12] generalized the class of rectangular b-metric spaces by introducing the class extended rectangular b-metric spaces.

In 2014, Asadi et al. [13] enlarged the class of the partial metric spaces (see [14]) by introducing M-metric spaces. In 2016, Mlaiki et al. [15] introduced the notion of an M_b-metric spaces and utilized the same to prove fixed point results. Later on, in an attempt to extend the classes of "rectangular metric spaces" and "M-metric spaces", Özgür [16] introduced the class of rectangular M-metric spaces. On the other hand, in 2018, Mlaiki et al. [17] generalized the class of M_b-metric spaces by introducing the the class extended M_b-metric spaces. Very recently, Asim et al. [18] introduced the class of rectangular M_b-metric space and utilized the same to prove an analogue of Banach contraction principle. Soon, Asim et al. [19] generalized the class of rectangular M_b-metric spaces by introducing the class of M_ν-metric spaces.

Inspired by foregoing observations, we introduce the class of extended rectangular $M_{r\xi}$-metric spaces and utilize the same to prove fixed point result in such spaces. We, also furnish an example to establish the genuineness of our newly proved result. Finally, we use our main result to examine the existence and uniqueness of solution for a Fredholm integral equation.

2. Preliminaries

In this section, we begin with some notions and definitions which are needed in our subsequent discussions.

Notation 1. *[13] The following notations will be utilized in our presentation:*

1. $m_{\varsigma,\sigma} = \min\{m(\varsigma,\varsigma), m(\sigma,\sigma)\}$,
2. $M_{\varsigma,\sigma} = \max\{m(\varsigma,\varsigma), m(\sigma,\sigma)\}$,
3. $m_{b_{\varsigma,\sigma}} = \min\{m_b(\varsigma,\varsigma), m_b(\sigma,\sigma)\}$,
4. $M_{b_{\varsigma,\sigma}} = \max\{m_b(\varsigma,\varsigma), m_b(\sigma,\sigma)\}$,
5. $m_{\xi_{\varsigma,\sigma}} = \min\{m_\xi(\varsigma,\varsigma), m_\xi(\sigma,\sigma)\}$,
6. $M_{\xi_{\varsigma,\sigma}} = \max\{m_\xi(\varsigma,\varsigma), m_\xi(\sigma,\sigma)\}$,
7. $m_{r_{\varsigma,\sigma}} = \min\{m_r(\varsigma,\varsigma), m_r(\sigma,\sigma)\}$,
8. $M_{r_{\varsigma,\sigma}} = \max\{m_r(\varsigma,\varsigma), m_r(\sigma,\sigma)\}$,
9. $m_{rb_{\varsigma,\sigma}} = \min\{m_{rb}(\varsigma,\varsigma), m_{rb}(\sigma,\sigma)\}$,
10. $M_{rb_{\varsigma,\sigma}} = \max\{m_{rb}(\varsigma,\varsigma), m_{rb}(\sigma,\sigma)\}$.

In 2014, Asadi et al. [13] introduced the following definition:

Definition 1. *[13] Let $\chi \neq \emptyset$. A mapping $m : \chi \times \chi \to \mathbb{R}_+$ is said to be an M-metric, if m satisfies the following (for all $\varsigma, \sigma, \rho \in \chi$):*

1. $m(\varsigma,\varsigma) = m(\varsigma,\sigma) = m(\sigma,\sigma)$ *if and only if* $\varsigma = \sigma$,
2. $m_{\varsigma,\sigma} \leq m(\varsigma,\sigma)$,
3. $m(\varsigma,\sigma) = m(\sigma,\varsigma)$,
4. $(m(\varsigma,\sigma) - m_{\varsigma,\sigma}) \leq (m(\varsigma,\rho) - m_{\varsigma,\rho}) + (m(\rho,\sigma) - m_{\rho,\sigma})$.

Then the pair (χ, m) is said to be an M-metric space.

In 2016, Mlaiki et al. [15] generalized the class of M-metric spaces by introducing the the class M_b-metric spaces.

Definition 2. *[15] Let $\chi \neq \emptyset$. A mapping $m_b : \chi \times \chi \to \mathbb{R}_+$ is said to be an M_b-metric with coefficient $s \geq 1$, if m_b satisfies the following (for all $\varsigma, \sigma, \rho \in \chi$):*

1. $m_b(\varsigma,\varsigma) = m_b(\varsigma,\sigma) = m_b(\sigma,\sigma)$ *if and only if* $\varsigma = \sigma$,
2. $m_{b_{\varsigma,\sigma}} \leq m_b(\varsigma,\sigma)$,
3. $m_b(\varsigma,\sigma) = m_b(\sigma,\varsigma)$,
4. $(m_b(\varsigma,\sigma) - m_{b_{\varsigma,\sigma}}) \leq s[(m_b(\varsigma,\rho) - m_{b_{\varsigma,\rho}}) + (m_b(\rho,\sigma) - m_{b_{\rho,\sigma}})] - m_b(\rho,\rho)$.

Then the pair (χ, m_b) is said to be an M_b-metric space.

In 2018, Mlaiki et al. [17] generalized the class of M_b-metric spaces by introducing the the class extended M_b-metric spaces.

Definition 3. *[17] Let $\chi \neq \emptyset$ and $\xi : \chi \times \chi \to [1, \infty)$. A mapping $m_\xi : \chi \times \chi \to \mathbb{R}_+$ is said to be an extended M_b-metric, if m_ξ satisfies the following (for all $\varsigma, \sigma, \rho \in \chi$):*

1. $m_\xi(\varsigma, \varsigma) = m_\xi(\varsigma, \sigma) = m_\xi(\sigma, \sigma)$ *if and only if* $\varsigma = \sigma$,
2. $m_{\xi_{\varsigma,\sigma}} \leq m_\xi(\varsigma, \sigma)$,
3. $m_\xi(\varsigma, \sigma) = m_\xi(\sigma, \varsigma)$,
4. $(m_\xi(\varsigma, \sigma) - m_{\xi_{\varsigma,\sigma}}) \leq \xi(\varsigma, \sigma)[(m_\xi(\varsigma, \rho) - m_{\xi_{\varsigma,\rho}}) + (m_\xi(\rho, \sigma) - m_{\xi_{\rho,\sigma}})]$.

Then the pair (χ, m_ξ) is said to be an extended M_b-metric space.

In 2018, Özgür et al. [16] introduced the notion of rectangular M-metric space as follows:

Definition 4. *[16] Let $\chi \neq \emptyset$. A mapping $m_r : \chi \times \chi \to \mathbb{R}_+$ is said to be a rectangular M-metric, if m_r satisfies the following (for all $\varsigma, \sigma \in \chi$ and all distinct $\rho, \varrho \in \chi \setminus \{\varsigma, \sigma\}$,):*

1. $m_r(\varsigma, \varsigma) = m_r(\varsigma, \sigma) = m_r(\sigma, \sigma)$ *if and only if* $\varsigma = \sigma$,
2. $m_{r_{\varsigma,\sigma}} \leq m_r(\varsigma, \sigma)$,
3. $m_r(\varsigma, \sigma) = m_r(\sigma, \varsigma)$,
4. $(m_r(\varsigma, \sigma) - m_{r_{\varsigma,\sigma}}) \leq (m_r(\varsigma, \rho) - m_{r_{\varsigma,\rho}}) + (m_r(\rho, v) - m_{r_{\rho,\varrho}}) + (m_r(\varrho, \sigma) - m_{r_{\varrho,\sigma}})$.

Then the pair (χ, m_r) is said to be a rectangular M-metric space.

Recently, Asim et al. [18] generalized the class of rectangular M-metric spaces by introducing the class of rectangular M_b-metric spaces. Now, we recall the definition of rectangular M_b-metric space.

Definition 5. *[18] Let $\chi \neq \emptyset$. A mapping $m_{rb} : \chi \times \chi \to \mathbb{R}_+$ is said to be a rectangular M_b-metric with coefficient $s \geq 1$, if m_{rb} satisfies the following (for all $\varsigma, \sigma \in \chi$ and all distinct $\rho, \varrho \in \chi \setminus \{\varsigma, \sigma\}$):*

1. $m_{rb}(\varsigma, \varsigma) = m_{rb}(\varsigma, \sigma) = m_{rb}(\sigma, \sigma)$ *if and only if* $\varsigma = \sigma$,
2. $m_{rb_{\varsigma,\sigma}} \leq m_{rb}(\varsigma, \sigma)$,
3. $m_{rb}(\varsigma, \sigma) = m_{rb}(\sigma, \varsigma)$,
4. $(m_{rb}(\varsigma, \sigma) - m_{rb_{\varsigma,\sigma}}) \leq s[(m_{rb}(\varsigma, \rho) - m_{rb_{\varsigma,\rho}}) + (m_{rb}(\rho, \varrho) - m_{rb_{\rho,\varrho}}) + (m_{rb}(\varrho, \sigma) - m_{rb_{\varrho,\sigma}})] - m_{rb}(\rho, \rho) - m_{rb}(\varrho, \varrho)$.

Then the pair (χ, m_{rb}) is said to be a rectangular M_b-metric space.

Asim et al. [18] proved the following:

Theorem 1. *Let (χ, m_{rb}) be a rectangular M_b-metric space with coefficient $s \geq 1$. Suppose, $f : \chi \to \chi$ satisfies the following conditions:*

1. *for all $\varsigma, \sigma \in \chi$, we have*
$$m_{rb}(f\varsigma, f\sigma) \leq \lambda m_{rb}(\varsigma, \sigma)$$
where $\lambda \in [0, \frac{1}{s})$,
2. (χ, m_{rb}) *is complete.*

Then f has a unique fixed point ς such that $m_r(\varsigma, \varsigma) = 0$.

Very recently, Asim et al. [12] introduced the notion of an extended rectangular b-metric space as a generalization of a rectangular b-metric space which runs as follows:

Definition 6. *[12] Let $\chi \neq \emptyset$ and $\xi : \chi \times \chi \to [1, \infty)$. A mapping $r_\xi : \chi \times \chi \to \mathbb{R}^+$ is said to be an extended rectangular b-metric on χ if, r_ξ satisfies the following (for all $\varsigma, \sigma \in \chi$ and all distinct $\rho, \varrho \in \chi \setminus \{\varsigma, \sigma\}$):*

1. $r_\xi(\varsigma,\sigma)=0$, *if and only if* $\varsigma=\sigma$,
2. $r_\xi(\varsigma,\sigma)=r_\xi(\sigma,\varsigma)$,
3. $r_\xi(\varsigma,\sigma)\leq\xi(\varsigma,\sigma)\left[r_b(\varsigma,\rho)+r_\xi(\rho,\varrho)+r_\xi(\varrho,\sigma)\right]$.

Then the pair (χ,r_ξ) *is said to be an extended rectangular b-metric space.*

3. Results

In this section, we introduce definition of an extended rectangular $M_{r\xi}$-metric space. We also establish a fixed point theorem besides deducing natural corollaries. But first we introduce the following notation:

Notation 2.

1. $m_{r\xi_{\varsigma,\sigma}}=\min\{m_{r\xi}(\varsigma,\varsigma),m_{r\xi}(\sigma,\sigma)\}$,
2. $M_{r\xi_{\varsigma,\sigma}}=\max\{m_{r\xi}(\varsigma,\varsigma),m_{r\xi}(\sigma,\sigma)\}$.

Definition 7. *Let* $\chi\neq\emptyset$ *and* $\xi:\chi\times\chi\to[1,\infty)$. *A mapping* $m_{r\xi}:\chi\times\chi\to\mathbb{R}_+$ *is said to be an extended rectangular* M_ξ*-metric, if* $m_{r\xi}$ *satisfies the following (for all* $\varsigma,\sigma\in\chi$ *and all distinct* $\varrho,\rho\in\chi\setminus\{\varsigma,\sigma\}$*):*

1. $m_{r\xi}(\varsigma,\varsigma)=m_{r\xi}(\varsigma,\sigma)=m_{r\xi}(\sigma,\sigma)$ *if and only if* $\varsigma=\sigma$,
2. $m_{r\xi_{\varsigma,\sigma}}\leq m_{r\xi}(\varsigma,\sigma)$,
3. $m_{r\xi}(\varsigma,\sigma)=m_{r\xi}(\sigma,\varsigma)$,
4. $(m_{r\xi}(\varsigma,\sigma)-m_{r\xi_{\varsigma,\sigma}})\leq\xi(\varsigma,\sigma)[(m_{r\xi}(\varsigma,\rho)-m_{r\xi_{\varsigma,\rho}})+(m_{r\xi}(\rho,\varrho)-m_{r\xi_{\rho,\varrho}})+(m_{r\xi}(\varrho,\sigma)-m_{r\xi_{\varrho,\sigma}})]-m_{r\xi}(\varrho,\varrho)-m_{r\xi}(\rho,\rho)$.

Then the pair $(\chi,m_{r\xi})$ *is said to be a extended rectangular* $M_{r\xi}$*-metric space.*

Remark 1. *If* $\xi(x,y)=s\geq1$, *then* $(X,m_{r\xi})$ *remains a sharpened version of rectangular b-metric space (see [11]).*

Now, we furnish an example in support of Definition 7 which runs as follows:

Example 1. *Let* $\chi=\{0\}\cup\mathbb{N}$ *and* p *a positive even integer. Define a mapping* $\xi:\chi\times\chi\to[1,\infty)$ *by (for all* $\varsigma,\sigma\in\chi$*):*

$$\xi(\varsigma,\sigma)=\begin{cases}|\varsigma-\sigma|^{p-1} & \text{if } \varsigma\neq\sigma\\ 1 & \text{if } \varsigma=\sigma.\end{cases}$$

Define $m_{r\xi}:\chi\times\chi\to\mathbb{R}_+$ *by:*

$$m_{r\xi}(\varsigma,\sigma)=|\varsigma-\sigma|^p,\ \text{for all } \varsigma,\sigma\in\chi.$$

Then $(\chi,m_{r\xi})$ *is an extended rectangular* $M_{r\xi}$*-metric space.*

Proof. By routine calculation, one can easily check that conditions $(1m_{r\xi})-(3m_{r\xi})$ are trivially satisfied. Now, we give the following inequality (for all $\alpha,\beta,\gamma\in\chi$):

$$(\alpha+\beta+\gamma)^p\leq|\alpha+\beta+\gamma|^{p-1}(\alpha^p+\beta^p+\gamma^p).$$

Above inequality is trivial for $\alpha=\beta=\gamma=0$. For $|\alpha|\geq1$ or $|\beta|\geq1$ or $|\gamma|\geq1$, we obtain

$$\begin{aligned}(\alpha+\beta+\gamma)^p &= \frac{(\alpha+\beta+\gamma)^p}{(\alpha^p+\beta^p+\gamma^p)}(\alpha^p+\beta^p+\gamma^p)\\ &\leq \frac{(\alpha+\beta+\gamma)^p}{(\alpha+\beta+\gamma)}(\alpha^p+\beta^p+\gamma^p)\\ &= |\alpha+\beta+\gamma|^{p-1}(\alpha^p+\beta^p+\gamma^p).\end{aligned}$$

Finally, we set $\alpha = \varsigma - \rho,\ \beta = \rho - \varrho,\ \gamma = \varrho - \sigma$ and obtain

$$(\varsigma - \sigma)^p \leq |\varsigma - \sigma|^{p-1}\big((\varsigma - \rho)^p + (\rho - \varrho)^p + (\varrho - \sigma)^p\big).$$

Therefore,

$$(m_{r\xi}(\varsigma, \sigma) - m_{r\xi_{\varsigma,\sigma}}) \leq (m_{r\xi}(\varsigma, \rho) - m_{r\xi_{\varsigma,\rho}}) + (m_{r\xi}(\rho, \varrho) - m_{r\xi_{\rho,\varrho}}) + (m_{r\xi}(\varrho, \sigma) - m_{r\xi_{\varrho,\sigma}}).$$

Hence, $(\chi, m_{r\xi})$ is an extended rectangular $M_{r\xi}$-metric space. □

Notice that $\sup\{\xi(\varsigma, \sigma);\ \varsigma, \sigma \in \chi\} = \infty$. Thus, $m_{r\xi}$ is not a rectangular M_b-metric space.

Let $(\chi, m_{r\xi})$ be an extended rectangular $M_{r\xi}$-metric space. The $m_{r\xi}$-open ball with center $\varsigma \in \chi$ and radius $\epsilon > 0$ is defined by:

$$B_{m_{r\xi}}(\varsigma, \epsilon) = \{\sigma \in \chi : m_{r\xi}(\varsigma, \sigma) < m_{r\xi_{\varsigma,\sigma}} + \epsilon\}.$$

Similarly, the $m_{r\xi}$-closed ball with center $\varsigma \in \chi$ and radius $\epsilon > 0$ is defined by:

$$B_{m_{r\xi}}[\varsigma, \epsilon] = \{\sigma \in \chi : m_{r\xi}(\varsigma, \sigma) \leq m_{r\xi_{\varsigma,\sigma}} + \epsilon\}.$$

The family of $m_{r\xi}$-open balls (for all $\varsigma \in \chi$ and $\epsilon > 0$)

$$\mathcal{U}_{m_{r\xi}} = \{B_{m_{r\xi}}(\varsigma, \epsilon) : \varsigma \in \chi, \epsilon > 0\},$$

forms a basis of some topology $\tau_{m_{r\xi}}$ on χ.

Definition 8. *A sequence $\{\varsigma_n\}$ in $(\chi, m_{r\xi})$ is said to be $m_{r\xi}$-convergent to $\varsigma \in \chi$ if and only if*

$$\lim_{n\to\infty}(m_{r\xi}(\varsigma_n, \varsigma) - m_{r\xi_{\varsigma_n,\varsigma}}) = 0.$$

Definition 9. *A sequence $\{\varsigma_n\}$ in $(\chi, m_{r\xi})$ is said to be $m_{r\xi}$-Cauchy if and only if*

$$\lim_{n,m\to\infty}(m_{r\xi}(\varsigma_n, \varsigma_m) - m_{r\xi_{\varsigma_n,\varsigma_m}}), \qquad \lim_{n,m\to\infty}(M_{r\xi_{\varsigma_n,\varsigma_m}} - m_{r\xi_{\varsigma_n,\varsigma_m}})$$

exist and are finite.

Definition 10. *An extended rectangular $M_{r\xi}$-metric space $(\chi, m_{r\xi})$ is said to be $m_{r\xi}$-complete if every $m_{r\xi}$-Cauchy in χ is $m_{r\xi}$-convergent to some point in χ.*

Now, we furnish two examples by which one can obtain an extended rectangular b-generalized metric space from extended rectangular $M_{r\xi}$-metric space.

Example 2. *Let $(\chi, m_{r\xi})$ be an extended rectangular $M_{r\xi}$-metric space. Define a function $m^*_v : \chi \times \chi \to \mathbb{R}_+$ by (for all $\varsigma, \sigma \in \chi$):*

$$m^*_{r\xi}(\varsigma, \sigma) = m_{r\xi}(\varsigma, \sigma) - 2m_{r\xi_{\varsigma,\sigma}} + M_{r\xi_{\varsigma,\sigma}}.$$

Then $m^*_{r\xi}$ is an extended rectangular b-metric and the pair $(\chi, m^*_{r\xi})$ is an extended rectangular b-metric space.

Proof. To verify condition ($1m_{r\xi}$), for any $\varsigma, \sigma \in \chi$, we have

$$\begin{aligned} m^*_{r\xi}(\varsigma,\sigma) &= 0 \\ &\Longleftrightarrow m_{r\xi}(\varsigma,\sigma) - 2m_{r\xi_{\varsigma,\sigma}} + M_{r\xi_{\varsigma,\sigma}} = 0 \\ &\Longleftrightarrow m_{r\xi}(\varsigma,\sigma) = 2m_{r\xi_{\varsigma,\sigma}} - M_{r\xi_{\varsigma,\sigma}}. \end{aligned}$$

Also,

$$\begin{aligned} m_{r\xi_{\varsigma,\sigma}} &\leq 2m_{r\xi_{\varsigma,\sigma}} - M_{r\xi_{\varsigma,\sigma}} \\ &\Longleftrightarrow M_{\nu_{\varsigma,\sigma}} \leq m_{\nu_{\varsigma,\sigma}} \\ &\Longleftrightarrow m_{r\xi}(\varsigma,\sigma) = m_{r\xi_{\varsigma,\sigma}} = M_{r\xi_{\varsigma,\sigma}} \\ &\Longleftrightarrow \varsigma = \sigma. \end{aligned}$$

Now, for condition ($2m_{r\xi}$), for any $\varsigma, \sigma \in \chi$, we have

$$\begin{aligned} m^*_{r\xi}(\varsigma,\sigma) &= m_{r\xi}(\varsigma,\sigma) - 2m_{r\xi_{\varsigma,\sigma}} + M_{r\xi_{\varsigma,\sigma}} \\ &= m_{r\xi}(\sigma,\varsigma) - 2m_{r\xi_{\sigma,\varsigma}} + M_{r\xi_{\sigma,\varsigma}} \\ &= m^*_{r\xi}(\sigma,\varsigma). \end{aligned}$$

Finally, we show that the condition ($3m_{r\xi}$) holds. For all distinct $\varsigma, \rho, \varrho, \sigma \in \chi$, we have

$$\begin{aligned} m^*_{r\xi}(\varsigma,\sigma) &= m_{r\xi}(\varsigma,\sigma) - 2m_{r\xi_{\varsigma,\sigma}} + M_{r\xi_{\varsigma,\sigma}} \\ &= (m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}}) + (M_{r\xi_{\varsigma,\sigma}} - m_{r\xi_{\varsigma,\sigma}}) \\ &\leq \big[(m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}}) + (m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}}) + (m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}})\big] \\ &\quad - m_{r\xi}(\rho,\rho) - m_{r\xi}(\varrho,\varrho) \\ &\quad + \big[(M_{r\xi_{\varsigma,\rho}} - m_{r\xi_{\varsigma,\rho}}) + (M_{r\xi_{\rho,\varrho}} - m_{r\xi_{\rho,\varrho}}) + (M_{r\xi_{\varrho,\sigma}} - m_{r\xi_{\varrho,\sigma}})\big] \\ &\quad - m_{r\xi_{\rho,\rho}} - m_{r\xi_{\varrho,\varrho}} \\ &\leq \big[(m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}}) + (m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}}) + (m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}})\big] \\ &\quad + \big[(M_{r\xi_{\varsigma,\rho}} - m_{r\xi_{\varsigma,\rho}}) + (M_{r\xi_{\rho,\varrho}} - m_{r\xi_{\rho,\varrho}}) + (M_{r\xi_{\varrho,\sigma}} - m_{r\xi_{\varrho,\sigma}})\big] \\ &= m^*_{r\xi}(\varsigma,\rho) + m^*_{r\xi}(\rho,\varrho) + m^*_{r\xi}(\varrho,\sigma). \end{aligned}$$

Thus, $(\chi, m^*_{r\xi})$ is an extended rectangular b-metric space. □

Example 3. *Let $(\chi, m_{r\xi})$ be an extended rectangular $M_{r\xi}$-metric space. Define a mapping $m^{**}_{r\xi} : \chi \times \chi \to \mathbb{R}_+$ by (for all $\varsigma, \sigma \in \chi$):*

$$m^{**}_{r\xi}(\varsigma,\sigma) = m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}}.$$

Then $m^{**}_{r\xi}$ is an extended rectangular b-metric and the pair $(\chi, m^{**}_{r\xi})$ is an extended rectangular b-metric space.

Proof. By similar arguments as in Example 2, one can easily show that $m^{**}_{r\xi}$ is an extended rectangular b-metric. □

Example 4. *Let $(\chi, m_{r\xi})$ be an extended rectangular $M_{r\xi}$-metric space. Then we have*

$$m_{r\xi}(\varsigma,\sigma) - M_{r\xi_{\varsigma,\sigma}} \leq m^{**}_{r\xi}(\varsigma,\sigma) \leq m_{r\xi}(\varsigma,\sigma) + M_{r\xi_{\varsigma,\sigma}}.$$

Proof. From Example 3, the proof follows easily. □

Now, we present the following lemma which is needed in the sequel.

Lemma 1. *Let $(\chi, m_{r\xi})$ be an extended rectangular $M_{r\xi}$-metric space and $f : \chi \to \chi$ a self mapping on χ. Suppose there exists $\lambda \in [0, \frac{1}{\xi})$ such that*

$$m_{r\xi}(f\varsigma, f\sigma) \leq \lambda m_{r\xi}(\varsigma, \sigma) \tag{1}$$

and consider the sequence $\{\varsigma_n\}_{n\geq 0}$ defined by $\varsigma_{n+1} = f\varsigma_n$. If $\varsigma_n \to \varsigma$ as $n \to \infty$, then $f\varsigma_n \to f\varsigma$ as $n \to \infty$.

Proof. If $m_{r\xi}(f\varsigma_n, f\varsigma) = 0$, then

$$m_{r\xi_{f\varsigma_n,f\varsigma}} \leq m_{r\xi}(f\varsigma_n, f\varsigma) = 0,$$

implies

$$m_{r\xi}(f\varsigma_n, f\varsigma) - m_{r\xi_{f\varsigma_n,f\varsigma}} \to 0 \text{ as } n \to \infty \text{ and } f\varsigma_n \to f\varsigma \text{ as } n \to \infty.$$

Otherwise, if $m_{r\xi}(f\varsigma_n, f\varsigma) > 0$, then by (1), we have

$$m_{r\xi}(f\varsigma_n, f\varsigma) \leq \lambda m_{r\xi}(\varsigma_n, \varsigma).$$

Now, we have the following two cases:
Case 1. If $m_{r\xi}(\varsigma, \varsigma) \leq m_{r\xi}(\varsigma_n, \varsigma_n)$, then by using (1), we get

$$\lim_{n\to\infty} m_{r\xi}(\varsigma_n, \varsigma_n) = 0 \Rightarrow m_{r\xi}(\varsigma, \varsigma) = 0$$

and

$$m_{r\xi}(f\varsigma, f\varsigma) < m_{r\xi}(\varsigma, \varsigma) = 0 \Rightarrow m_{r\xi}(f\varsigma, f\varsigma) = 0.$$

By the definition of $m_{r\xi}$-convergent of a sequence $\{\varsigma_n\}$, which converges to ς, we have

$$\lim_{n\to\infty} (m_{r\xi}(\varsigma_n, \varsigma) - m_{r\xi_{\varsigma_n,\varsigma}}) = 0.$$

Since $m_{r\xi_{\varsigma_n,\varsigma}} = \min\{m_{r\xi}(\varsigma_n, \varsigma_n), m_{r\xi}(\varsigma_n, \varsigma_n)\}$ so that $m_{r\xi_{\varsigma_n,\varsigma}} \to 0$ as $n \to \infty$ and henceforth $m_{r\xi}(\varsigma_n, \varsigma) \to 0$ as $n \to \infty$. Thus

$$m_{r\xi}(f\varsigma_n, f\varsigma) < m_{r\xi}(\varsigma_n, \varsigma) \to 0 \text{ as } n \to \infty.$$

Therefore, $m_{r\xi}(f\varsigma_n, f\varsigma) - m_{r\xi f\varsigma_n, f\varsigma} \to 0$ as $n \to \infty$ and thus $f\varsigma_n \to f\varsigma$ as $n \to \infty$.
Case 2. If $m_{r\xi}(\varsigma, \varsigma) \geq m_{r\xi}(\varsigma_n, \varsigma_n)$, then again

$$\lim_{n\to\infty} m_{r\xi}(\varsigma_n, \varsigma_n) = 0 \Rightarrow \lim_{n\to\infty} m_{r\xi\varsigma_n,\varsigma} = 0,$$

Hence, $m_{r\xi}(\varsigma_n, \varsigma) \to 0$ as $n \to \infty$. Since

$$m_{r\xi}(f\varsigma_n, f\varsigma) < m_{r\xi}(\varsigma_n, \varsigma) \to 0 \text{ as } n \to \infty,$$

then

$$m_{r\xi}(f\varsigma_n, f\varsigma) - m_{r\xi f\varsigma_n, f\varsigma} \to 0 \text{ as } n \to \infty$$

so that $f\varsigma_n \to f\varsigma$ as $n \to \infty$. This completes the proof. □

Now, we state and prove our main result as follows:

Theorem 2. *Let $(\chi, m_{r\xi})$ be an extended rectangular $M_{r\xi}$-metric space. Suppose $f : \chi \to \chi$ satisfies the following conditions:*

1. *for all $\varsigma, \sigma \in \chi$, we have*
$$m_{r\xi}(f\varsigma, f\sigma) \leq \lambda m_{r\xi}(\varsigma, \sigma) \tag{2}$$
where $\lambda \in [0,1)$,
2. $\lim\limits_{n,m\to\infty} \xi(\varsigma_n, \varsigma_m) < \frac{1}{\lambda}$,
3. *$(\chi, m_{r\xi})$ is $m_{r\xi}$-complete.*

Then f has a unique fixed point ς such that $m_{r\xi}(\varsigma, \varsigma) = 0$.

Proof. Assume that $\varsigma_0 \in \chi$ and construct an iterative sequence $\{\varsigma_n\}$ by:

$$\varsigma_1 = f\varsigma_0,\ \varsigma_2 = f^2\varsigma_0,\ \varsigma_3 = f^3\varsigma_0, \ldots, \varsigma_n = f^n\varsigma_0, \ldots$$

Now, we assert that $\lim\limits_{n\to\infty} m_{r\xi}(\varsigma_n, \varsigma_{n+1}) = 0$. On setting $\varsigma = \varsigma_n$ and $\sigma = \varsigma_{n+1}$ in (2), we get

$$\begin{aligned} m_{r\xi}(\varsigma_n, \varsigma_{n+1}) &= m_{r\xi}(f\varsigma_{n-1}, f\varsigma_n) \\ &\leq \lambda m_{r\xi}(\varsigma_{n-2}, \varsigma_{n-1}) \\ &\leq \lambda^{n-1} m_{r\xi}(\varsigma_0, \varsigma_1), \end{aligned}$$

which on making $n \to \infty$, gives rise

$$\lim_{n\to\infty} m_{r\xi}(\varsigma_n, \varsigma_{n+1}) = 0.$$

Similarly, from condition (2), we get

$$m_{r\xi}(\varsigma_n, \varsigma_n) = m_{r\xi}(f\varsigma_{n-1}, f\varsigma_{n-1}) \leq \lambda m_{r\xi}(\varsigma_{n-1}, \varsigma_{n-1}) \leq \ldots \leq \lambda^{n-1} m_{r\xi}(\varsigma_0, \varsigma_0).$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} m_{r\xi}(\varsigma_n, \varsigma_n) = 0. \tag{3}$$

Firstly, we show that $\varsigma_n \neq \varsigma_m$ for any $n \neq m$. Let on contrary that, $\varsigma_n = \varsigma_m$ for some $n > m$, then we have $\varsigma_{n+1} = f\varsigma_n = f\varsigma_m = \varsigma_{m+1}$. On using (2) with $\varsigma = \varsigma_m$ and $\sigma = \varsigma_{m+1}$, we have

$$m_{r\xi}(\varsigma_m, \varsigma_{m+1}) = m_{r\xi}(\varsigma_n, \varsigma_{n+1}) < m_{r\xi}(\varsigma_{n-1}, \varsigma_n) < \ldots < m_{r\xi}(\varsigma_m, \varsigma_{m+1}),$$

a contradiction. This in turn yields that $\varsigma_n \neq \varsigma_m$ for all $n \neq m$.

Now, we show that $\{\varsigma_n\}$ is a $m_{r\xi}$-Cauchy sequence in $(\chi, m_{r\xi})$. In doing so, we distinguish two cases as below:

Case 1. *Firstly, let p is odd, that is $p = 2m+1$ for any $m \geq 1$. Now using $(4m_{r\xi})$ for any $n \in \mathbb{N}$, we have*

$$\begin{aligned}
m_{r\xi}(\varsigma_n, \varsigma_{n+p}) &\leq \xi(\varsigma_n, \varsigma_{n+p})[m_{r\xi}(\varsigma_n, \varsigma_{n+1}) + m_{r\xi}(\varsigma_{n+1}, \varsigma_{n+2}) + m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+p})] \\
&\quad - m_{r\xi}(\varsigma_{n+1}, \varsigma_{n+1}) - m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+2}) \\
&\leq \xi(\varsigma_n, \varsigma_{n+p})[\lambda^n m_{r\xi}(\varsigma_0, \varsigma_1) + \lambda^{n+1} m_{r\xi}(\varsigma_0, \varsigma_1)] \\
&\quad + \xi(\varsigma_n, \varsigma_{n+p}) m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+2m+1}) - \lambda^{n+1} m_{r\xi}(\varsigma_0, \varsigma_0) - \lambda^{n+2} m_{r\xi}(\varsigma_0, \varsigma_0) \\
&= \xi(\varsigma_n, \varsigma_{n+p})(\lambda^n + \lambda^{n+1}) m_{r\xi}(\varsigma_0, \varsigma_1) + \xi(\varsigma_n, \varsigma_{n+p}) m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+2m+1}) \\
&\quad - (\lambda^{n+1} + \lambda^{n+2}) m_{r\xi}(\varsigma_0, \varsigma_0) \\
&\leq \xi(\varsigma_n, \varsigma_{n+j})(\lambda^n + \lambda^{n+1}) m_{r\xi}(\varsigma_0, \varsigma_1) + \xi(\varsigma_n, \varsigma_{n+p}) \xi(\varsigma_{n+2}, \varsigma_{n+p}) \times \\
&\quad (\lambda^{n+2} + \lambda^{n+3}) m_{r\xi}(\varsigma_0, \varsigma_1) + \ldots + \xi(\varsigma_n, \varsigma_{n+p}) \ldots . \xi(\varsigma_{n+p-2}, \varsigma_{n+p}) \times \\
&\quad (\lambda^{n+2m-2} + \lambda^{n+2m-1}) m_{r\xi}(\varsigma_0, \varsigma_1) + \xi(\varsigma_n, \varsigma_{n+p}) \ldots . \xi(\varsigma_{n+p-2}, \varsigma_{n+p}) \times \\
&\quad \lambda^{n+2m} m_{r\xi}(\varsigma_0, \varsigma_1) - (\lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + \ldots) m_{r\xi}(\varsigma_0, \varsigma_0) \\
&= \lambda^n (1+\lambda) m_{r\xi}(\varsigma_0, \varsigma_1) \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{n+2j}, \varsigma_{n+2m+1}) + \\
&\quad \lambda^{n+2m} \prod_{j=0}^{m-1} \xi(\varsigma_{n+2j}, \varsigma_{n+2m+1}) m_{r\xi}(\varsigma_0, \varsigma_1) - \frac{\lambda^{n+1}}{1-\lambda} m_{r\xi}(\varsigma_0, \varsigma_0),
\end{aligned}$$

yielding thereby

$$\sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{n+2j}, \varsigma_{n+2m+1}) \leq \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{2j}, \varsigma_{n+2m+1}).$$

In view of condition (ii), we have $\lim_{n,m\to\infty} \xi(\varsigma_n, \varsigma_m)\lambda < 1$, therefore utilizing the ratio test, we conclude that the series $\sum_{i=0}^{\infty} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{2j}, \varsigma_{n+2m+1})$ is convergent for each $m \in \mathbb{N}$. Assume that

$$S = \sum_{i=0}^{\infty} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{2j}, \varsigma_{n+2m+1}), \; S_n = \sum_{i=0}^{n} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{2j}, \varsigma_{n+2m+1}).$$

Therefore, from the above inequality, we have

$$\begin{aligned}
m_{r\xi}(\varsigma_n, \varsigma_{n+2m+1}) &\leq \lambda^n (1+\lambda) m_{r\xi}(\varsigma_0, \varsigma_1)[S_{m-1} - S_{n-1}] + \\
&\quad \lambda^{n+2m} \prod_{j=0}^{m-1} \xi(\varsigma_{n+2j}, \varsigma_{n+2m+1}) m_{r\xi}(\varsigma_0, \varsigma_1) \\
&\quad - \frac{\lambda^{n+1}}{1-\lambda} m_{r\xi}(\varsigma_0, \varsigma_0).
\end{aligned} \tag{4}$$

Letting $n \to \infty$ in Equation (4), we conclude that

$$\lim_{n,m\to\infty} m_{r\xi}(\varsigma_n, \varsigma_{n+2m+1}) = 0.$$

Case 2. *Secondly, assume that p is even, that is $p = 2m$ for any $m \geq 1$. Then*

$$\begin{aligned}
m_{r\xi}(\varsigma_n, \varsigma_{n+p}) &\leq \xi(\varsigma_n, \varsigma_{n+p})[m_{r\xi}(\varsigma_n, \varsigma_{n+1}) + m_{r\xi}(\varsigma_{n+1}, \varsigma_{n+2}) + m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+p})] \\
&\quad - m_{r\xi}(\varsigma_{n+1}, \varsigma_{n+1}) - m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+2}) \\
&\leq \xi(\varsigma_n, \varsigma_{n+p})[\lambda^n m_{r\xi}(\varsigma_0, \varsigma_1) + \lambda^{n+1} m_{r\xi}(\varsigma_0, \varsigma_1)] + \xi(\varsigma_n, \varsigma_{n+p}) \times \\
&\quad m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+2m}) - \lambda^{n+1} m_{r\xi}(\varsigma_0, \varsigma_0) - \lambda^{n+2} m_{r\xi}(\varsigma_0, \varsigma_0) \\
&= \xi(\varsigma_n, \varsigma_{n+p})(\lambda^n + \lambda^{n+1}) m_{r\xi}(\varsigma_0, \varsigma_1) + \xi(\varsigma_n, \varsigma_{n+p}) m_{r\xi}(\varsigma_{n+2}, \varsigma_{n+2m}) \\
&\quad -(\lambda^{n+1} + \lambda^{n+2}) m_{r\xi}(\varsigma_0, \varsigma_0) \\
&\leq \xi(\varsigma_n, \varsigma_{n+p})(\lambda^n + \lambda^{n+1}) m_{r\xi}(\varsigma_0, \varsigma_1) + \xi(\varsigma_n, \varsigma_{n+p})\xi(\varsigma_{n+2}, \varsigma_{n+p}) \times \\
&\quad (\lambda^{n+2} + \lambda^{n+3}) m_{r\xi}(\varsigma_0, \varsigma_1) + \ldots + \xi(\varsigma_n, \varsigma_{n+p}) \ldots \xi(\varsigma_{n+p-2}, \varsigma_{n+p}) \times \\
&\quad (\lambda^{n+2m-4} + \lambda^{n+2m-3}) m_{r\xi}(\varsigma_0, \varsigma_1) + \xi(\varsigma_n, \varsigma_{n+p}) \ldots \xi(\varsigma_{n+p-2}, \varsigma_{n+p}) \times \\
&\quad \lambda^{n+2m-2} m_{r\xi}(\varsigma_0, \varsigma_2) + s^{m-1} \lambda^{n+2m-2} m_{r\xi}(\varsigma_0, \varsigma_2) \\
&\quad -(\lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + \ldots) m_{r\xi}(\varsigma_0, \varsigma_0) \\
&= \lambda^n (1+\lambda) m_{r\xi}(\varsigma_0, \varsigma_1) \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \xi(\varsigma_{n+2j}, \varsigma_{n+2m}) + \\
&\quad \lambda^{n+2m-2} \prod_{j=0}^{m-1} \xi(\varsigma_{n+2j}, \varsigma_{n+2m}) m_{r\xi}(\varsigma_0, \varsigma_2) - \frac{\lambda^{n+1}}{1-\lambda} m_{r\xi}(\varsigma_0, \varsigma_0),
\end{aligned}$$

so that

$$\begin{aligned}
m_{r\xi}(\varsigma_n, \varsigma_{n+2m}) &\leq \lambda^n (1+\lambda) m_{r\xi}(\varsigma_0, \varsigma_1)[S_{m-1} - S_{n-1}] + \\
&\quad \lambda^{n+2m-2} \prod_{j=0}^{m-1} \xi(\varsigma_{n+2j}, \varsigma_{n+2m}) m_{r\xi}(\varsigma_0, \varsigma_2) - \frac{\lambda^{n+1}}{1-\lambda} m_{r\xi}(\varsigma_0, \varsigma_0). \qquad (5)
\end{aligned}$$

Taking the limit as $n \to \infty$, in (5), we conclude that

$$\lim_{n,m \to \infty} m_{r\xi}(\varsigma_n, \varsigma_{n+2m}) = 0.$$

Therefore, in both the cases, we have

$$\lim_{n,m \to \infty} (m_{r\xi}(\varsigma_n, \varsigma_m) - m_{r\xi_{\varsigma_n, \varsigma_m}}) = 0.$$

On the other hand, without loss of generality we may assume that

$$M_{r\xi_{\varsigma_n, \varsigma_m}} = m_{r\xi}(\varsigma_n, \varsigma_n).$$

Hence, we obtain

$$\begin{aligned}
M_{r\xi_{\varsigma_n, \varsigma_m}} - m_{r\xi_{\varsigma_n, \varsigma_m}} &\leq M_{r\xi_{\varsigma_n, \varsigma_m}} \\
&= m_{r\xi}(\varsigma_n, \varsigma_n) \\
&\leq \lambda^n m_{r\xi}(\varsigma_0, \varsigma_0).
\end{aligned}$$

Taking the limit of the above inequality as $n \to \infty$, we deduce that

$$\lim_{n,m \to \infty} (M_{r\xi_{\varsigma_n, \varsigma_m}} - m_{r\xi_{\varsigma_n, \varsigma_m}}) = 0.$$

Therefore, the sequence $\{\varsigma_n\}$ is $m_{r\xi}$-Cauchy in χ. Since χ is $m_{r\xi}$-complete, then there exists $\varsigma \in \chi$ such that $\varsigma_n \to \varsigma$. Now, we show that $f\varsigma = \varsigma$. By Lemma 1, we have

$$\begin{aligned}
\lim_{n,m\to\infty}(m_{r\xi}(\varsigma_n,\varsigma) - m_{r\xi_{\varsigma_n,\varsigma}}) &= 0 \\
&= \lim_{n,m\to\infty}(m_{r\xi}(\varsigma_{n+1},\varsigma) - m_{r\xi_{\varsigma_{n+1},\varsigma}}) \\
&= \lim_{n,m\to\infty}(m_{r\xi}(f\varsigma_n,\varsigma) - m_{r\xi_{f\varsigma_n,\varsigma}}) \\
&= \lim_{n,m\to\infty}(m_{r\xi}(f\varsigma,\varsigma) - m_{r\xi_{f\varsigma,\varsigma}}).
\end{aligned}$$

Hence, we find $m_{r\xi}(f\varsigma,\varsigma) = m_{r\xi_{\varsigma,f\varsigma}}$. Since $m_{r\xi_{\varsigma,f\varsigma}} = \min\{m_{r\xi}(\varsigma,\varsigma), m_{r\xi}(f\varsigma,f\varsigma)\}$. Therefore, $m_{r\xi_{\varsigma,f\varsigma}} = m_{r\xi}(\varsigma,\varsigma)$ or $m_{r\xi_{\varsigma,f\varsigma}} = m_{r\xi}(f\varsigma,f\varsigma)$ which implies that $f\varsigma = \varsigma$.

Next, we show the uniqueness of the fixed point of f. Assume that f has two fixed points $\varsigma, \sigma \in \chi$, that is, $f\varsigma = \varsigma$ and $f\sigma = \sigma$. Thus

$$m_{r\xi}(\varsigma,\sigma) = m_{r\xi}(f\varsigma,f\sigma) \le \lambda m_{r\xi}(\varsigma,\sigma) < m_{r\xi}(\varsigma,\sigma),$$

which implies that $m_{r\xi}(\varsigma,\sigma) = 0$ and hence, $\varsigma = \sigma$.

Finally, we show that if ς is a fixed point, then $m_{r\xi}(\varsigma,\varsigma) = 0$. To accomplish this, let ς be a fixed point of f then

$$\begin{aligned}
m_{r\xi}(\varsigma,\varsigma) &= m_{r\xi}(f\varsigma,f\varsigma) \\
&\le \lambda m_{r\xi}(\varsigma,\varsigma) \\
&< m_{r\xi}(\varsigma,\varsigma),
\end{aligned}$$

yielding thereby $m_{r\xi}(\varsigma,\varsigma) = 0$. This concludes the proof. □

Now, we present an example which illustrates the utility of our newly proved result:

Example 5. *Let $\chi = \{1,3,5,7\}$. Define*

$$m_{r\xi}(\varsigma,\sigma) = \left(\frac{\varsigma+\sigma}{2}\right)^3 \text{ and } \xi(\varsigma,\sigma) = \varsigma\sigma, \text{ for all } \varsigma,\sigma \in \chi.$$

Let us first show that $(\chi, m_{r\xi})$ is an extended rectangular $M_{r\xi}$-metric space. It is easy to check that the conditions $(1m_{r\xi})$-$(3m_{r\xi})$ are hold for all $\varsigma,\sigma \in \chi$. Now, to verify condition $(4m_{r\xi})$, we have following cases:

Case 3. *If $\varsigma = 1, \sigma = 3, \rho = 5$ and $\varrho = 7$, then we have*

$$m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}} = (2)^3 - 1^3 = 7,$$

$$3[m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}} + m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}} + m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}}] = 645,$$

$$m_{r\xi}(\rho,\rho) + m_{r\xi}(\varrho,\varrho) = (5)^3 + (7)^3 = 468.$$

Case 4. *If $\varsigma = 1, \sigma = 5, \rho = 3$ and $\varrho = 7$, then we have*

$$m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}} = (3)^3 - 1^3 = 26,$$

$$5[m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}} + m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}} + m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}}] = 980,$$

$$m_{r\xi}(\rho,\rho) + m_{r\xi}(\varrho,\varrho) = (3)^3 + (7)^3 = 370.$$

Case 5. *If $\varsigma = 1, \sigma = 7, \rho = 3$ and $\varrho = 5$, then we have*

$$m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}} = (4)^3 - 1^3 = 63,$$

$$7[m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}} + m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}} + m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}}] = 945,$$
$$m_{r\xi}(\rho,\rho) + m_{r\xi}(\varrho,\varrho) = (3)^3 + (5)^3 = 152.$$

Case 6. *If $\varsigma = 3, \sigma = 5, \rho = 7$ and $\varrho = 1$, then we have*

$$m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}} = (4)^3 - (3)^3 = 37,$$

$$15[m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}} + m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}} + m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}}] = 2805,$$
$$m_{r\xi}(\rho,\rho) + m_{r\xi}(\varrho,\varrho) = (7)^3 + (1)^3 = 344.$$

Case 7. *If $\varsigma = 3, \sigma = 7, \rho = 1$ and $\varrho = 5$, then we have*

$$m_{r\xi}(\varsigma,\sigma) - m_{r\xi_{\varsigma,\sigma}} = (3)^3 + (2)^3 = 35,$$

$$21[m_{r\xi}(\varsigma,\rho) - m_{r\xi_{\varsigma,\rho}} + m_{r\xi}(\rho,\varrho) - m_{r\xi_{\rho,\varrho}} + m_{r\xi}(\varrho,\sigma) - m_{r\xi_{\varrho,\sigma}}] = 2604,$$
$$m_{r\xi}(\rho,\rho) + m_{r\xi}(\varrho,\varrho) = (1)^3 + (5)^3 = 126.$$

Then $(\chi, m_{r\xi})$ is an $m_{r\xi}$-complete extended rectangular $M_{r\xi}$-metric space. Consider a mapping $f : \chi \to \chi$ defined by:

$$f = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 1 & 1 & 3 & 3 \end{pmatrix}.$$

In particular, if we take $\varsigma, \sigma \in \{1,3\}$ (or $\varsigma, \sigma \in \{5,7\}$), then $f\varsigma = f\sigma = 1$ (or $f\varsigma = f\sigma = 3$) and hence one can easily check that condition (2) satisfy with $\xi(\varsigma,\sigma) = \varsigma\sigma$,

$$m_{r\xi}(f\varsigma, f\sigma) \leq \varsigma\sigma m_{r\xi}(\varsigma,\sigma).$$

Now, by taking $\varsigma \in \{1,3\}$ and $\sigma \in \{5,7\}$ then (by (2)), we have

$$\begin{aligned} m_{r\xi}(f\varsigma, f\sigma) &= \left(\frac{1+3}{2}\right)^3 \\ &= 2^3 \\ &< \varsigma\sigma\left(\frac{\varsigma+\sigma}{2}\right)^3 \\ &= \xi(\varsigma,\sigma) m_{r\xi}(\varsigma,\sigma). \end{aligned}$$

Therefore

$$m_{r\xi}(f\varsigma, f\sigma) \leq \xi(\varsigma,\sigma) m_{r\xi}(\varsigma,\sigma), \text{ for all } \varsigma,\sigma \in \chi.$$

Hence, all the requirements of Theorem 2 are fulfilled and $\varsigma = 1$ is a unique fixed point of f.

Corollary 1. *Theorem 1 of Asim at al. [18] is immediate from Theorem 2.*

The following corollary deduced form Theorem 2, which remains genuinely sharpened version of Theorem 4.2 of Özgür et al. [16].

Corollary 2. *Let (χ, m_{rb}) be a rectangular M_b-metric space with coefficient $\xi(\varsigma,\sigma) = 1$ and $f : \chi \to \chi$ satisfying the following condition:*

1. *for all $\varsigma, \sigma \in \chi$, we have*
$$m_{rb}(f\varsigma, f\sigma) \leq \lambda m_{rb}(\varsigma, \sigma)$$
where $\lambda \in [0,1)$,
2. *(χ, m_{rb}) is m_{rb}-complete.*

Then f has a unique fixed point ς such that $m_{rb}(\varsigma, \varsigma) = 0$.

4. An Application to an Integral Equation

In this section, we endeavor to apply Theorem 2 to investigate the existence and uniqueness of solution of the Fredholm integral equation.

Let $\chi = C([a,b], \mathbb{R})$ be the be the set of continuous real valued functions defined on $[a,b]$. Now, we consider the following Fredholm type integral equation:

$$\varsigma(p) = \int_a^b G(p, q, \varsigma(p))dq + h(p), \text{ for } p, q \in [a,b] \tag{6}$$

where $G, h \in C([a,b], \mathbb{R})$. Define $m_{r\xi} : \chi \times \chi \to \mathbb{R}^+$ and $\xi : \chi \times \chi \to [1, \infty)$ by:

$$m_{r\xi}(\varsigma(p), \sigma(p)) = \sup_{p \in [a,b]} \left(\frac{|\varsigma(p)| + |\sigma(p)|}{2} \right)^3 \text{ and } \xi(\varsigma, \sigma) = |\varsigma| + |\sigma| + 3, \text{ for all } \varsigma, \sigma \in \chi.$$

Then $(\chi, m_{r\xi})$ is an $m_{r\xi}$-complete $M_{r\xi}$-metric space.

Now, we are equipped to state and prove our result as follows:

Theorem 3. *Assume that (for all $\varsigma, \sigma \in C([a,b], \mathbb{R})$)*

$$|G(p, q, \varsigma(p)) + G(p, q, \sigma(p))| \leq \frac{1}{3(b-a)} |\varsigma(p) + \sigma(p)|, \ for\ all\ p, q \in [a,b]. \tag{7}$$

Then the integral Equation (6) has a unique solution.

Proof. Define $f : \chi \to \chi$ by

$$f\varsigma(p) = \int_a^b G(p, q, \varsigma(p))dq + h(p), \text{ for all } p, q \in [a,b].$$

Observe that, existence of a fixed point of the operator f is equivalent to the existence of a solution of the integral Equation (6). Now, for all $\varsigma, \sigma \in \chi$, we have

$$\begin{aligned} m_{r\xi}(f\varsigma, f\sigma) = \left| \frac{f\varsigma(p) + f\sigma(p)}{2} \right|^3 &= \left| \int_a^b \left(\frac{G(p,q,\varsigma(p)) + G(p,q,\sigma(p))}{2} \right) dq \right|^3 \\ &\leq \left(\int_a^b \left| \frac{G(p,q,\varsigma(p)) + G(p,q,\sigma(p))}{2} \right| dq \right)^3 \\ &\leq \left(\int_a^b \frac{1}{3(b-a)} \left| \frac{\varsigma(p) + \sigma(p)}{2} \right| dq \right)^3 \\ &\leq \frac{1}{27(b-a)^3} \int_a^b \left(\frac{|\varsigma(p)| + |\sigma(p)|}{2} \right) dq \\ &\leq \frac{1}{27(b-a)^3} \sup_{p \in [a,b]} \left(\frac{|\varsigma(p)| + |\sigma(p)|}{2} \right)^3 \left(\int_a^b dq \right)^3 \\ &\leq \frac{1}{27} m_{r\xi}(\varsigma, \sigma). \end{aligned}$$

Thus, the condition (7) is satisfied. Therefore, all the conditions of Theorem 2 are satisfied. Hence, the operator f has a unique fixed point, which means that the Fredholm integral Equation (6) has a unique solution. This completes the proof. □

5. Conclusions

As the rectangular M_b-metric space is a relatively new addition to the existing literature, therefore, in this note, we endeavor to further enrich this notion by introducing the idea of extended rectangular $M_{r\xi}$-metric spaces wherein we generalized the constant $s \geq 1$ by a function $\xi(\varsigma, \sigma)$ in quadrilateral inequality. Our main result (i.e., Theorem 2) is an analogue of Banach contraction principle. An example is also included to highlight the realized improvements in our newly proved result. Finally, we apply Theorem 2 to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

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