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The First Fundamental Equation and Generalized Wintgen-Type Inequalities for Submanifolds in Generalized Space Forms

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Abstract: In this work, we first derive a generalized Wintgen type inequality for a Lagrangian submanifold in a generalized complex space form. Further, we extend this inequality to the case of bi-slant submanifolds in generalized complex and generalized Sasakian space forms and derive some applications in various slant cases. Finally, we obtain obstructions to the existence of non-flat generalized complex space forms and non-flat generalized Sasakian space forms in terms of dimension of the vector space of solutions to the first fundamental equation on such spaces.

Keywords: Wintgen inequality; generalized complex space form; generalized Sasakian space form; Lagrangian submanifold; Legendrian submanifold

MSC: 53B05; 53B20; 53C40

1. Introduction

The classical Wintgen inequality is a sharp geometric inequality established in [1], according to which the Gaussian curvature $\mathcal{K}$ of any surface $\mathcal{N}^2$ in the Euclidean space $\mathbb{E}^4$, the normal curvature $\mathcal{K}^\perp$, and also the squared mean curvature $\|\mathcal{H}\|^2$ of $\mathcal{N}^2$, satisfy

$$\|\mathcal{H}\|^2 \geq \mathcal{K} + |\mathcal{K}^\perp|$$

and the equality is attained only in the case when the ellipse of curvature of $\mathcal{N}^2$ in $\mathbb{E}^4$ is a circle. Later, this inequality was extended independently by Rouxel [2] and Gaudalupe and Rodriguez [3] for surfaces of arbitrary codimension m in real space forms $\overline{\mathcal{N}}^{m+2}(c)$ with constant sectional curvature c as

$$\|\mathcal{H}\|^2 + c \geq \mathcal{K} + |\mathcal{K}^\perp|.$$

The generalized Wintgen inequality, also known as the DDVV-inequality or the DDVV-conjecture, is a natural extension of the above inequalities that was conjectured in 1999 by De Smet, Dillen, Verstraelen and Vrancken [4] and settled in the general case independently by Ge and Tang [5] and Lu [6]. The DDVV-conjecture generalizes the classical Wintgen inequality to the case of an isometric immersion $f : M^n \to N^{n+p}(c)$ from an n-dimensional Riemannian submanifold M^n into a real space form $N^{n+p}(c)$ of dimension $(n+p)$ and of constant sectional curvature c, stating that such an isometric immersion satisfies

$$\rho + \rho^{\perp} \leq \|\mathcal{H}\|^2 + c,$$

where ρ is the normalized scalar curvature, while $\rho^{\perp}$ denotes the normalized normal scalar curvature. Notice that there are many examples of submanifolds satisfying the equality case of the above inequality and these submanifolds are known as Wintgen ideal submanifolds [7].

Recently, the generalized Wintgen inequality was extended for several kinds of submanifolds in many ambient spaces, e.g., complex space forms [8], Sasakian space forms [9], quaternionic space forms [10], warped products [11], and Kenmotsu statistical manifolds [12]. In the first part of the present paper, we obtain generalized Wintgen-type inequalities for different types of submanifolds in generalized complex space forms and also in generalized Sasakian space forms, generalizing the main results in [8,9], and also discuss some applications. The last part of the paper is devoted to the investigation of the Hessian equation on both generalized complex space forms and generalized Sasakian space-forms. In particular, some obstructions to the existence of these spaces are established. Recall that the notion of generalized complex space form was introduced in differential geometry by Tricerri and Vanhecke [13], the authors proving that, if $n \geq 3$, a $2n$-dimensional generalized complex space form is either a real space form or a complex space form, a result partially extendable to four-dimensional manifolds. However, the existence of proper generalized complex space form in dimension 4 was obtained by Olszak [14], using some conformal deformations of four-dimensional flat Bochner–Kähler manifolds of non-constant scalar curvature. It is important to note that the generalized complex space forms are a particular kind of almost Hermitian manifolds with constant holomorphic sectional curvature and constant type in the sense of Gray [15].

On the other hand, Alegre, Blair and Carriazo [16] generalized the notions of Sasakian space form, Kenmotsu space form and cosymplectic space form, by introducing the concept of generalized Sasakian space form. Notice that several examples of non-trivial generalized Sasakian space-forms are given in [16] using different geometric constructions, such as Riemannian submersions, warped products, and D-conformal deformations. Afterwards, many interesting results have been proved in these ambient spaces (see, e.g., [17–27]). We only recall that, very recently, Bejan and Güler [28] obtained an unexpected link between the class of generalized Sasakian space-forms and the class of Kähler manifolds of quasi-constant holomorphic sectional curvature, providing conditions under which each of these structures induces the other one.

2. Preliminaries

An almost Hermitian manifold consists in a smooth manifold $\overline{\mathcal{N}}$ endowed with an almost complex structure J and a Riemannian metric g that is compatible with the structure J. Such a manifold is called Kähler if $\overline{\nabla} J = 0$, where $\overline{\nabla}$ is the Levi–Civita connection of the metric g.

On the other hand, an almost Hermitian manifold $\overline{\mathcal{N}}$ is called a generalized complex space form [13], denoted by $\overline{\mathcal{N}}(f_1, f_2)$, if the Riemannian curvature tensor $\overline{R}$ satisfies

$$\begin{aligned} \overline{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY \\ &\quad - g(Y,JZ)JX + 2g(X,JY)JZ\} \end{aligned} \tag{1}$$

for all vector fields X, Y and Z on $\overline{\mathcal{N}}$, where f_1 and f_2 are smooth functions on $\overline{\mathcal{N}}$. This name is motivated by the fact that, in the case of a complex space form, viz. a Kähler manifold with constant holomorphic sectional curvature $4c$, the curvature tensor field of the manifold satisfies Equation (1) with $f_1 = f_2 = c$.

Let $\mathcal{N}$ be a submanifold of real dimension n in a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ of complex dimension m. If ∇ and $\overline{\nabla}$ are the Levi–Civita connections on $\mathcal{N}$ and $\overline{\mathcal{N}}(f_1, f_2)$, respectively, then the fundamental formulas of Gauss and Weingarten are [29]

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\overline{\nabla}_X \xi = -S_\xi X + \nabla^{\perp}_X Y,$$

where X, Y are vector fields tangent to $\mathcal{N}$, ξ is a vector field normal to $\mathcal{N}$, and $\nabla^{\perp}$ represents the normal connection. Recall that, in the above basic formulas, h denotes the second fundamental form and S is the shape operator, they being connected by

$$g(h(X, Y), \xi) = g(S_\xi X, Y).$$

On the other hand, the Gauss' equation is expressed by [30]

$$\begin{aligned} \overline{R}(X, Y, Z, W) &= R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ &\quad - g(h(X, W), h(Y, Z)) \end{aligned} \tag{2}$$

for all vector fields X, Y, Z, W tangent to $\mathcal{N}$, where $\overline{R}$ denotes the curvature tensor of $\overline{\mathcal{N}}(f_1, f_2)$, while R represents the curvature tensors of $\mathcal{N}$. Let us point out now that the Ricci equation in our setting is expressed as

$$\begin{aligned} R^{\perp}(X, Y, \xi, \eta) &= f_2[g(X, J\xi)g(JY, \eta) - g(JX, \eta)g(Y, J\xi)] \\ &\quad - g([S_\xi, S_\eta]X, Y), \end{aligned} \tag{3}$$

for all vector fields X, Y tangent to $\mathcal{N}$ and ξ, η normal to $\mathcal{N}$.

If $\mathcal{N}$ is a submanifold of real dimension n in a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ of complex dimension m, then, for any $X \in T\mathcal{N}$, we have the decomposition $JX = PX + QX$, where P and Q denote the tangential component and the normal component of JX, respectively. We recall that, in the case $P = 0$, the submanifold $\mathcal{N}$ is called anti-invariant, while, in the case f $Q = 0$, the submanifold $\mathcal{N}$ is called invariant.

Now, let $\{e_1, \ldots, e_n\}$ be a tangent orthonormal frame on $\mathcal{N}$ and let $\{\xi_1, \ldots, \xi_{2m-n}\}$ be a normal orthonormal frame on $\mathcal{N}$. Then, the squared norm of P at $p \in \mathcal{N}$ is defined as

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j), \tag{4}$$

while the mean curvature vector field is given by

$$\mathcal{H} = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{5}$$

We also set

$$h_{ij}^r = g(h(e_i, e_j), \xi_r),\ i, j = 1, \ldots, n,\ r = 1, \ldots, 2m-n. \tag{6}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{7}$$

3. Generalized Wintgen Inequality for Lagrangian Submanifolds in Generalized Complex Space Form

Let $\mathcal{N}$ be a submanifold of real dimension n in a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ of complex dimension m. In the following, let $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_{2m-n}\}$ be tangent orthonormal frame and normal orthonormal frame on $\mathcal{N}$, respectively. If we denote by K the sectional curvature function and by τ the scalar curvature, then the normalized scalar curvature ρ of $\mathcal{N}$ can be expressed as [8]

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} K(e_i \wedge e_j). \tag{8}$$

On the other hand, the normalized normal scalar curvature of $\mathcal{N}$ is given by [8]

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n} \sum_{1\leq r<s\leq 2m-n} (R^{\perp}(e_i, e_j, \xi_r, \xi_s))^2}, \tag{9}$$

where $R^{\perp}$ denotes the normal curvature tensor on $\mathcal{N}$.

The scalar normal curvature of $\mathcal{N}$ can be defined following [31] as

$$\mathcal{K}_N = \frac{1}{4} \sum_{r,s=1}^{2m-n} (Trace[S_r, S_s])^2. \tag{10}$$

Now, the normalized scalar normal curvature can be defined with the help of $\mathcal{K}_N$ by [8]

$$\rho_N = \frac{2}{n(n-1)} \sqrt{\mathcal{K}_N}.$$

Obviously

$$\begin{aligned} \mathcal{K}_N &= \frac{1}{2} \sum_{1\leq r<s\leq 2m-n} (Trace[S_r, S_s])^2 \\ &= \sum_{1\leq r<s\leq 2m-n} \sum_{1\leq i<j\leq n} (g([S_r, S_s]e_i, e_j))^2. \end{aligned} \tag{11}$$

It is easy to verify now that $\mathcal{K}_N$ can be expressed by

$$\mathcal{K}_N = \sum_{1\leq r<s\leq 2m-n} \sum_{1\leq i<j\leq n} \Big(\sum_{k=1}^{n} h_{jk}^r h_{ik}^s - h_{jk}^s h_{ik}^r \Big)^2. \tag{12}$$

Among the classes of submanifolds in complex geometry, we can distinguish two fundamental families depending on the behavior of J: holomorphic and totally real submanifolds. A submanifold $\mathcal{N}$ of a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ is said to be a holomorphic submanifold if each tangent space of $\mathcal{N}$ is carried into itself by J, i.e., $J(T_p\mathcal{N}) \subset T_p\mathcal{N}$, for all $p \in \mathcal{N}$. Similarly, the submanifold $\mathcal{N}$ is called a totally real submanifold if J maps each tangent space of $\mathcal{N}$ into the normal space, i.e., $J(T_p\mathcal{N}) \subset T_p^{\perp}\mathcal{N}$, for all $p \in \mathcal{N}$. In particular, if $n = m$, then $\mathcal{N}$ is said to be a Lagrangian submanifold.

Next, we prove the following lemma, which is required in the proof of the main result of this section.

Lemma 1. *Let $\mathcal{N}$ be a totally real submanifold of dimension n in a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ of complex dimension m. Then, we have*

$$\rho_N \quad \leq \quad \|\mathcal{H}\|^2 - \rho + f_1, \tag{13}$$

and the equality holds at a point $p \in \mathcal{N}$ if and only if the shape operator S of $\mathcal{N}$ in $\overline{\mathcal{N}}(f_1, f_2)$ with respect to some suitable orthonormal bases $\{e_1, \ldots, e_n\}$ of $T_p\mathcal{N}$ and $\{\xi_1, \ldots, \xi_{2m-n}\}$ of $T_p^{\perp}\mathcal{N}$ takes the following forms

$$S_{\xi_1} = \begin{pmatrix} \gamma_1 & \nu & 0 & \ldots & 0 \\ \nu & \gamma_1 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_1 \end{pmatrix},$$

$$S_{\xi_2} = \begin{pmatrix} \gamma_2 + \nu & 0 & 0 & \ldots & 0 \\ 0 & \gamma_2 - \nu & 0 & \ldots & 0 \\ 0 & 0 & \gamma_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_2 \end{pmatrix},$$

$$S_{\xi_3} = \begin{pmatrix} \gamma_3 & 0 & 0 & \ldots & 0 \\ 0 & \gamma_3 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_3 \end{pmatrix}, \quad S_{\xi_4} = \cdots = S_{\xi_{2m-n}} = 0,$$

where $\gamma_1, \gamma_2, \gamma_3$, and ν are real functions on $\mathcal{N}$.

Proof. We know that

$$n^2\|\mathcal{H}\|^2 = \sum_{r=1}^{2m-n}\Big(\sum_{i=1}^{n} h_{ii}^r\Big)^2 = \frac{1}{n-1}\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}(h_{ii}^r - h_{jj}^r)^2 + \frac{2n}{n-1}\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r. \tag{14}$$

Further, from [6], we have

$$\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}(h_{ii}^r - h_{jj}^r)^2 + 2n\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}(h_{ij}^r)^2 \geq 2n\left[\sum_{1\le r<s\le 2m-n}\sum_{1\le i<j\le n}\Big(\sum_{k=1}^{n}(h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s)\Big)^2\right]^{\frac{1}{2}}. \tag{15}$$

Now, combining Equations (12), (14) and (15), we find

$$n^2\|\mathcal{H}\|^2 - n^2\rho_N \geq \frac{2n}{n-1}\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{16}$$

In addition, due to the fact that $\mathcal{N}$ is a totally real submanifold, we get from Equation (2):

$$\tau = \frac{n(n-1)}{2} f_1 + \sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{17}$$

Next, using Equations (8) and (17) in Equation (16), we obtain the inequality in Equation (13). Moreover, it follows easily that the equality case holds in Equation (13) if and only if the shape operator takes the above stated forms. □

Now, we prove the following.

Theorem 1. *Let $\mathcal{N}$ be a Lagrangian submanifold of a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ of complex dimension n. Then,*

$$(\rho^{\perp})^2 \leq \big(\|\mathcal{H}\|^2 - \rho + f_1\big)^2 + \frac{2}{n(n-1)} f_2^2 + \frac{4 f_2}{n(n-1)}(\rho - f_1) \tag{18}$$

and the equality in Equation (18) holds at a point $p \in \mathcal{N}$ if and only if the shape operator takes similar forms as in Lemma 1 with respect to some suitable tangent and normal orthonormal bases.

Proof. Let $\mathcal{N}$ be a Lagrangian submanifold of a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$. We choose $\{e_1, \ldots, e_n\}$ and $\{\xi_1 = Je_1, \ldots, \xi_n = Je_n\}$ as orthonormal frame and orthonormal normal frame on $\mathcal{N}$, respectively. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ in Equation (1), we obtain

$$\overline{R}(e_i, e_j, e_j, e_i) = f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\}. \tag{19}$$

Combining Equations (2) and (19), we derive

$$R(e_i,e_j,e_j,e_i) = f_1\{\delta_{ii}\delta_{jj}-\delta_{ij}^2\} - g(h(e_i,e_j),h(e_j,e_i)) + g(h(e_i,e_i),h(e_j,e_j)). \quad (20)$$

By taking summation for $1 \le i,j \le n$ in Equation (20) and making use of Equations (5) and (7), we obtain

$$2\tau = n(n-1)f_1 + n^2\|\mathcal{H}\|^2 - \|h\|^2. \quad (21)$$

Using Equation (8) in Equation (21), we get

$$\rho = f_1 + \frac{n}{n-1}\|\mathcal{H}\|^2 - \frac{1}{n(n-1)}\|h\|^2, \quad (22)$$

which implies

$$n^2\|\mathcal{H}\|^2 - \|h\|^2 = n(n-1)(\rho - f_1). \quad (23)$$

Further, Equation (3) gives

$$R^{\perp}(e_i,e_j,\xi_r,\xi_s) = f_2\{-(\delta_{ir}\delta_{js}-\delta_{jr}\delta_{is})\} - g([S_{\xi_r},S_{\xi_s}]e_i,e_j), \quad (24)$$

for any indices $i,j,r,s \in \{1,\dots,n\}$.

Next, by taking summation for $1 \le r < s \le n$ and $1 \le i < j \le n$ in Equation (24), we derive easily the following relation:

$$(\tau^{\perp})^2 = \frac{n(n-1)}{2}f_2^2 + \frac{n^2(n-1)^2}{4}\rho_N^2 - f_2\|h\|^2 + f_2n^2\|\mathcal{H}\|^2. \quad (25)$$

However, the above Equation (25) can be rewritten as

$$(\rho^{\perp})^2 = \frac{2}{n(n-1)}f_2^2 + \rho_N^2 - \frac{4f_2}{n^2(n-1)^2}\|h\|^2 + \frac{4f_2}{(n-1)^2}\|\mathcal{H}\|^2. \quad (26)$$

Now, from Equations (23) and (26), we have

$$(\rho^{\perp})^2 = \frac{2}{n(n-1)}f_2^2 + \rho_N^2 + \frac{4f_2}{n(n-1)}(\rho - f_1). \quad (27)$$

Combining now Equations (13) and (27), we obtain the required inequality and the equality case of the inequality is also clear from Lemma 1. □

Remark 1. *Theorem 2 generalizes the main result of [8], namely the generalized Wintgen inequality for the class of Lagrangian submanifolds in a complex space form. Indeed, if in the statement of Theorem 2 one particularizes the generalized complex space form by putting $f_1 = f_2 = c$, then $\overline{N}$ reduces to a complex space form and one arrives at ([8] Theorem 2.3).*

4. Generalized Wintgen Inequality for bi-Slant Submanifolds in Generalized Complex Space Form

A submanifold $\mathcal{N}$ of an almost Hermitian manifold $(\overline{\mathcal{N}}, J, g)$ is said to be a slant submanifold if for any point $p \in \mathcal{N}$ and any non-zero vector $X \in T_p\mathcal{N}$, the angle θ between the vector JX and the tangent space $T_p\mathcal{N}$ is constant, i.e., this angle does not depend on the choice of $p \in \mathcal{N}$ and $X \in T_p\mathcal{N}$. Moreover, $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of $\mathcal{N}$ in $\overline{\mathcal{N}}$. Recall that both invariant and anti-invariant submanifolds are particular examples of slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. Moreover, if $0 < \theta < \frac{\pi}{2}$, then $\mathcal{N}$ is said to be a θ-slant submanifold or a proper slant submanifold. It is known that any proper slant submanifold has even dimension. The concept of slant submanifold originally introduced by Chen [32,33] was later generalized as follows.

Definition 1. *([34]) A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a bi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2, such that:*

(i) $T\mathcal{N}$ admits the orthogonal direct decomposition:

$$T\mathcal{N} = D_1 \oplus D_2.$$

(ii) $JD_1 \perp D_2$ and $JD_2 \perp D_1$.

(iii) For $i = 1, 2$, the distribution D_i is slant with slant angle θ_i.

It is easy to see that the class of bi-slant submanifolds of almost Hermitian manifolds naturally englobes not only the class of slant submanifolds, but also the classes of semi-slant submanifolds [35], hemi-slant submanifolds [36], and CR-submanifolds [37], as synthesized in ([38] Table 1).

In the following, let us denote $d_1 = dimD_1$ and $d_2 = dimD_2$. We say that a bi-slant submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ with slant angles θ_1 and θ_2, respectively, is a proper bi-slant submanifold if $d_1 d_2 \neq 0$ and $0 < \theta_i < \frac{\pi}{2}$, for $i = 1, 2$. If $\mathcal{N}$ is a proper bi-slant submanifold in a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$, then one can check that

$$\sum_{i,j=1}^{n} g^2(Je_i, e_j) = (d_1 cos^2\theta_1 + d_2 cos^2\theta_2). \tag{28}$$

Now, we state and prove the generalized Wintgen inequality for proper bi-slant submanifolds in generalized complex space forms.

Theorem 2. *Let $\mathcal{N}$ be a proper bi-slant submanifold of dimension n in a generalized complex space form $\overline{\mathcal{N}}(f_1, f_2)$ of complex dimension m, with slant angles θ_1, θ_2 and $d_i = dimD_i$, $i = 1, 2$. Then,*

$$\begin{aligned} \rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho + f_1 \\ & + \frac{3f_2}{n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2). \end{aligned} \tag{29}$$

Proof. Let $\{e_1, \ldots, e_{n-1}, e_n\}$ be an orthonormal frame on $\mathcal{N}$ and $\{\xi_1, \ldots, \xi_{2m-n}\}$ be a normal orthonormal frame on $\mathcal{N}$.

Equation (2) can be re-written in view of Equation (1) as

$$\begin{aligned} R(X,Y,Z,W) &= f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ f_2\{g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) \\ &+ 2g(X,JY)g(JZ,W)\} \\ &- g(h(X,Z),h(Y,W)) + g(h(X,W),h(Y,Z)) \end{aligned} \tag{30}$$

and this implies

$$\begin{aligned} \tau &= \sum_{1\le i<j\le n} R(e_i,e_j,e_j,e_i) \\ &= \frac{n(n-1)}{2} f_1 + \frac{3}{2} f_2 \sum_{1\le i<j\le n} g^2(Je_j,e_i) \\ &+ \sum_{r=1}^{2m-n} \sum_{1\le i<j\le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] \\ &= \frac{n(n-1)}{2} f_1 + \frac{3}{2} f_2 (d_1 cos^2\theta_1 + d_2 cos^2\theta_2) \\ &+ \sum_{r=1}^{2m-n} \sum_{1\le i<j\le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \end{aligned} \tag{31}$$

However, we know from the proof of Lemma 1 that

$$n^2\|\mathcal{H}\|^2 - n^2\rho_N \ge \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1\le i<j\le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{32}$$

Combining Equations (31) and (32), we find

$$\begin{aligned} \rho_N &\le \|\mathcal{H}\|^2 - (\rho - f_1) \\ &+ \frac{3f_2}{n(n-1)} (d_1 cos^2\theta_1 + d_2 cos^2\theta_2) \end{aligned} \tag{33}$$

and the proof is now complete. □

Remark 2. *If in the statement of the above theorem one takes $f_1 = f_2 = c$, then $\overline{\mathcal{N}}$ reduces to a complex space form and we can immediately see that Theorem 2 generalizes the generalized Wintgen inequality for the class of proper slant submanifolds in a complex space form, namely ([8] Theorem 3.1).*

5. Generalized Wintgen Inequalities for Submanifolds in Generalized Sasakian Space Form

Let $\overline{\mathcal{N}}$ be an almost contact metric manifold of dimension $(2m+1)$, equipped with the almost contact structure (ϕ, ξ, η, g). Then, it is known that the $(1,1)$ tensor field ϕ, the structure vector field ξ, the 1-form η, and the Riemannian metric g on $\overline{\mathcal{N}}$ verify the compatibility relations

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

These conditions also imply that [39]

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(X) = g(X, \xi)$$

and

$$g(\phi X, Y) + g(X, \phi Y) = 0,$$

for all vector fields X, Y on $\overline{\mathcal{N}}$.

Let $(\overline{\mathcal{N}}, \phi, \xi, \eta, g)$ be an almost contact metric manifold whose curvature tensor satisfies

$$\begin{aligned} \overline{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &\quad - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{34}$$

for all vector fields X, Y, Z on $\overline{\mathcal{N}}$, where f_1, f_2, f_3 are differentiable functions on $\overline{\mathcal{N}}$. Then, $\overline{\mathcal{N}}(f_1, f_2, f_3)$ is said to be a generalized Sasakian space form. It is important to outline that the generalized Sasakian space forms are an umbrella of the following well known spaces:

i. Sasakian space forms, i.e., Sasakian manifolds with constant ϕ-sectional curvature c. In this case, $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$.
ii. Kenmotsu space forms, i.e., Kenmotsu manifolds of constant ϕ-sectional curvature c. In this case, $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$).
iii. cosymplectic space forms, i.e., cosymplectic manifolds of constant ϕ-sectional curvature c. In this case, $f_1 = f_2 = f_3 = \frac{c}{4}$.

For definitions, basic results, and examples of such spaces, the readers are referred to the monographs [39,40].

A Riemannian manifold $\mathcal{N}$ isometrically immersed in an almost contact metric manifold $(\overline{\mathcal{N}}, \phi, \xi, \eta, g))$ is called a C-totally real submanifold of $\overline{\mathcal{N}}$ if the structure vector field ξ is a normal vector field on $\mathcal{N}$. As an immediate consequence of the definition of a C-totally real submanifold, we deduce that ϕ maps any tangent space of $\overline{\mathcal{N}}$ into the normal space. We recall that, if the dimension of the C-totally real submanifold $\mathcal{N}$ is $n = \frac{dim\overline{\mathcal{N}}-1}{2}$, then $\mathcal{N}$ is said to be a Legendrian submanifold. Notice that Legendrian submanifolds are the counterpart in odd dimension of Lagrangian submanifolds investigated in Section 3.

The first aim of this section is to obtain the generalized Wintgen inequality for Legendrian submanifolds in generalized Sasakian space forms. Similar to the case of Lemma 1, we can prove the following.

Lemma 2. *Let $\mathcal{N}$ be a C-totally real submanifold of dimension n in a generalized Sasakian space form $\overline{\mathcal{N}}(f_1, f_2, f_3)$ of dimension $(2m+1)$. Then, we have*

$$\rho_N \leq \|\mathcal{H}\|^2 - \rho + f_1, \tag{35}$$

with the equality case holding at $p \in \mathcal{N}$ if and only if the shape operator S of $\mathcal{N}$ in $\overline{\mathcal{N}}(f_1, f_2, f_3)$ with respect to some suitable orthonormal bases $\{e_1, \ldots, e_n\}$ of $T_p\mathcal{N}$ and $\{\xi_1, \ldots, \xi_{2m-n+1}\}$ of $T_p^{\perp}\mathcal{N}$ takes the following forms

$$S_{\xi_1} = \begin{pmatrix} \gamma_1 & \nu & 0 & \ldots & 0 \\ \nu & \gamma_1 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_1 \end{pmatrix},$$

$$S_{\xi_2} = \begin{pmatrix} \gamma_2 + \nu & 0 & 0 & \ldots & 0 \\ 0 & \gamma_2 - \nu & 0 & \ldots & 0 \\ 0 & 0 & \gamma_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_2 \end{pmatrix},$$

$$S_{\xi_3} = \begin{pmatrix} \gamma_3 & 0 & 0 & \ldots & 0 \\ 0 & \gamma_3 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_3 \end{pmatrix}, \quad S_{\xi_4} = \cdots = S_{\xi_{2m-n+1}} = 0,$$

where $\gamma_1, \gamma_2, \gamma_3$, and ν are real functions on $\mathcal{N}$.

Next, we can state a generalized Wintgen-type inequality for Legendrian submanifolds in a generalized Sasakian ambient.

Theorem 3. *If $\mathcal{N}$ is a Legendrian submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\overline{\mathcal{N}}(f_1, f_2, f_3)$, then*

$$\begin{aligned} (\rho^{\perp})^2 \quad & \leq \quad \left(\|\mathcal{H}\|^2 - \rho + f_1\right)^2 + \frac{2}{n(n-1)} f_2^2 \\ & \quad + \frac{4 f_2}{n(n-1)} (\rho - f_1) \end{aligned} \tag{36}$$

and the equality holds at a point $p \in \mathcal{N}$ if and only if the shape operator takes the forms as in Lemma 2 with respect to some suitable tangent and normal orthonormal bases.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame on $\mathcal{N}$. Due to the fact that $\mathcal{N}$ is a Legendrian submanifold of $\overline{\mathcal{N}}$, it follows that $\{\xi_1 = \phi e_1, \ldots, \xi_n = \phi e_n, \xi_{n+1} = \xi\}$ is an orthonormal frame in the normal bundle of $\mathcal{N}$. Next, the proof is similar to the one of Theorem 2, being based on Lemma 2 instead of Lemma 1, so we omit it. □

Remark 3. *We note that function f_3 does not appear in the generalized Wintgen inequality in Equation (36) for a Legendrian submanifold $\mathcal{N}$ in a generalized Sasakian space form $\overline{\mathcal{N}}(f_1, f_2, f_3)$. This is a consequence of the fact that ξ is normal to $\mathcal{N}$. However, for a submanifold tangent to the structure vector field ξ, the corresponding generalized Wintgen inequality will depend on f_3, as we can see in the second part of this section.*

Remark 4. *Theorem 3 generalizes the main result of [9], namely the generalized Wintgen inequality for the class of Legendrian submanifolds in a Sasakian space form. Actually, if in the statement of Theorem 3, one considers $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, then $\overline{\mathcal{N}}$ reduces to a Sasakian space form and Theorem 3 becomes nothing but ([9] Theorem 3.2).*

Corollary 1. *Let $\mathcal{N}$ be a Legendrian submanifold of a $(2n+1)$-dimensional Kenmotsu space form $\overline{\mathcal{N}}(c)$. Then*

$$\begin{aligned}(\rho^{\perp})^2 \quad \leq \quad & \left(\|\mathcal{H}\|^2 - \rho + \frac{c-3}{4}\right)^2 + \frac{(c+1)^2}{8n(n-1)} \\ & + \frac{c+1}{n(n-1)}\left(\rho - \frac{c-3}{4}\right) \end{aligned} \tag{37}$$

and the equality holds at a point $p \in \mathcal{N}$ if and only if the shape operator takes the forms as in Lemma 2 with respect to some suitable tangent and normal orthonormal bases.

Proof. The proof follows immediately from Theorem 3 by replacing $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$. □

Corollary 2. *Let $\mathcal{N}$ be a Legendrian submanifold of a $(2n+1)$-dimensional cosymplectic space form $\overline{\mathcal{N}}(c)$. Then,*

$$\begin{aligned}(\rho^{\perp})^2 \quad \leq \quad & \left(\|\mathcal{H}\|^2 - \rho + \frac{c}{4}\right)^2 + \frac{c^2}{8n(n-1)} \\ & + \frac{c}{n(n-1)}\left(\rho - \frac{c}{4}\right) \end{aligned} \tag{38}$$

and the equality holds at a point $p \in \mathcal{N}$ if and only if the shape operator takes the forms as in Lemma 2 with respect to some suitable tangent and normal orthonormal bases.

Proof. The proof follows immediately from Theorem 3 by putting $f_1 = f_2 = f_3 = \frac{c}{4}$. □

Remark 5. *We note that the proof of Theorem 3.3 of [41] contains an error. Consequently, Theorem 3.3 of [41] must be replaced by Corollary 1 of the present article.*

In 1996, Lotta [42] introduced the notion of slant submanifold in almost contact geometry as follows. A submanifold $\mathcal{N}$ of an almost contact metric manifold $(\overline{\mathcal{N}}, \phi, \xi, \eta, g)$ tangent to the structure vector field ξ is said to be a contact slant submanifold if, for any point $p \in \mathcal{N}$ and any vector $X \in T_p\mathcal{N}$ linearly independent on ξ_p, the angle between the vector ϕX and the tangent space $T_p\mathcal{N}$ is constant. This constant, usually denoted by θ, is said to be the slant angle of $\mathcal{N}$. We recall that invariant and anti-invariant submanifolds are particular examples of slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A contact slant submanifold is said to be θ-slant or proper if $0 < \theta < \frac{\pi}{2}$. Notice that ([42] Theorem 3.3) implies the dimension of a contact slant submanifold tangent to the structure vector field ξ and with slant angle $\theta \neq \frac{\pi}{2}$ is odd. The concept of contact slant submanifold is further generalized as follows.

Definition 2. *[43] A submanifold $\mathcal{N}$ of an almost contact metric manifold $\overline{\mathcal{N}}$ is said to be a bi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2 on $\mathcal{N}$, such that:*

(i) $T\mathcal{N}$ admits the orthogonal direct decomposition $T\mathcal{N} = D_1 \oplus D_2 \oplus \xi$.
(ii) $JD_1 \perp D_2$ and $JD_2 \perp D_1$.
(iii) For i=1,2, the distribution D_i is slant with slant angle θ_i.

In the following, we denote by d_i the dimension of the distribution D_i, $i = 1, 2$. It is easy to check that, similar to in the case of complex geometry, the class of bi-slant submanifolds of almost contact metric manifolds naturally includes not only the class of slant submanifolds, but also the classes of semi-slant submanifolds [44], hemi-slant submanifolds (also named pseudo-slant submanifolds) [45], and contact CR-submanifolds (also known as semi-invariant submanifolds) [46]. For definitions and basic properties of the above classes of submanifolds, see also [47]. We only recall here that a bi-slant submanifold is called proper if $d_1 d_2 \neq 0$ and the slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$. Notice that various examples of proper bi-slant submanifolds in almost contact metric manifolds can be found in [43,44,48].

Next, we focus on the second aim of this section, that is to derive a generalized Wintgen-type inequality for bi-slant submanifolds in generalized Sasakian space form.

Theorem 4. *Let $\mathcal{N}$ be a proper bi-slant submanifold of dimension n in a generalized Sasakian space form $\overline{\mathcal{N}}(f_1, f_2, f_3)$ of dimension $(2m+1)$, with slant angles θ_1, θ_2 and $dim D_i = d_i$, $i = 1, 2$. Then,*

$$\begin{aligned} \rho_N &\leq \|\mathcal{H}\|^2 - \rho + f_1 \\ &\quad + \frac{3f_2}{n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - \frac{2}{n} f_3. \end{aligned} \tag{39}$$

Proof. First, we remark that the definition of a bi-slant submanifold implies that $d_1 + d_2 + 1 = n$. Next, let $\{e_1, \ldots, e_{d_1}, e_{d_1+1}, \ldots, e_{d_1+d_2}, e_n = \xi\}$ be an orthonormal frame on $\mathcal{N}$ and $\{\xi_1, \ldots, \xi_{2m-n+1}\}$ be a normal orthonormal frame on $\mathcal{N}$.

Using Equations (2) and (34), we obtain

$$\begin{aligned} \tau &= \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_j, e_i) \\ &= \frac{n(n-1)}{2} f_1 + \frac{3}{2} f_2 (d_1 cos^2\theta_1 + d_2 cos^2\theta_2) \\ &\quad + (1-n) f_3 + \sum_{r=1}^{2m-n+1} \sum_{1 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \end{aligned} \tag{40}$$

However, as in the proof of Lemma 1, we get

$$n^2 \|\mathcal{H}\|^2 - n^2 \rho_N \geq \frac{2n}{n-1} \sum_{r=1}^{2m-n+1} \sum_{1 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{41}$$

Combining now Equations (40) and (41), we obtain Equation (44) and the conclusion follows. □

As immediate consequences of Theorem 4, we derive the following results.

Corollary 3. *Let $\mathcal{N}$ be a proper bi-slant submanifold of dimension n in a Sasakian space form $\overline{\mathcal{N}}(c)$ of dimension $(2m+1)$, with slant angles θ_1, θ_2 and $dim D_i = d_i$, $i = 1, 2$. Then,*

$$\begin{aligned} \rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho + \frac{c+3}{4} \\ & + \frac{3(c-1)}{4n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - \frac{c-1}{2n}. \end{aligned} \tag{42}$$

Corollary 4. *Let $\mathcal{N}$ be a proper bi-slant submanifold of dimension n in a Kenmotsu space form $\overline{\mathcal{N}}(c)$ of dimension $(2m+1)$, with slant angles θ_1, θ_2 and $dim D_i = d_i$, $i = 1, 2$. Then,*

$$\begin{aligned} \rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho + \frac{c-3}{4} \\ & + \frac{3(c+1)}{4n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - \frac{c+1}{2n}. \end{aligned} \tag{43}$$

Corollary 5. *Let $\mathcal{N}$ be a proper bi-slant submanifold of dimension n in a cosymplectic space form $\overline{\mathcal{N}}(c)$ of dimension $(2m+1)$, with slant angles θ_1, θ_2 and $dim D_i = d_i$, $i = 1, 2$. Then,*

$$\begin{aligned} \rho_N \quad \leq \quad & \|\mathcal{H}\|^2 - \rho + \frac{c}{4} \\ & + \frac{3c}{4n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - \frac{c}{2n}. \end{aligned} \tag{44}$$

Remark 6. *Corollary 3 generalizes Theorem 4.1 of [9].*

Remark 7. *We note that the authors of [8,9] provided non-trivial examples of Lagrangian and Legendrian submanifolds satisfying the equality case of the corresponding Wintgen-type inequalities stated in this paper, because the shape operators have the appropriate form (see also [49]).*

6. The First Fundamental Equation of Generalized Space Forms

For a given Riemannian manifold $(\overline{\mathcal{N}}, g)$, let us denote by $\overline{\nabla}$ the Levi–Civita connection of the metric g and by $\overline{R}$ the curvature tensor of $\overline{\nabla}$. We consider the differential operator $D_{\overline{\nabla}}$ defined in the tangent vector bundle $T\overline{\mathcal{N}}$ with values belonging to the vector bundle $hom(\otimes^2 T\overline{\mathcal{N}}, T\overline{\mathcal{N}})$. Hence, for a given vector field X on $\overline{\mathcal{N}}$, we have that $D_{\overline{\nabla}}(X)$ is a section of the vector bundle $T\overline{\mathcal{N}} \otimes T^{*\otimes 2}\overline{\mathcal{N}}$ defined by

$$D_{\overline{\nabla}}(X) = \overline{\nabla}^2 X.$$

Obviously, the complete expression is

$$D_{\overline{\nabla}}(X)(Y, Z) = \overline{\nabla}_Y \overline{\nabla}_Z X - \overline{\nabla}_{\overline{\nabla}_Y Z} X, \; \forall Y, Z \in \mathcal{X}(M).$$

We recall now that the first fundamental equation of $(\overline{\mathcal{N}}, \overline{\nabla})$ is the second-order differential equation [50]

$$D_{\overline{\nabla}}(X) = 0. \tag{45}$$

In the following, we denote by $\mathcal{J}_{\overline{\nabla}}$ the sheaf of germs of solutions to Equation (45) and by $J_{\overline{\nabla}}$ the vector space of sections of $\mathcal{J}_{\overline{\nabla}}$.

We would like to investigate next the consequences of the condition $dim \mathcal{J}_{\overline{\nabla}} > 0$, i.e., the first fundamental in Equation (45) admits non-null solutions, on the geometry and topology of generalized complex space forms and generalized Sasakian space forms. Before answering the above question, we need the following.

Proposition 1. *Let $\overline{\nabla}$ be the Levi–Civita connection of a Riemannian metric g on a manifold $\overline{\mathcal{N}}$. If Z is a solution to the first fundamental equation of $(\overline{\mathcal{N}}, \overline{\nabla})$, then one has*

(i) $\overline{R}(X, Y)Z = 0$,
(ii) $\overline{R}(X, Z)Y = 0$,

for all vector fields X, Y on M.

Proof. (i) If Z is a solution to the first fundamental equation of $(\overline{\mathcal{N}}, \overline{\nabla})$, then

$$D_{\overline{\nabla}}(Z)(X, Y) = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_{\overline{\nabla}_X Y} Z = 0, \tag{46}$$

for all vector fields X, Y on M.

However, since the connection $\overline{\nabla}$ is torsion-free, we can express its Riemann curvature tensor $\overline{R}$ by

$$\overline{R}(X, Y)Z = D_{\overline{\nabla}}(Z)(X, Y) - D_{\overline{\nabla}}(Z)(Y, X). \tag{47}$$

Consequently, from Equations (46) and (47), we derive $\overline{R}(X, Y)Z = 0$.
(ii) Using (i) and the Bianchi identity, one has

$$\overline{R}(Z, X)Y = \overline{R}(Z, Y)X. \tag{48}$$

Then, we have

$$\begin{aligned} g(\overline{R}(Z, X)Y, W) &= -g(Y, \overline{R}(Z, X)W) \\ &= -g(Y, \overline{R}(Z, W)X) \\ &= g(\overline{R}(Z, W)Y, X) \\ &= g(\overline{R}(Z, Y)W, X) \\ &= -g(W, \overline{R}(Z, Y)X) \\ &= -g(W, \overline{R}(Z, X)Y) \\ &= -g(\overline{R}(Z, X)Y, W), \end{aligned} \tag{49}$$

which implies

$$g(\overline{R}(Z, X)Y, W) = 0$$

and the conclusion is now clear. □

Theorem 5. *Let $\overline{\mathcal{N}}(f_1, f_2)$ be a generalized complex space form of real dimension $2m > 2$. If $\dim \mathcal{J}_{\overline{\nabla}} > 0$, then $\overline{\mathcal{N}}$ is flat. Moreover, $\overline{\mathcal{N}}$ admits a normal Riemannian covering by a flat $2m$-dimensional torus, provided that the manifold is compact and connected.*

Proof. Let Z be a non-null solution of the first fundamental equation of $(\overline{\mathcal{N}}(f_1, f_2), \overline{\nabla})$, where $\overline{\nabla}$ is the Levi–Civita connection on $\overline{\mathcal{N}}(f_1, f_2)$. Then, using Equation (1) and Proposition 1 (i), we get

$$\begin{aligned} &f_1\{g(Z, Y)X - g(Z, X)Y\} \\ &+ f_2\{g(JZ, X)JY - g(JZ, Y)JX\} = -2f_2 g(X, JY)JZ, \end{aligned} \tag{50}$$

for all vector fields X, Y on $\overline{\mathcal{N}}$.

In addition, using Equation (1) and Proposition 1 (ii), we obtain

$$\begin{aligned} & f_1 g(Z,Y)X + f_2\{2g(X,JZ)JY - g(Z,JY)JX\} \\ & = f_1 g(X,Y)Z - f_2 g(X,JY)JZ. \end{aligned} \tag{51}$$

Replacing now $X = Z$ and $Y = JZ$ in Equation (50), we derive:

$$(f_1 + 3f_2)g(Z,Z)JZ = 0$$

and therefore we obtain

$$3f_2 + f_1 = 0. \tag{52}$$

Combining Equations (51) and (52), we get

$$\begin{aligned} & f_2\{-3g(X,Y)Z - g(X,JY)JZ\} \\ & +f_2\{3g(Z,Y)X - 2g(X,JZ)JY + g(Z,JY)JX\} = 0 \end{aligned} \tag{53}$$

and choosing $Y = X$ in Equation (53) we derive

$$f_2\{-3g(X,X)Z + 3g(Z,X)X - 3g(X,JZ)JX\} = 0. \tag{54}$$

Now, because $m > 1$, we can choose a vector field X on $\overline{\mathcal{N}}$ subjected to

1. $g(X,JZ) = 0$,
2. $g(X,Z) = 0$,

and therefore Equation (54) yields

$$3f_2 g(X,X)Z = 0. \tag{55}$$

Thereby,

$$f_2 = 0$$

and, from Equation (52), we also derive

$$f_1 = 0.$$

Thus, Equation (1) implies that $\overline{\mathcal{N}}$ is flat and the conclusion follows immediately (see ([51] Theorem 3.3.1)). □

Theorem 6. *Let $\overline{\mathcal{N}}(f_1, f_2, f_3)$ be a generalized Sasakian space form of dimension $2m+1 > 3$. If the first fundamental equation admits solutions linearly independent on the structure vector field ξ, then $\overline{\mathcal{N}}$ is flat. Moreover, $\overline{\mathcal{N}}$ admits a normal Riemannian covering by a flat $(2m+1)$-dimensional torus, provided that the manifold is compact and connected.*

Proof. Let Z be a solution to the first fundamental equation of $(\overline{\mathcal{N}}(f_1, f_2, f_3), \overline{\nabla})$ linearly independent on the structure vector field ξ, where $\overline{\nabla}$ is the Levi–Civita connection on $\overline{\mathcal{N}}(f_1, f_2, f_3)$. Then, using Equation (34) and Proposition 1, we get the following identities:

$$\begin{aligned}
&f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\} \\
&+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\
&= -2f_2 g(X,\phi Y)\phi Z - f_3\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi, \qquad (56)\\
&f_1 g(Z,Y)X + f_2\{2g(X,\phi Z)\phi Y - g(Z,\phi Y)\phi X\} \\
&- f_3\eta(Z)\eta(Y)X = f_1 g(X,Y)Z - f_2 g(X,\phi Y)\phi Z \\
&- f_3\{\eta(X)\eta(Y)Z + g(X,Y)\eta(Z)\xi - g(Z,Y)\eta(X)\xi\} \qquad (57)
\end{aligned}$$

for all vector fields X, Y on $\overline{\mathcal{N}}$.

Choosing now in Equation (56) the vector field X to be orthogonal to Z, ϕZ, and ξ, we derive

$$\{f_1 g(Y,Z) - f_3\eta(Y)\eta(Z)\}X - f_2 g(Y,\phi Z)\phi X + 2f_2 g(X,\phi Y)\phi Z = 0 \qquad (58)$$

and, particularizing $Y = \phi Z$ in Equation (58), one immediately gets

$$f_2 = 0. \qquad (59)$$

Therefore, Equations (56) and (57) become

$$\begin{aligned}
&f_1\{g(Y,Z)X - g(X,Z)Y\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\
&= -f_3\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi, \qquad (60)\\
&f_1 g(Z,Y)X - f_3\eta(Z)\eta(Y)X = f_1 g(X,Y)Z \\
&- f_3\{\eta(X)\eta(Y)Z + g(X,Y)\eta(Z)\xi - g(Z,Y)\eta(X)\xi\}, \qquad (61)
\end{aligned}$$

for all vector fields X, Y on $\overline{\mathcal{N}}$.

Similarly, considering in Equation (61) the vector field X to be orthogonal to Z, ϕZ, and ξ, we deduce

$$f_1 g(Z,Y)X + f_3\eta(Z)\eta(Y)X = f_1 g(X,Y)Z - f_3 g(X,Y)\eta(Z)\xi \qquad (62)$$

and, particularizing $Y = X$ in Equation (58), one obtains

$$f_1 g(X,X)Z - f_3 g(X,X)\eta(Z)\xi = 0. \qquad (63)$$

As Z and ξ are linearly independent, Equation (63) implies

$$f_1 = 0 \qquad (64)$$

and

$$f_3\eta(Z) = 0. \qquad (65)$$

Now, we have to distinguish two cases.

Case I: $\eta(Z) \neq 0$. Then, it follows from Equation (65) that

$$f_3 = 0 \qquad (66)$$

and replacing Equations (59), (64), and (66) in Equation (1), we conclude that $\overline{\mathcal{N}}$ is flat.
Case II: $\eta(Z) = 0$. Then, taking account of Equation (64), we obtain from Equation (60) that

$$f_3\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi = 0. \quad (67)$$

Particularizing now $X = Z$ and $Y = \xi$ in Equation (67), one obtains also Equation (66) and therefore we reach again the required conclusion. □

Remark 8. *Theorems 5 and 6 provide obstructions to the existence of non-flat generalized space forms. Therefore, the existence of non-null solutions for the first fundamental equation of a generalized complex space form* $\overline{\mathcal{N}}(f_1, f_2)$ *implies the flatness of this space. On the other hand, the existence of solutions linearly independent on the structure vector field for the first fundamental equation of a generalized Sasakian space form* $\overline{\mathcal{N}}(f_1, f_2, f_3)$ *also implies that its Riemannian curvature tensor vanishes identically.*

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