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On the Study of Fixed Points for a New Class of α-Admissible Mappings

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Abstract: In this paper, we discuss the existence of fixed points for new classes of mappings. Some examples are presented to illustrate our results.

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1. Introduction

The Banach contraction principle is one of the most famous and important results in metric fixed point theory. It is a useful tool in establishing existence results in nonlinear analysis. This principle has been extended and generalized by several authors in many directions (see e.g., [1–15], and the references therein).

In [16], the author introduced the class of F-contractions, and established a fixed point result for this class of mappings, which generalizes the Banach contraction principle. The main result in [16] can be stated as follows.

Theorem 1. *Let (X,d) be a complete metric space, and let $T: X \to X$ be a mapping satisfying*

$$\tau + F(d(Tx,Ty)) \le F(d(x,y)), \quad (1)$$

for all $(x,y) \in X \times X$ with $d(Tx,Ty) > 0$, where $\tau > 0$ is a constant and $F: (0,+\infty) \to \mathbb{R}$ is a function satisfying

(a) F is nondecreasing.
(b) For every sequence $\{t_n\} \subset (0,+\infty)$, we have

$$\lim_{n\to+\infty} F(t_n) = -\infty \iff \lim_{n\to+\infty} t_n = 0.$$

(c) There exists $k \in (0,1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$.

Then T has a unique fixed point. Moreover, for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to this fixed point.

Observe that, if $T: X \to X$ is a q-contraction for some $0 < q < 1$, i.e.,

$$d(Tx,Ty) \le q d(x,y), \ (x,y) \in X \times X,$$

then T satisfies (1) with $F(t) = \ln t$, $t > 0$, and $\tau = -\ln q$. Therefore, the Banach contraction principle follows from Theorem 1.

For different extensions and generalizations of Theorem 1, we refer the reader to [17–27], and the references therein.

In [5], Ćirić introduced a class of mappings with a non-unique fixed point and he established the following fixed point result.

Theorem 2. *Let (X,d) be a complete metric space, and let $T : X \to X$ be a continuous mapping satisfying*

$$\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} - \min\{d(x,Ty), d(y,Tx)\} \leq qd(x,y), \tag{2}$$

for all $(x,y) \in X \times X$, where $0 < q < 1$ is a constant. Then, for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T.

An example was presented in [5] to show that the set of fixed points of mappings satisfying the condition of Theorem 2 contains in general more than one element.

In this paper, we first introduce the class of generalized Ćirić-contractions by combining the ideas in [5,16]. Next, a fixed point result is established for this class of mappings. Our result generalizes Theorem 2 and extends Theorem 1. Next, we introduce a more general class of mappings using the concept of α-admissibility introduced in [28] (see also [29]). Our fixed point result for this class of mappings has several consequences. It is not only a generalization of Theorems 1 and 2, but generalizes most fixed point theorems dealing with F-contractions, linear contractions, and many others. Several examples are presented to illustrate this fact.

Throughout this paper, we denote by $\mathbb{N}$ the set of natural numbers, that is, $\mathbb{N} = \{0,1,2,\cdots\}$. We denote by $\mathbb{N}^*$ the set $\mathbb{N}\backslash\{0\}$. Let $T : X \to X$ be a certain self-mapping on X. For $n \in \mathbb{N}$, we denote by T^n the nth-iterate of T (we suppose that T^0 is the identity mapping on X).

2. The Class of Generalized Ćirić-Contractions

Let Ψ be the set of functions $\psi : [0,+\infty) \to (-\infty, 0)$ such that ψ is upper semi-continuous from the right. We denote by Φ the set of functions $\varphi : (0,+\infty) \to \mathbb{R}$ such that

(Φ_1) φ is non-decreasing, i.e., $0 < t < s \implies \varphi(t) \leq \varphi(s)$.

(Φ_2) For every sequence $\{t_n\} \subset (0,+\infty)$,

$$\lim_{n\to+\infty} \varphi(t_n) = -\infty$$

if and only if

$$\lim_{n\to+\infty} t_n = 0.$$

(Φ_3) There exists $k \in (0,1)$ such that $\lim_{t\to 0^+} t^k \varphi(t) = 0$.

Let (X,d) be a metric space. For a given mapping $T : X \to X$, let

$$M_T(x,y) = \min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} - \min\{d(x,Ty), d(y,Tx)\}, \ (x,y) \in X \times X.$$

Definition 1. *A mapping $T : X \to X$ is said to be a generalized Ćirić-contraction, if there exists $(\varphi,\psi) \in \Phi \times \Psi$ such that*

$$\varphi(M_T(x,y)) \leq \varphi(d(x,y)) + \psi(d(x,y)), \tag{3}$$

for all $(x,y) \in X \times X$ with $M_T(x,y) > 0$.

We have the following fixed point result.

Theorem 3. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a continuous mapping. If T is a generalized Ćirić-contraction for some $(\varphi, \psi) \in \Phi \times \Psi$, then for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T.*

Proof. Let $x \in X$ be fixed, and let $\{x_n\} \subset X$ be the sequence defined by

$$x_n = T^n x, \quad n \in \mathbb{N}.$$

If $x_{p+1} = x_p$ for some $p \in \mathbb{N}$, then x_p will be a fixed point of T. Therefore, we may assume that

$$d(x_n, x_{n+1}) > 0, \quad n \in \mathbb{N}. \tag{4}$$

On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} M_T(x_n, x_{n+1}) &= M_T(T^n x, T^{n+1} x) \\ &= \min\{d(T^{n+1}x, T^{n+2}x), d(T^n x, T^{n+1}x), d(T^{n+1}x, T^{n+2}x)\} \\ &\quad - \min\{d(T^n x, T^{n+2}x), d(T^{n+1}x, T^{n+1}x)\} \\ &= \min\{d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\}. \end{aligned}$$

Therefore, from (4), we have

$$M_T(x_n, x_{n+1}) > 0, \quad n \in \mathbb{N}.$$

From (3), we obtain

$$\varphi(M_T(x_n, x_{n+1})) \le \varphi(d(x_n, x_{n+1})) + \psi(d(x_n, x_{n+1})), \quad n \in \mathbb{N}.$$

If for some $n \in \mathbb{N}$, we have $M_T(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then we obtain

$$\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_n, x_{n+1})) + \psi(d(x_n, x_{n+1})),$$

that is,

$$0 \le \psi(d(x_n, x_{n+1})),$$

which is a contradiction with the fact that $\psi(t) < 0$, for all $t > 0$. As a consequence, we have

$$M_T(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2}), \quad n \in \mathbb{N}.$$

Hence, we find

$$\varphi(d(x_{n+1}, x_{n+2})) \le \varphi(d(x_n, x_{n+1})) + \psi(d(x_n, x_{n+1})), \quad n \in \mathbb{N}. \tag{5}$$

Taking $n = 0$ in (5), we obtain

$$\varphi(d(x_1, x_2)) \le \varphi(d(x_0, x_1)) + \psi(d(x_0, x_1)).$$

Taking $n = 1$ in (5) and using the above inequality, we obtain

$$\begin{aligned} \varphi(d(x_2, x_3)) &\le \varphi(d(x_1, x_2)) + \psi(d(x_1, x_2)) \\ &\le \varphi(d(x_0, x_1)) + \psi(d(x_0, x_1)) + \psi(d(x_1, x_2)). \end{aligned}$$

Continuing this process, by induction we have

$$\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_0, x_1)) + \sum_{i=0}^{n-1} \psi(d(x_i, x_{i+1})), \quad n \in \mathbb{N}^*. \tag{6}$$

Next, let us denote by $\{u_n\}$ the real sequence defined by

$$u_n = d(x_n, x_{n+1}), \quad n \in \mathbb{N}.$$

Observe that from (5), and using (Φ_1) and the fact that $\psi(t) < 0$ for all $t > 0$, we deduce that $\{u_n\}$ is a decreasing sequence. Therefore, there exists some $r \ge 0$ such that

$$u_n \downarrow r \text{ as } n \to +\infty.$$

Since ψ is upper semi-continuous from the right, there exists some $N \in \mathbb{N}$ such that

$$\psi(u_p) < \psi(r) - \frac{\psi(r)}{2} = \frac{\psi(r)}{2}, \quad p \ge N. \tag{7}$$

Further, using (6) and the fact that $\psi(t) < 0$ for all $t > 0$, we obtain

$$\varphi(u_n) \le \varphi(u_0) + \sum_{i=N}^{n-1} \psi(u_i), \quad n \ge N+1.$$

Therefore, from (7) we deduce that

$$\varphi(u_n) \le \varphi(u_0) + \frac{(n-N)}{2}\psi(r), \quad n \ge N+1. \tag{8}$$

Let $n \to +\infty$ in (8) and we obtain

$$\lim_{n\to+\infty} \varphi(u_n) = -\infty,$$

which implies from (Φ_2) that

$$\lim_{n\to+\infty} u_n = 0 = r. \tag{9}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. From (Φ_3) and (9), there exists some $k \in (0,1)$ such that

$$\lim_{n\to+\infty} u_n^k \varphi(u_n) = 0. \tag{10}$$

Using (8), we obtain

$$u_n^k \varphi(u_n) - u_n^k \varphi(u_0) \le \frac{(n-N)}{2}\psi(r) u_n^k \le 0, \quad n \ge N+1.$$

Let $n \to +\infty$, and using (9) and (10), we deduce that

$$\lim_{n\to+\infty} n u_n^k = 0.$$

Then there exists some $q \in \mathbb{N}$ such that

$$u_n < \frac{1}{n^{1/k}}, \quad n \ge q. \tag{11}$$

Using (11) and the triangle inequality, for $n \geq q$ and $m \in \mathbb{N}^*$, we have

$$d(x_n, x_{n+m}) \leq \sum_{i=n}^{n+m-1} u_i \leq \sum_{i=n}^{+\infty} \frac{1}{i^{1/k}}.$$

The convergence of the Riemann series $\sum_n \frac{1}{n^{1/k}}$ (since $0 < k < 1$) yields $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists some $\omega \in X$ such that

$$\lim_{n\to+\infty} d(T^n x, \omega) = \lim_{n\to+\infty} d(x_n, \omega) = 0.$$

The continuity of T yields

$$\lim_{n\to+\infty} d(T^{n+1} x, T\omega) = 0.$$

Finally, the uniqueness of the limit implies that $\omega = T\omega$, i.e., ω is a fixed point of T. □

Let us give some examples to illustrate the result given by Theorem 3.

Example 1. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a continuous mapping. Let $F : (0, +\infty) \to \mathbb{R}$ be a function that belongs to Φ. Suppose that there exists a constant $\tau > 0$ such that*

$$\tau + F(M_T(x, y)) \leq F(d(x, y)), \tag{12}$$

for all $(x, y) \in X \times X$ with $M_T(x, y) > 0$. Then for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T. In order to prove this result, we apply Theorem 3 with $(\varphi, \psi) = (F, -\tau)$.

Example 2. *Suppose that all the assumptions of Theorem 2 are satisfied. Then T satisfies (3) with $\varphi(t) = \ln t$, $t > 0$, and $\psi \equiv \ln q$. Therefore, the result of Theorem 2 follows from Theorem 3.*

Example 3. *Let*

$$X = \left\{ x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}^* \right\}.$$

We endow X with the metric

$$d(x, y) = |x - y|, \quad (x, y) \in X \times X.$$

Then (X, d) is a complete metric space. Consider the mapping $T : X \to X$ defined by

$$Tx_1 = x_1 \quad \text{and} \quad Tx_{n+1} = x_n, \ n \in \mathbb{N}^*.$$

One observes easily that

$$\{(x, y) \in X \times X : M_T(x, y) > 0\} = \{(x_n, x_{n+1}) : n \in \mathbb{N}^*\}.$$

Furthermore, for all $n \in \mathbb{N}^*$, one has

$$\frac{M_T(x_n, x_{n+1})}{d(x_n, x_{n+1})} = \frac{n}{n+1} \to 1 \text{ as } n \to \infty,$$

which shows that (2) is not satisfied. Hence Theorem 2 cannot be applied in this case. On the other hand, taking $\tau = 1$ and

$$F(t) = t + \ln t, \quad t > 0,$$

one obtains

$$\begin{aligned}\tau + F(M_T(x_n, x_{n+1})) &= 1 + F(n) \\ &= 1 + n + \ln n \\ &\le 1 + n + \ln(n+1) \\ &= F(d(x_n, x_{n+1})),\end{aligned}$$

for all $n \in \mathbb{N}^*$. *Hence* (12) *is satisfied for all* $(x, y) \in X \times X$ *with* $M_T(x, y) > 0$. *Therefore, by Example* 1, *one deduces that* T *has a fixed point* $x^* \in X$. *In this case, one observes that* $x^* = x_1 = 1$.

3. A Larger Class of Mappings

In this part, we discuss the existence of fixed points for a larger class of mappings than the one studied in the previous section. First, let us recall some concepts introduced recently by Samet in [29] (see also [28]).

Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a given function.

Definition 2. *Let* $\{x_n\} \subset X$ *be a given sequence. We say that* $\{x_n\}$ *is* α*-regular if*

$$\alpha(x_n, x_{n+1}) \ge 1, \quad n \in \mathbb{N}.$$

Definition 3. *We say that* $T : X \to X$ *is* α*-admissible if*

$$(x, y) \in X \times X,\ \alpha(x, y) \ge 1 \implies \alpha(Tx, Ty) \ge 1.$$

Definition 4. *We say that* $T : X \to X$ *is* α*-continuous if for every* α*-regular sequence* $\{x_n\} \subset X$ *and* $u \in X$,

$$\lim_{n \to +\infty} d(x_n, u) = 0$$

implies that there exists a sub-sequence $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that*

$$\lim_{k \to +\infty} d(Tx_{n_k}, Tu) = 0.$$

Definition 5. *Let* $\{x_n\} \subset X$ *be a given sequence. We say that* $\{x_n\}$ *is* α*-Cauchy if*

(i) $\{x_n\}$ *is* α*-regular.*
(ii) $\{x_n\}$ *is a Cauchy sequence.*

Definition 6. *We say that* (X, d) *is* α*-complete if every* α*-Cauchy sequence is convergent.*

Next, we introduce the following class of mappings.

Let $\mathcal{T}_\alpha$ be the class of mappings $T : X \to X$ satisfying the following conditions:

($\mathcal{T}_1$) T is α-continuous.
($\mathcal{T}_2$) There exists $(\varphi, \psi) \in \Phi \times \Psi$ such that for all $(x, y) \in X \times X$ with $d(Tx, Ty) > 0$,

$$\alpha(x, y) \exp\left(\varphi(d(Tx, Ty))\right) \le \exp\left(\varphi(d(x, y)) + \psi(d(x, y))\right).$$

We now give some examples of mappings $T : X \to X$ that belong to the set $\mathcal{T}_\alpha$, for some $\alpha : X \times X \to \mathbb{R}$. Let (X, d) be a metric space.

Proposition 1 (The class of generalized Ćirić-contractions)**.** *Let* $T : X \to X$ *be a continuous mapping. If* T *is a generalized Ćirić-contraction, then there exists a function* $\alpha : X \times X \to \mathbb{R}$ *such that* $T \in \mathcal{T}_\alpha$.

Proof. Let us consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if} \quad y = Tx, \\ 0 & \text{if} \quad y \neq Tx. \end{cases} \tag{13}$$

Let $(x,y) \in X \times X$ be such that $d(Tx,Ty) > 0$. We discuss two possible cases.
Case 1: $y \neq Tx$. In this case,

$$\alpha(x,y) \exp\left(\varphi(d(Tx,Ty))\right) = 0 \leq \exp\left(\varphi(d(x,y)) + \psi(d(x,y))\right).$$

Case 2: $y = Tx$. In this case, we have

$$\begin{aligned} M_T(x,y) &= M_T(x,Tx) \\ &= \min\{d(Tx,T^2x), d(x,Tx)\}. \end{aligned}$$

Since $d(Tx,T^2x) = d(Tx,Ty) > 0$, we have $d(x,Tx) > 0$. Therefore, $M_T(x,y) > 0$. Using the fact that T is a generalized Ćirić-contraction, we deduce that

$$\varphi(M_T(x,Tx)) \leq \varphi(d(x,Tx)) + \psi(d(x,Tx)),$$

that is,

$$\varphi(\min\{d(Tx,T^2x), d(x,Tx)\}) \leq \varphi(d(x,Tx)) + \psi(d(x,Tx)),$$

which yields (since $\psi(t) < 0$, for all $t > 0$)

$$\varphi(d(Tx,T^2x)) \leq \varphi(d(x,Tx)) + \psi(d(x,Tx)).$$

Hence, we obtain

$$\alpha(x,Tx) \exp\left(\varphi(d(Tx,T^2x))\right) \leq \exp\left(\varphi(d(x,Tx)) + \psi(d(x,Tx))\right).$$

Therefore, T satisfies ($\mathcal{T}_2$) with α given by (13). Obviously, since T is continuous, then T is α-continuous. Then T satisfies ($\mathcal{T}_1$). As a consequence, we have $T \in \mathcal{T}_\alpha$. □

Proposition 2 (The class of F-contractions)**.** *Let $T : X \to X$ be an F-contraction, for some $F \in \Phi$, that is, there exists a constant $\tau > 0$ such that*

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

for all $(x,y) \in X \times X$ with $d(Tx,Ty) > 0$. Then there exists a function $\alpha : X \times X \to \mathbb{R}$ such that $T \in \mathcal{T}_\alpha$.

Proof. Let

$$\alpha(x,y) = 1, \quad (x,y) \in X \times X. \tag{14}$$

Let $\varphi = F$ and $\psi \equiv -\tau$. Then $(\varphi,\psi) \in \Phi \times \Psi$. Let $(x,y) \in X \times X$ be such that $d(Tx,Ty) > 0$. Then

$$\varphi(d(Tx,Ty)) \leq \varphi(d(x,y)) + \psi(d(x,y)),$$

which yields

$$\alpha(x,y) \exp\left(\varphi(d(Tx,Ty))\right) \leq \exp\left(\varphi(d(x,y)) + \psi(d(x,y))\right).$$

Then T satisfies $\mathcal{T}_2$ with α given by (14). On the other hand, it can be easily seen that any F-contraction is continuous, so it is α-continuous. Then T satisfies also $\mathcal{T}_1$. As a consequence, we have $T \in \mathcal{T}_\alpha$. □

Proposition 3. *Let $T : X \to X$ be an orbitally continuous mapping, that is, for every $x \in X$, if*

$$\lim_{n\to+\infty} d(T^n x, u) = 0,\ u \in X,$$

then

$$\lim_{n\to+\infty} d(TT^n x, Tu) = 0.$$

Suppose that there exist $F \in \Phi$ and a constant $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \le F(N_T(x, y)), \tag{15}$$

for all $(x, y) \in X \times X$ with $d(Tx, Ty) > 0$, where

$$N_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Then there exists a function $\alpha : X \times X \to \mathbb{R}$ such that $T \in \mathcal{T}_\alpha$.

Proof. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by (13). Let $\varphi = F$ and $\psi \equiv -\tau$. Then $(\varphi, \psi) \in \Phi \times \Psi$. Let $(x, y) \in X \times X$ be such that $d(Tx, Ty) > 0$. We discuss two possible cases.
Case 1. $y \ne Tx$. In this case,

$$\alpha(x, y) \exp\left(\varphi(d(Tx, Ty))\right) = 0 \le \exp\left(\varphi(d(x, y)) + \psi(d(x, y))\right).$$

Case 2. $y = Tx$. In this case,

$$N_T(x, y) = \max\left\{d(x, Tx), d(Tx, T^2x), \frac{d(x, T^2x)}{2}\right\}.$$

On the other hand, by the triangle inequality, we have

$$\frac{d(x, T^2x)}{2} \le \frac{d(x, Tx) + d(Tx, T^2x)}{2} \le \max\{d(x, Tx), d(Tx, T^2x)\}.$$

Therefore,

$$N_T(x, y) = \max\left\{d(x, Tx), d(Tx, T^2x)\right\}.$$

Suppose that $N_T(x, y) = d(Tx, T^2x)$. Then by (15), we have

$$\tau + \varphi(d(Tx, T^2x)) \le \varphi(d(Tx, T^2x)),$$

which yields $\tau \le 0$, which is a contradiction. Then we have $N_T(x, y) = d(x, Tx)$. Again, by (15), we deduce that

$$\varphi(d(Tx, T^2x)) \le \varphi(d(x, Tx)) + \psi(d(x, Tx)),$$

which yields

$$\alpha(x, Tx) \exp\left(\varphi(d(Tx, T^2x))\right) \le \exp\left(\varphi(d(x, Tx)) + \psi(d(x, Tx))\right).$$

Then T satisfies $\mathcal{T}_2$ with α given by (13). Next, we prove that T is α-continuous. Let $\{x_n\} \subset X$ be an α-regular sequence. By the definition of α, this means that

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N},$$

that is,

$$x_n = T^n x_0, \quad n \in \mathbb{N}.$$

Suppose that there exists $u \in X$ such that

$$\lim_{n\to+\infty} d(x_n, u) = \lim_{n\to+\infty} d(T^n x_0, u) = 0.$$

Since T is orbitally continuous, we obtain

$$\lim_{n\to+\infty} d(Tx_n, Tu) = 0.$$

Then T is α-continuous, and it satisfies $(\mathcal{T}_1)$. As a consequence, we have $T \in \mathcal{T}_\alpha$. □

Remark 1. *Let $T : X \to X$ be a given mapping. Suppose that there exists a constant $0 < q < 1$ such that*

$$d(Tx, Ty) \leq qN_T(x, y), \quad (x, y) \in X \times X.$$

It can be easily seen that T is orbitally continuous mapping, and it satisfies (15) with $\tau = -\ln q$ and $F(t) = \ln t$, $t > 0$. Therefore, $T \in \mathcal{T}_\alpha$, where α is given by (13) and $(\phi, \psi) = (F, -\ln q)$.

Proposition 4. *Let $T : X \to X$ be an orbitally continuous mapping. Suppose that there exist $F \in \Phi$ and a constant $\tau > 0$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(\mu_T(x, y)), \tag{16}$$

for all $(x, y) \in X \times X$ with $d(Tx, Ty) > 0$, where

$$\mu_T(x, y) = \max\left\{d(x, y), d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}\right\}.$$

Then there exists a function $\alpha : X \times X \to \mathbb{R}$ such that $T \in \mathcal{T}_\alpha$.

Proof. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by (13). Let $\varphi = F$ and $\psi \equiv -\tau$. Then $(\varphi, \psi) \in \Phi \times \Psi$. Let $(x, y) \in X \times X$ be such that $d(Tx, Ty) > 0$. We discuss two possible cases.
Case 1. $y \neq Tx$. In this case, we have

$$\alpha(x, y) \exp\left(\varphi(d(Tx, Ty))\right) = 0 \leq \exp\left(\varphi(d(x, y)) + \psi(d(x, y))\right).$$

Case 2. $y = Tx$. In this case,

$$\mu_T(x, y) = \max\left\{d(x, Tx), d(Tx, T^2x)\right\}.$$

If $\mu_T(x, y) = d(Tx, T^2x)$, then by (16), we have

$$\tau + F(d(Tx, T^2x)) \leq F(d(Tx, T^2x)),$$

that is

$$\tau \leq 0,$$

which is a contradiction. Therefore, $\mu_T(x, y) = d(x, Tx)$. Again, by (16), we deduce that

$$\varphi(d(Tx, T^2x)) \leq \varphi(d(x, Tx)) + \psi(d(x, Tx)),$$

which yields

$$\alpha(x,Tx)\exp\left(\varphi(d(Tx,T^2x))\right) \le \exp\left(\varphi(d(x,Tx)) + \psi(d(x,Tx))\right).$$

Then T satisfies $\mathcal{T}_2$ with α given by (13). Since T is orbitally continuous, from the proof of Proposition 3, T is α-continuous, and it satisfies $\mathcal{T}_1$. As a consequence, we have $T \in \mathcal{T}_\alpha$. □

Remark 2. *Let $T : X \to X$ be a given mapping. Suppose that there exists a constant $0 < q < 1$ such that*

$$d(Tx,Ty) \le q\mu_T(x,y), \quad (x,y) \in X \times X.$$

It can be easily seen that T is orbitally continuous mapping, and it satisfies (16) with $\tau = -\ln q$ and $F(t) = \ln t$, $t > 0$. Therefore, $T \in \mathcal{T}_\alpha$, where α is given by (13) and $(\phi,\psi) = (F, -\ln q)$.

Proposition 5 (The class of almost F-contractions)**.** *Let $T : X \to X$ be an almost F-contraction (see [22]), that is, there exist $F \in \Phi$, $\tau > 0$ and $L \ge 0$ such that*

$$\tau + F(d(Tx,Ty)) \le F(d(x,y) + Ld(y,Tx)), \tag{17}$$

for all $(x,y) \in X \times X$ with $d(Tx,Ty) > 0$. Then there exists a function $\alpha : X \times X \to \mathbb{R}$ such that $T \in \mathcal{T}_\alpha$.

Proof. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by (13). Let $\varphi = F$ and $\psi \equiv -\tau$. Then $(\varphi,\psi) \in \Phi \times \Psi$. Let $(x,y) \in X \times X$ be such that $d(Tx,Ty) > 0$. We discuss two possible cases.
Case 1. $y \ne Tx$. In this case, we have

$$\alpha(x,y)\exp\left(\varphi(d(Tx,Ty))\right) = 0 \le \exp\left(\varphi(d(x,y)) + \psi(d(x,y))\right).$$

Case 2. $y = Tx$. In this case, from (17), we have

$$\varphi(d(Tx,T^2x)) \le \varphi(d(x,y)) + \psi(d(x,y)),$$

which yields

$$\alpha(x,Tx)\exp\left(\varphi(d(Tx,T^2x))\right) \le \exp\left(\varphi(d(x,Tx)) + \psi(d(x,Tx))\right).$$

Then T satisfies $\mathcal{T}_2$ with α given by (13). Next, we shall prove that T is α-continuous. Let $\{x_n\} \subset X$ be an α-regular sequence, i.e.,

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}.$$

Suppose that there exists $u \in X$ such that

$$\lim_{n\to+\infty} d(x_n,u) = 0.$$

Let us define the set

$$\mathbb{I} = \{n \in \mathbb{N} : d(x_n,Tu) = 0\}.$$

If $|\mathbb{I}| < +\infty$, then there exists some $N \in \mathbb{N}$ such that

$$d(x_{n+1},Tu) > 0, \quad n \ge N.$$

From (17) and (Φ_1), we have

$$d(x_{n+1},Tu) \le d(x_n,u) + Ld(u,x_{n+1}), \quad n \ge N.$$

Let $n \to +\infty$ and we obtain

$$\lim_{n\to+\infty} d(x_{n+1},Tu) = 0.$$

If $|\mathbb{I}| = +\infty$, then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, Tu) = 0, \quad k \in \mathbb{N}.$$

Therefore, we have

$$\lim_{k\to+\infty} d(Tx_{n_k}, Tu) = \lim_{k\to+\infty} d(x_{n_k+1}, Tu) = 0.$$

Then T is α-continuous, and it satisfies $\mathcal{T}_1$. As a consequence, we have $T \in \mathcal{T}_\alpha$. □

Remark 3. *Let $T : X \to X$ be a mapping that belongs to the class of Berinde mappings (see [2]), that is, there exist $0 < q < 1$ and $\ell \geq 0$ such that*

$$d(Tx, Ty) \leq qd(x, y) + \ell d(y, Tx), \quad (x, y) \in X \times X.$$

It can be easily seen that T is an almost F-contraction with $F(t) = \ln t$, $t > 0$, and $(\tau, L) = (-\ln q, \ell/q)$. Therefore, $T \in \mathcal{T}_\alpha$, where α is given by (13) and $(\phi, \psi) = (F, -\ln q)$.

Now, we state and prove the main result of this section.

Theorem 4. *Let (X, d) be a metric space, and let $T : X \to X$ be a given mapping. Suppose that*

(i) There exists $\alpha : X \times X \to \mathbb{R}$ such that (X, d) is α-complete.
(ii) There exists $(\varphi, \psi) \in \Phi \times \Psi$ such that $T \in \mathcal{T}_\alpha$.
(iii) T is α-admissible.
(iv) There exists some $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Then there exists a sub-sequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ that converges to a fixed point of T.

Proof. Let $\{x_n\}$ be the Picard sequence defined by

$$x_n = T^n x_0, \quad n \in \mathbb{N}.$$

Without loss of generality, we may suppose that

$$d(x_n, x_{n+1}) > 0, \quad n \in \mathbb{N}.$$

From $(\mathcal{T}_2)$, we have

$$\alpha(x_{n-1}, x_n) \exp\left(\varphi(d(x_n, x_{n+1}))\right) \leq \exp\left(\varphi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n))\right), \quad n \in \mathbb{N}^*.$$

On the other hand, from (iii) and (iv), we have

$$\alpha(x_{n-1}, x_n) \geq 1, \quad n \in \mathbb{N}^*. \tag{18}$$

Therefore, we obtain

$$\exp\left(\varphi(d(x_n, x_{n+1}))\right) \leq \exp\left(\varphi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n))\right), \quad n \in \mathbb{N}^*,$$

which yields

$$\varphi(d(x_n, x_{n+1})) \leq \varphi(d(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n)), \quad n \in \mathbb{N}^*.$$

Next, following the same argument as in the proof of Theorem 3, we can prove that $\{x_n\}$ is a Cauchy sequence. Moreover, from (18), $\{x_n\}$ is α-Cauchy. Since (X, d) is α-complete, there exists some $\omega \in X$ such that

$$\lim_{n\to+\infty} d(x_n, \omega) = 0.$$

From $(\mathcal{T}_1)$, there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to+\infty} d(x_{n_k+1}, T\omega) = 0.$$

The uniqueness of the limit yields $T\omega = \omega$, i.e., ω is a fixed point of T. □

Remark 4. *From the proof of Theorem 4, it can be easily seen that if we replace $(\mathcal{T}_1)$ by the continuity of T, then the Picard sequence $\{T^n x_0\}$ converges to a fixed point of T.*

Next, we will show that most fixed point results from the literature involving F-contraction mappings follow easily from Theorem 4.

The following lemma will be used later.

Lemma 1. *Let $T : X \to X$ be a given mapping. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by (13). Then T is α-admissible.*

Proof. Let $(x, y) \in X \times X$ be such that $\alpha(x, y) \geq 1$. By the definition of α, this means that $y = Tx$. Then $Ty = T^2 x$, which yields $\alpha(Tx, Ty) = 1$. This proves that T is α-admissible. □

Corollary 1. *Theorem 4 $\Longrightarrow$ Theorem 3.*

Proof. Suppose that all the assumptions of Theorem 3 are satisfied. By Proposition 1, we know that $T \in \mathcal{T}_\alpha$, where $\alpha : X \times X \to \mathbb{R}$ is given by (13). Since (X, d) is complete, then it is α-complete. From Lemma 1, T is α-admissible. From the definition of α, we have $\alpha(x, Tx) = 1$, for all $x \in X$. Therefore, all the assumptions of Theorem 4 are satisfied. In particular (iv) is satisfied for every $x \in X$. Taking in consideration Remark 4, we obtain that for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T. □

Corollary 2. *Theorem 4 $\Longrightarrow$ Theorem 1.*

Proof. It follows from Proposition 2, Lemma 1 and Remark 4. □

Corollary 3. *Let (X, d) be a complete metric space, and let $T : X \to X$ be an orbitally continuous mapping. Suppose that there exist $F \in \Phi$ and a constant $\tau > 0$ such that (15) is satisfied. Then, for any $x \in X$, there exists a sub-sequence $\{T^{n_k} x\}$ of $\{T^n x\}$ such that $\{T^n x\}$ converges to a fixed point of T.*

Proof. It follows from Proposition 3, Lemma 1, and Theorem 4. □

Remark 5. *By Remark 4, if we replace the assumption T is orbitally continuous with T is continuous, then for any $x \in X$, the Picard sequence $\{T^n x\}$ converges to a fixed point of T. Such a result was established by Wardowski and Van Dung in [27].*

Corollary 4. *Let (X, d) be a complete metric space, and let $T : X \to X$ be an orbitally continuous mapping. Suppose that there exist $F \in \Phi$ and a constant $\tau > 0$ such that (16) is satisfied. Then, for any $x \in X$, there exists a sub-sequence $\{T^{n_k} x\}$ of $\{T^n x\}$ such that $\{T^n x\}$ converges to a fixed point of T.*

Proof. It follows from Proposition 4, Lemma 1, and Theorem 4. □

The next result was established by Minak et al. [22].

Corollary 5. *Let (X, d) be a complete metric space, and let $T : X \to X$ be an almost F-contraction, that is, there exit $F \in \Phi$, $\tau > 0$ and $L \geq 0$ such that (17) is satisfied. Then, for any $x \in X$, there exists a sub-sequence $\{T^{n_k}x\}$ of $\{T^n x\}$ such that $\{T^n x\}$ converges to a fixed point of T.*

Proof. It follows from Proposition 5, Lemma 1, and Theorem 4. □

Next, we will show that we can deduce easily from Theorem 4 several fixed point results in partially ordered metric spaces.

Corollary 6. *Let (X, d) be a complete metric space, and let $T : X \to X$ be continuous mapping. Suppose that X is partially ordered by a certain binary relation $\preceq$. Suppose that*

(i) T is non-decreasing with respect to $\preceq$, i.e.,

$$Tx \preceq Ty,$$

for all $(x, y) \in X \times X$ with $x \preceq y$.

(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.

(iii) There exist $F \in \Phi$ and $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

for all $(x, y) \in X \times X$ with $x \preceq y$ and $d(Tx, Ty) > 0$.

Then $\{T^n x_0\}$ converges to a fixed point of T.

Proof. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \preceq y, \\ 0 & \text{if} \quad x \npreceq y. \end{cases} \tag{19}$$

From (i) and the definition of α, it can be easily seen that T is α-admissible. Since T is continuous, it is α-continuous. Since (X, d) is complete, it is α-complete. On the other hand, from (iii), we have

$$\exp\left(F(d(Tx, Ty))\right) \leq \exp\left(F(d(x, y)) - \tau\right),$$

for all $(x, y) \in X \times X$ with $x \preceq y$ and $d(Tx, Ty) > 0$. Let $(\varphi, \psi) = (F, -\tau)$. Then $(\varphi, \psi) \in \Phi \times \Psi$. Further, by the definition of α, for all $(x, y) \in X \times X$ with $(d(Tx, Ty) > 0$, we have

$$\alpha(x, y) \exp\left(\varphi(d(Tx, Ty))\right) \leq \exp\left(\varphi(d(x, y)) + \psi(d(x, y))\right).$$

Therefore, $T \in \mathcal{T}_\alpha$, where α is given by (19). Note that by (ii), we have $\alpha(x_0, Tx_0) = 1$. Applying Theorem 4 and taking in consideration Remark 4, we obtain the desired result. □

Corollary 7 (Ran–Reurings fixed point theorem [13])**.** *Let (X, d) be a complete metric space, and let $T : X \to X$ be continuous mapping. Suppose that X is partially ordered by a certain binary relation $\preceq$. Suppose that*

(i) T is non-decreasing with respect to $\preceq$.

(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.

(iii) There exists $0 < q < 1$ such that for all $(x, y) \in X \times X$ with $x \preceq y$,

$$d(Tx, Ty) \leq q d(x, y).$$

Then $\{T^n x_0\}$ converges to a fixed point of T.

Proof. We have observe that T satisfies the condition (iii) of Corollary 6 with $F(t) = \ln t$, $t > 0$, and $\tau = -\ln q$. Therefore, the result follows immediately from Corollary 6. □

Remark 6. *Note that several other fixed point results can be deduced from Theorem 4. For example, we mention the Banach fixed point theorem, the Berinde fixed point theorem [2], the Dass–Gupta fixed point theorem [7], the Chatterjea fixed point theorem [4], the Kannan fixed point theorem [11], the Reich fixed point theorem [14], the Hardy–Rogers fixed point theorem [8], etc.*

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