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# Extension of Extragradient Techniques for Variational Inequalities

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**Abstract:** An extragradient type method for finding the common solutions of two variational inequalities has been proposed. The convergence result of the algorithm is given under mild conditions on the algorithm parameters.

**Keywords:** variational inequality; extragradient-type method; inverse-strongly-monotone; projection; strong convergence

**MSC:** 47H05; 47J05; 47J25.

## 1. Introduction

Let  $H$  be a real Hilbert space equipped with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $\emptyset \neq C \subset H$  be a closed and convex set. Let  $A: C \rightarrow H$  be a mapping. Recall that the variational inequality (VI) seeks an element  $x^* \in C$  such that

$$\langle Az^*, z - z^* \rangle \geq 0, \quad \forall z \in C. \quad (1)$$

The solution set of (1) is denoted by  $VI(C, A)$ .

The problem (1) introduced and studied by Stampacchia [1] is being applied as a useful tool and model to refine a multitude of problems. A large number of methods for solving VI (1) are projection methods that implement projections onto the feasible set of the VI (1), or onto another set in order to achieve a solution. Several iterative methods for solving the VI (1) have been proposed. See, e.g., [2–30]. A basic one is the natural extension of the gradient projection algorithm for solving the optimization problem  $\min_{x \in C} f(x)$ . For  $x_0 \in C$ , calculate iteratively the sequence  $\{x_n\}$  through

$$x_{n+1} = \text{proj}_C[x_n - \alpha_n \nabla f(x_n)], \quad n \geq 0, \quad (2)$$

where  $\text{proj}$  is the metric projection and  $\alpha_n > 0$  is the step-size.

Korpelevich [31] introduced an iterative method for solving the VI (1), known as the extragradient method ([7]). In Korpelevich's method, two projections are used for computing the next iteration. For the current iteration  $x_n$ , compute

$$\begin{cases} y_n = \text{proj}_C[x_n - \zeta Ax_n], \\ x_{n+1} = \text{proj}_C[x_n - \zeta Ay_n], \end{cases} \quad n \geq 0, \quad (3)$$

where  $\xi > 0$  is a fixed number.

Korpelevich’s method has received so much attention by a range of scholars, who improved it in several ways; see, e.g., [32–36]. Now, we know that Korpelevich’s method (3) can only achieve weak convergence in a large dimensional spaces ([37,38]). In order to reach strong convergence, Korpelevich’s method was adapted by many mathematicians. For example, in [32], it is shown that several extragradient-type methods converge strongly to an element in  $VI(C, A)$ .

Very recently, Censor, Gibali and Reich [39] presented an alternating method for finding common solutions of two variational inequalities. In [40], Zaslavski studied an extragradient method for finding common solutions of a finite family of variational inequalities.

Inspired by the work given above, in this article, we present an extragradient type method for finding the common solutions of two variational inequalities. We prove the strong convergence of the proposed method under the mild assumptions on the parameters.

## 2. Preliminaries

Let  $H$  be a real Hilbert space. Let  $C \subset H$  be a nonempty, closed and convex set.

**Definition 1.** An operator  $A: C \rightarrow H$  is called Lipschitz if

$$\|Au^\dagger - Av^\dagger\| \leq L\|u^\dagger - v^\dagger\|, \forall u^\dagger, v^\dagger \in C,$$

where  $L > 0$  is a constant.

If  $L = 1$ , we call  $A$  is nonexpansive.

**Definition 2.** An operator  $A: C \rightarrow H$  is called inverse strongly monotone if

$$\alpha\|Au^\dagger - Av^\dagger\|^2 \leq \langle Au^\dagger - Av^\dagger, u^\dagger - v^\dagger \rangle, \forall u^\dagger, v^\dagger \in C,$$

where  $\alpha \geq 0$  is a constant.

In this case, we call  $A$  is  $\alpha$ -inverse-strongly-monotone.

**Proposition 1** ([41]). If  $C$  is a bounded closed convex subset of a real Hilbert space  $H$  and  $A: C \rightarrow H$  is an inverse strongly monotone operator, then  $VI(C, A) \neq \emptyset$ .

For fixed  $z \in H$ , there exists a unique  $z^\dagger \in C$  satisfying

$$\|z - z^\dagger\| = \inf\{\|z - \tilde{z}\| : \tilde{z} \in C\}.$$

We denote  $z^\dagger$  by  $proj_C z$ . The following inequality is an important property of projection  $proj_C$ : for given  $x \in H$ ,

$$\langle x - proj_C x, y - proj_C x \rangle \leq 0, \forall y \in C, \tag{4}$$

which is equivalent to

$$\langle x - y, proj_C x - proj_C y \rangle \geq \|proj_C x - proj_C y\|^2, \forall x, y \in H.$$

It follows that  $proj_C$  is nonexpansive. We also know that  $2proj_C - I$  is nonexpansive.

**Lemma 1 ([41]).** *If  $\emptyset \neq C$  is a closed convex subset of a real Hilbert space  $H$  and  $A : C \rightarrow H$  is an  $\alpha$ -inverse strongly monotone operator, then*

$$\|(I - \mu A)u - (I - \mu A)v\|^2 \leq \|u - v\|^2 + \mu(\mu - 2\alpha)\|Au - Av\|^2, \forall u, v \in C.$$

*Especially,  $I - \mu A$  is nonexpansive provided  $0 \leq \mu \leq 2\alpha$ .*

**Lemma 2 ([42]).** *Suppose that  $\{u_n\}$  and  $\{v_n\}$  are two bounded sequences in Banach spaces. Let  $\{\lambda_n\} \subset [0, 1]$  be a sequence satisfying  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$ . Suppose  $u_{n+1} = (1 - \lambda_n)v_n + \lambda_n u_n, \forall n \geq 0$  and  $\limsup_{n \rightarrow \infty} (\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leq 0$ . Then  $\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0$ .*

**Lemma 3 ([43]).** *Let  $\{\mu_n\} \subset (0, \infty), \{\gamma_n\} \subset (0, 1)$  and  $\{\delta_n\}$  be three real number sequences. If  $\mu_{n+1} \leq (1 - \gamma_n)\mu_n + \delta_n$  for all  $n \geq 0$  with  $\sum_{n=1}^{\infty} \gamma_n = \infty$  and  $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ , then  $\lim_{n \rightarrow \infty} \mu_n = 0$ .*

### 3. Main Results

Let  $\emptyset \neq C$  be a convex and closed subset of a real Hilbert space  $H$ . Let the operators  $A, B : C \rightarrow H$  be  $\alpha$ -inverse strongly monotone and  $\beta$ -inverse strongly monotone, respectively. Let  $\{\alpha_n\} \subset [0, 1], \{\beta_n\} \subset [0, 1], \{\lambda_n\} \subset [0, 2\alpha]$  and  $\{\mu_n\} \subset [0, 2\beta]$  be four sequences. In the sequel, assume  $\Omega = VI(C, A) \cap VI(C, B) \neq \emptyset$ .

Motivated by the algorithms presented in [31,39,40], we present the following iterative Algorithm 1 for finding the common solution of two variational inequalities.

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**Algorithm 1:**

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For  $u, x_0 \in C$ . Assume the sequence  $\{x_n\}$  has been constructed. Compute the next iteration  $\{x_{n+1}\}$  by the following manner

$$\begin{cases} y_n = \text{proj}_C[x_n - (1 - \alpha_n)\lambda_n Ax_n + \alpha_n(u - x_n)], \\ x_{n+1} = \text{proj}_C[x_n - \mu_n \beta_n B y_n + \beta_n(y_n - x_n)], \quad n \geq 0. \end{cases} \tag{5}$$

Suppose that the control parameters  $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$  and  $\{\mu_n\}$  satisfy the following assumptions:

- (C1):  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
  - (C2):  $\liminf_{n \rightarrow \infty} \beta_n > 0$  and  $\lim_{n \rightarrow \infty} (\beta_{n+1} - \beta_n) = 0$ ;
  - (C3):  $\lambda_n \in [a, b] \subset (0, 2\alpha)$  and  $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$ ;
  - (C4):  $\mu_n \in [c, d] \subset (0, 2\beta)$  and  $\lim_{n \rightarrow \infty} (\mu_{n+1} - \mu_n) = 0$ .
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We will divide our main result into several propositions.

**Proposition 2.** *The sequence  $\{x_n\}$  generated by (5) is bounded.*

**Proof.** Choose any  $x^* \in \Omega$ . Note that  $x^* = \text{proj}_C[(I - \delta A)x^*]$  for any  $\delta > 0$ . Hence,

$$x^* = \text{proj}_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda_n A x^*)], \quad \forall n \geq 0. \tag{6}$$

Thus, by (5) and (6), we have

$$\begin{aligned} \|y_n - x^*\| &= \|\text{proj}_C[\alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n] - \text{proj}_C[\alpha_n x^* + (1 - \alpha_n)(I - \lambda_n A)x^*]\| \\ &\leq \|\alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n - [\alpha_n x^* + (1 - \alpha_n)(I - \lambda_n A)x^*]\| \\ &\leq \|\alpha_n(u - x^*) + (1 - \alpha_n)[(I - \lambda_n A)x_n - (I - \lambda_n A)x^*]\| \\ &\leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|. \end{aligned} \tag{7}$$

From Lemma 1, we know that  $I - \lambda_n A$  and  $I - \mu_n B$  are nonexpansive. Thus, from (7), we get

$$\|y_n - x^*\| \leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\|.$$

So,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\text{proj}_C[(1 - \beta_n)x_n + \beta_n(y_n - \mu_n B y_n)] - \text{proj}_C[(1 - \beta_n)x^* + \beta_n(x^* - \mu_n B x^*)]\| \\ &\leq \|[(1 - \beta_n)x_n + \beta_n(y_n - \mu_n B y_n)] - [(1 - \beta_n)x^* + \beta_n(x^* - \mu_n B x^*)]\| \\ &\leq \beta_n \|(y_n - \mu_n B y_n) - (x^* - \mu_n B x^*)\| + (1 - \beta_n) \|x_n - x^*\| \\ &\leq \beta_n \|y_n - x^*\| + (1 - \beta_n) \|x_n - x^*\| \\ &\leq (1 - \beta_n \alpha_n) \|x_n - x^*\| + \beta_n \alpha_n \|u - x^*\|. \end{aligned} \tag{8}$$

It follows that

$$\|x_{n+1} - x^*\| \leq \max\{\|x_0 - x^*\|, \|u - x^*\|\}.$$

Then  $\{x_n\}$  is bounded, and hence the sequences  $\{y_n\}$ ,  $\{Ax_n\}$  and  $\{By_n\}$  are all bounded.  $\square$

**Proposition 3.** *The following two conclusions hold*

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

**Proof.** Let  $S = 2\text{proj}_C - I$ . It is clear that  $S$  is nonexpansive. Set  $v_n = (1 - \beta_n)x_n + \beta_n(I - \mu_n B)y_n$  for all  $n \geq 0$ . Then, we can rewrite  $x_{n+1}$  in (5) as

$$\begin{aligned} x_{n+1} &= \frac{1 - \beta_n}{2} x_n + \frac{\beta_n}{2} (I - \mu_n B)y_n + \frac{1}{2} S v_n \\ &= \frac{1 + \beta_n}{2} z_n + \frac{1 - \beta_n}{2} x_n, \end{aligned}$$

where  $z_n = \frac{\beta_n}{1 + \beta_n} (I - \mu_n B)y_n + \frac{1}{1 + \beta_n} S v_n$  for all  $n \geq 0$ .

Hence,

$$z_{n+1} - z_n = \frac{1}{1 + \beta_{n+1}} S v_{n+1} + \frac{\beta_{n+1}}{1 + \beta_{n+1}} (I - \mu_{n+1} B)y_{n+1} - \frac{\beta_n}{1 + \beta_n} (I - \mu_n B)y_n - \frac{1}{1 + \beta_n} S v_n.$$

Hence,

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \frac{\beta_{n+1}}{1 + \beta_{n+1}} \|(I - \mu_{n+1} B)y_{n+1} - (I - \mu_n B)y_n\| + \frac{1}{1 + \beta_{n+1}} \|S v_{n+1} - S v_n\| \\ &\quad + \left| \frac{\beta_{n+1}}{1 + \beta_{n+1}} - \frac{\beta_n}{1 + \beta_n} \right| \|(I - \mu_n B)y_n\| + \left| \frac{1}{1 + \beta_{n+1}} - \frac{1}{1 + \beta_n} \right| \|S v_n\| \\ &\leq \frac{\beta_{n+1}}{1 + \beta_{n+1}} \|(I - \mu_{n+1} B)y_{n+1} - (I - \mu_{n+1} B)y_n\| \\ &\quad + \frac{\beta_{n+1}}{1 + \beta_{n+1}} |\mu_{n+1} - \mu_n| \|B y_n\| + \left| \frac{\beta_{n+1}}{1 + \beta_{n+1}} - \frac{\beta_n}{1 + \beta_n} \right| \|y_n - \mu_n B y_n\| \\ &\quad + \frac{1}{1 + \beta_{n+1}} \|S v_{n+1} - S v_n\| + \left| \frac{1}{1 + \beta_{n+1}} - \frac{1}{1 + \beta_n} \right| \|S v_n\|. \end{aligned}$$

By using the nonexpansivity of  $I - \mu_n B$  and  $S$  to deduce

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \frac{\beta_{n+1}}{1 + \beta_{n+1}} \|y_{n+1} - y_n\| + \left| \frac{\beta_{n+1}}{1 + \beta_{n+1}} - \frac{\beta_n}{1 + \beta_n} \right| \|y_n - \mu_n B y_n\| \\ &\quad + \beta_{n+1} [(I - \mu_{n+1} B)y_{n+1} - (I - \mu_{n+1} B)y_n] + \left| \frac{1}{1 + \beta_n} - \frac{1}{1 + \beta_n} \right| \|Sv_n\| \\ &\quad + \frac{\beta_{n+1}}{1 + \beta_{n+1}} |\mu_{n+1} - \mu_n| \|B y_n\| + \frac{1}{1 + \beta_{n+1}} \|(1 - \beta_{n+1})(x_{n+1} - x_n) \\ &\quad + (\beta_{n+1} - \beta_n)(y_n - x_n) + (\beta_n \mu_n - \beta_{n+1} \mu_{n+1}) B y_n\| \tag{9} \\ &\leq \frac{|\beta_{n+1} - \beta_n|}{1 + \beta_{n+1}} (\|x_n\| + \|y_n\|) + \frac{1 - \beta_{n+1}}{1 + \beta_{n+1}} \|x_{n+1} - x_n\| + \left| \frac{1}{1 + \beta_n} - \frac{1}{1 + \beta_n} \right| \|Sv_n\| \\ &\quad + \frac{2\beta_{n+1}}{1 + \beta_{n+1}} \|y_{n+1} - y_n\| + \left| \frac{\beta_{n+1}}{1 + \beta_{n+1}} - \frac{\beta_n}{1 + \beta_n} \right| \|y_n - \mu_n B y_n\| \\ &\quad + \frac{\beta_{n+1}}{1 + \beta_{n+1}} |\mu_{n+1} - \mu_n| \|B y_n\| + \frac{|\beta_{n+1} \mu_{n+1} - \beta_n \mu_n|}{1 + \beta_{n+1}} \|B y_n\|. \end{aligned}$$

Next, we estimate  $\|y_{n+1} - y_n\|$ . By (5), we get

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|\text{proj}_C[\alpha_{n+1} u + (1 - \alpha_{n+1})(I - \lambda_{n+1} A)x_{n+1}] - \text{proj}_C[\alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n]\| \\ &\leq |\alpha_{n+1} - \alpha_n| \|x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\| + \alpha_{n+1} \|u\| + \alpha_n \|u\| + \|x_{n+1} - x_n\| \tag{10} \\ &\quad + |\lambda_{n+1} - \lambda_n| \|A x_n\| + |\lambda_{n+1} \alpha_{n+1} - \lambda_n \alpha_n| \|A x_n\|. \end{aligned}$$

Substituting (10) into (9) to get

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \left| \frac{\beta_{n+1}}{1 + \beta_{n+1}} - \frac{\beta_n}{1 + \beta_n} \right| \|y_n - \mu_n B y_n\| + \frac{\beta_{n+1}}{1 + \beta_{n+1}} |\mu_{n+1} - \mu_n| \|B y_n\| \\ &\quad + \frac{|\beta_{n+1} - \beta_n|}{1 + \beta_{n+1}} (\|x_n\| + \|y_n\|) + \frac{|\beta_{n+1} \mu_{n+1} - \beta_n \mu_n|}{1 + \beta_{n+1}} \|B y_n\| + \|x_{n+1} - x_n\| \\ &\quad + \left| \frac{1}{1 + \beta_n} - \frac{1}{1 + \beta_n} \right| \|Sv_n\| + 2|\lambda_{n+1} - \lambda_n| \|A x_n\| + 2\alpha_{n+1} \|u\| + 2\alpha_n \|u\| \\ &\quad + 2|\alpha_{n+1} - \alpha_n| \|x_n\| + 2|\lambda_{n+1} - \lambda_n| \|A x_n\| + 2|\lambda_{n+1} \alpha_{n+1} - \lambda_n \alpha_n| \|A x_n\|. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} (\beta_{n+1} - \beta_n) = 0$  and  $\lim_{n \rightarrow \infty} (\mu_{n+1} - \mu_n) = 0$ , we derive that  $\lim_{n \rightarrow \infty} \left| \frac{\beta_{n+1}}{1 + \beta_{n+1}} - \frac{\beta_n}{1 + \beta_n} \right| = 0$ ,  $\lim_{n \rightarrow \infty} |\beta_{n+1} \mu_{n+1} - \beta_n \mu_n| = 0$  and  $\lim_{n \rightarrow \infty} \left| \frac{1}{1 + \beta_n} - \frac{1}{1 + \beta_n} \right| = 0$ . At the same time, note that  $\{x_n\}$ ,  $\{A x_n\}$ ,  $\{y_n\}$  and  $\{B y_n\}$  are bounded. Therefore,

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Applying Lemma 2 to derive

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \frac{1 + \beta_n}{2} \|z_n - x_n\| = 0.$$

By virtue of (7), (8) and Lemma 1, we deduce

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|(y_n - \mu_n B y_n) - (x^* - \mu_n B x^*)\|^2 \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2 + \beta_n\mu_n(\mu_n - 2\beta)\|B y_n - B x^*\|^2 \\ &\leq \beta_n\|\alpha_n(u - x^*) + (1 - \alpha_n)[(x_n - \lambda_n A x_n) - (x^* - \lambda_n A x^*)]\|^2 \\ &\quad + (1 - \beta_n)\|x_n - x^*\|^2 + \beta_n\mu_n(\mu_n - 2\beta)\|B y_n - B x^*\|^2 \\ &\leq \beta_n[\alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2] \\ &\quad + \beta_n\mu_n(\mu_n - 2\beta)\|B y_n - B x^*\|^2 + (1 - \beta_n)\|x_n - x^*\|^2 \\ &\leq (1 - \beta_n)\|x_n - x^*\|^2 + \alpha_n\beta_n\|u - x^*\|^2 + \beta_n\mu_n(\mu_n - 2\beta)\|B y_n - B x^*\|^2 \\ &\quad + (1 - \alpha_n)\beta_n[\|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A x_n - A x^*\|^2] \\ &\leq \alpha_n\beta_n\|u - x^*\|^2 + \|x_n - x^*\|^2 + \beta_n\mu_n(\mu_n - 2\beta)\|B y_n - B x^*\|^2 \\ &\quad + (1 - \alpha_n)\beta_n\lambda_n(\lambda_n - 2\alpha)\|A x_n - A x^*\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &(1 - \alpha_n)\beta_n\lambda_n(2\alpha - \lambda_n)\|A x_n - A x^*\|^2 + \beta_n\mu_n(2\beta - \mu_n)\|B y_n - B x^*\|^2 \\ &\leq \alpha_n\beta_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq \alpha_n\beta_n\|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times \|x_n - x_{n+1}\|. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|A x_n - A x^*\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|B y_n - B x^*\| = 0$$

According to (4) and (5), we get

$$\begin{aligned} \|y_n - x^*\|^2 &= \|\text{proj}_C[\alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n] - \text{proj}_C(I - \lambda_n A)x^*\|^2 \\ &\leq \langle \alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n - (I - \lambda_n A)x^*, y_n - x^* \rangle \\ &= \frac{1}{2} \left\{ \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^* + \alpha_n(u - x_n + \lambda_n A x^*)\|^2 + \|y_n - x^*\|^2 \right. \\ &\quad \left. - \|\alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n - (I - \lambda_n A)x^* - (y_n - x^*)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2 \right. \\ &\quad \left. + 2\alpha_n\|u - x_n + \lambda_n A x^*\| \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^* + \alpha_n(u - x_n + \lambda_n A x^*)\| \right. \\ &\quad \left. + \|y_n - x^*\|^2 - \|(x_n - y_n) - \lambda_n(A x_n - A x^*) + \alpha_n(u - x_n + \lambda_n A x^*)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^*\|^2 + \alpha_n M + \|y_n - x^*\|^2 \right. \\ &\quad \left. - \|(x_n - y_n) - \lambda_n(A x_n - A x^*) + \alpha_n(u - x_n + \lambda_n A x^*)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + \alpha_n M + \|y_n - x^*\|^2 + 2\lambda_n \langle x_n - y_n, A x_n - A x^* \rangle \right. \\ &\quad \left. - 2\alpha_n \langle u - x_n + \lambda_n A x^*, x_n - y_n \rangle - \|\lambda_n(A x_n - A x^*) - \alpha_n(u - x_n + \lambda_n A x^*)\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|x_n - x^*\|^2 + \alpha_n M + \|y_n - x^*\|^2 - \|x_n - y_n\|^2 \right. \\ &\quad \left. + 2\alpha_n\|u - x_n + \lambda_n A x^*\| \|x_n - y_n\| + 2\lambda_n \|x_n - y_n\| \|A x_n - A x^*\| \right\}, \end{aligned}$$

where  $M > 0$  is some constant such that

$$\sup_n \{2\|u - x_n + \lambda_n Ax^*\| \|(I - \lambda_n A)x_n - (I - \lambda_n A)x^* + \alpha_n(u - x_n + \lambda_n Ax^*)\|\} \leq M.$$

Thus

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + \alpha_n M - \|x_n - y_n\|^2 + 2\lambda_n \|x_n - y_n\| \|Ax_n - Ax^*\| \\ &\quad + 2\alpha_n \|u - x_n + \lambda_n Ax^*\| \|x_n - y_n\|, \end{aligned}$$

and it follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + \alpha_n M - \beta_n \|x_n - y_n\|^2 + 2\lambda_n \|x_n - y_n\| \|Ax_n - Ax^*\| \\ &\quad + 2\alpha_n \|u - x_n + \lambda_n Ax^*\| \|x_n - y_n\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \beta_n \|x_n - y_n\|^2 &\leq 2\lambda_n \|x_n - y_n\| \|Ax_n - Ax^*\| + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\ &\quad + \alpha_n M + 2\alpha_n \|u - x_n + \lambda_n Ax^*\| \|x_n - y_n\|. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$  and  $\lim_{n \rightarrow \infty} \|Ax_n - Ax^*\| = 0$ , we derive

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

This concludes the proof.  $\square$

**Proposition 4.**  $\limsup_{n \rightarrow \infty} \langle \tilde{x} - u, \tilde{x} - y_n \rangle \leq 0$ , where  $\tilde{x} = P_\Omega u$ .

**Proof.** Let  $\{y_{n_i}\}$  be a subsequence of  $\{y_n\}$  satisfying

$$\limsup_{n \rightarrow \infty} \langle \tilde{x} - u, \tilde{x} - y_n \rangle = \lim_{i \rightarrow \infty} \langle \tilde{x} - u, \tilde{x} - y_{n_i} \rangle.$$

By the boundedness of  $\{y_{n_i}\}$ , we can choose a subsequence  $\{y_{n_{ij}}\}$  of  $\{y_{n_i}\}$  such that  $y_{n_{ij}} \rightharpoonup z$ .

Next, we demonstrate that  $z \in \Omega$ . First, we prove that  $z \in VI(C, A)$ . Let  $N_C v$  be the normal cone of  $C$  at  $v \in C$ ; i.e.,  $N_C v = \{w \in H : 0 \leq \langle v - u, w \rangle, \forall u \in C\}$ . Define a mapping  $T$  by the formula

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Let  $(v, w) \in G(T)$ . Since  $y_n \in C$  and  $w - Av \in N_C v$ , we deduce  $0 \leq \langle v - y_n, w - Av \rangle$ . According to  $y_n = \text{proj}_C[\alpha_n u + (1 - \alpha_n)(I - \lambda_n A)x_n]$ , we obtain

$$0 \leq \langle v - y_n, y_n - \alpha_n u - (1 - \alpha_n)(x_n - \lambda_n Ax_n) \rangle,$$

that is,

$$0 \leq \langle v - y_n, \frac{y_n - x_n}{\lambda_n} + Ax_n - \frac{\alpha_n}{\lambda_n}(u - x_n + \lambda_n Ax_n) \rangle.$$

Thus,

$$\begin{aligned}
 \langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \\
 &\geq \langle v - y_{n_i}, Av \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} - \frac{\alpha_{n_i}}{\lambda_{n_i}}(u - x_{n_i} + \lambda_{n_i}Ax_{n_i}) \rangle \\
 &= \langle v - y_{n_i}, Av - Ax_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + \frac{\alpha_{n_i}}{\lambda_{n_i}}(u - x_{n_i} + \lambda_{n_i}Ax_{n_i}) \rangle \\
 &= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle \\
 &\quad - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{\alpha_{n_i}}{\lambda_{n_i}}(u - x_{n_i} + \lambda_{n_i}Ax_{n_i}) \rangle \\
 &\geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{\alpha_{n_i}}{\lambda_{n_i}}(u - x_{n_i} + \lambda_{n_i}Ax_{n_i}) \rangle.
 \end{aligned}$$

Noting that  $\alpha_{n_i} \rightarrow 0$  and  $\|y_{n_i} - x_{n_i}\| \rightarrow 0$ , we deduce  $0 \leq \langle v - z, w \rangle$ . Hence,  $z \in T^{-1}(0)$  and thus  $z \in VI(C, A)$ .

Next, we show that  $z \in VI(C, B)$ . Define a mapping  $T^\dagger$  as follows

$$T^\dagger v = \begin{cases} Bv^\dagger + N_C v^\dagger, & v^\dagger \in C, \\ \emptyset, & v^\dagger \notin C. \end{cases}$$

Let  $(v^\dagger, w^\dagger) \in G(T^\dagger)$ . Since  $w^\dagger - Bv^\dagger \in N_C v$  and  $x_{n+1} \in C$ , we obtain  $0 \leq \langle v^\dagger - x_{n+1}, w^\dagger - Bv^\dagger \rangle$ . By virtue of  $x_{n+1} = \text{proj}_C[x_n - \mu_n \beta_n B y_n + \beta_n(y_n - x_n)]$ , we obtain

$$0 \leq \langle v^\dagger - x_{n+1}, \mu_n \beta_n B y_n + x_{n+1} - x_n - \beta_n(y_n - x_n) \rangle,$$

that is,

$$0 \leq \langle v^\dagger - x_{n+1}, B y_n + \frac{x_{n+1} - x_n}{\beta_n \mu_n} - \frac{y_n - x_n}{\mu_n} \rangle.$$

Therefore, we have

$$\begin{aligned}
 \langle v^\dagger - x_{n_i+1}, w^\dagger \rangle &\geq \langle v^\dagger - x_{n_i+1}, Bv^\dagger \rangle \\
 &\geq \langle v^\dagger - x_{n_i+1}, Bv^\dagger \rangle - \langle v^\dagger - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\beta_{n_i} \mu_{n_i}} + B y_{n_i} - \frac{y_{n_i} - x_{n_i}}{\mu_{n_i}} \rangle \\
 &= \langle v^\dagger - x_{n_i+1}, Bv^\dagger - B y_{n_i} - \frac{x_{n_i+1} - x_{n_i}}{\beta_{n_i} \mu_{n_i}} + \frac{y_{n_i} - x_{n_i}}{\mu_{n_i}} \rangle \\
 &= \langle v^\dagger - x_{n_i+1}, Bv^\dagger - B y_{n_i} \rangle + \langle v^\dagger - x_{n_i+1}, B x_{n_i+1} - B y_{n_i} \rangle \\
 &\quad - \langle v^\dagger - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\beta_{n_i} \mu_{n_i}} - \frac{y_{n_i} - x_{n_i}}{\mu_{n_i}} \rangle \\
 &\geq \langle v^\dagger - x_{n_i+1}, B x_{n_i+1} - B y_{n_i} \rangle - \langle v - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\beta_{n_i} \mu_{n_i}} - \frac{y_{n_i} - x_{n_i}}{\mu_{n_i}} \rangle.
 \end{aligned}$$

Noting that  $\|y_{n_i} - x_{n_i}\| \rightarrow 0$  and  $\|x_{n_i+1} - x_{n_i}\| \rightarrow 0$ , we get  $0 \leq \langle v^\dagger - z, w^\dagger \rangle$ . Hence,  $z \in T^{\dagger^{-1}}(0)$  and  $z \in VI(C, B)$ . Thus,  $z \in \Omega$  and we have

$$\limsup_{n \rightarrow \infty} \langle \tilde{x} - u, \tilde{x} - y_n \rangle = \lim_{i \rightarrow \infty} \langle \tilde{x} - u, \tilde{x} - y_{n_i} \rangle = \langle \tilde{x} - u, \tilde{x} - z \rangle \leq 0.$$

□

Finally, we prove our main result.



**Theorem 1.** Suppose that  $\Omega = VI(C, A) \cap VI(C, B) \neq \emptyset$ . Assume that  $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$  and  $\{\mu_n\}$  satisfy the following restrictions (C1)–(C4). Then  $\{x_n\}$  defined by (5) converges strongly to  $\tilde{x} = \text{proj}_\Omega(u)$ .

**Proof.** First, we have Propositions 2–4 in hand. In terms of (4), we have

$$\begin{aligned} \|y_n - \tilde{x}\|^2 &= \|\text{proj}_C[\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n Ax_n)] - \text{proj}_C[x^* - (1 - \alpha_n)\lambda_n Ax^*]\|^2 \\ &\leq \langle \alpha_n(u - x^*) + (1 - \alpha_n)[(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)], y_n - x^* \rangle \\ &\leq \alpha_n \langle \tilde{x} - u, \tilde{x} - y_n \rangle + (1 - \alpha_n) \|(x_n - \lambda_n Ax_n) - (\tilde{x} - \lambda_n A\tilde{x})\| \|y_n - \tilde{x}\| \\ &\leq \alpha_n \langle \tilde{x} - u, \tilde{x} - y_n \rangle + (1 - \alpha_n) \|x_n - \tilde{x}\| \|y_n - \tilde{x}\| \\ &\leq \alpha_n \langle \tilde{x} - u, \tilde{x} - y_n \rangle + \frac{1 - \alpha_n}{2} \|x_n - \tilde{x}\|^2 + \frac{1}{2} \|y_n - \tilde{x}\|^2. \end{aligned}$$

It follows that

$$\|y_n - \tilde{x}\|^2 \leq (1 - \alpha_n) \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle \tilde{x} - u, \tilde{x} - y_n \rangle.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &\leq (1 - \beta_n) \|x_n - \tilde{x}\|^2 + \beta_n \|y_n - \tilde{x}\|^2 \\ &\leq (1 - \alpha_n \beta_n) \|x_n - \tilde{x}\|^2 + 2\alpha_n \beta_n \langle \tilde{x} - u, \tilde{x} - y_n \rangle. \end{aligned}$$

By Lemma 3 and the above inequality, we deduce  $x_n \rightarrow \tilde{x}$ . This completes the proof.  $\square$

If we take  $u = 0$ , then we have the following Algorithm 2.

---

**Algorithm 2:**

For initial value  $x_0 \in C$ . Assume the sequence  $\{x_n\}$  has been constructed. Compute the next iteration  $\{x_{n+1}\}$  by the following manner

$$\begin{cases} y_n = \text{proj}_C[(1 - \alpha_n)(x_n - \lambda_n Ax_n)], \\ x_{n+1} = \text{proj}_C[x_n - \mu_n \beta_n B y_n + \beta_n (y_n - x_n)], \quad n \geq 0. \end{cases} \tag{11}$$


---

**Corollary 1.** Suppose that  $\Omega = VI(C, A) \cap VI(C, B) \neq \emptyset$ . Assume that  $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$  and  $\{\mu_n\}$  satisfy the following restrictions (C1)–(C4). Then  $\{x_n\}$  defined by (11) converges strongly to the minimum norm element  $\tilde{x}$  in  $\Omega$ .

**4. Conclusions**

In this paper, we investigated the variational inequality problem. We suggest an extragradient type method for finding the common solutions of two variational inequalities. We prove the strong convergence of the method under the mild conditions. Noting that in our suggested iterative sequence (Equation (5)), the involved operators  $A$  and  $B$  require some form of strong monotonicity. A natural question arises, i.e., how to weaken these assumptions?

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