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Fixed Point Results for Multi-Valued Contractions in b –Metric Spaces and an Application

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Abstract: In this paper, by characterizing a weak contractive condition based on using C –functions and α –admissible multi-valued mapping of type S , we present some fixed point results for (α, F) –admissible multi-valued mappings in the setting of b –metric spaces. Some examples and an application are added in order to show the reliability of our obtained results. Our results amend, unify, and generalize some existing results in the literature. The scientific novelty of our main results is to take new contraction self-mappings in b –metric spaces for multi-valued mappings.

Keywords: α –admissible mapping of type S ; multi-valued mapping; Fixed point; b –metric space; Hausdorff metric

1. Introduction

The Banach contraction principle is considered to be one of the most useful tools in fixed point theory. It has been extended or generalized in different directions and many (common) fixed point theorems have been provided (see References [1–13]). The theory of fixed points for multi-valued mappings has been developed after the famous paper of Nadler [14]. Several authors have given many forms of multi-valued mapping conditions by using the concept of Hausdorff–Pompiou metric. Among them, we cite References [15–17].

First, we recite some notations, definitions, and elementary results needed in the sequel. Denote by \mathbb{N} and \mathbb{R} the sets of positive integers and real numbers, respectively. Consider a metric space (X, d) . Denote by 2^X (resp. $CL(X)$) the family of subsets (resp. of closed subsets) of X . Let $CB(X)$ be the class of all nonempty closed bounded subsets of X and $F(f)$ be the set of all fixed points of f .

Definition 1 ([14]). A point $u \in X$ is called a fixed point of the multi-valued map $T : X \rightarrow 2^X$ if $u \in Tu$.

For $B, C \in CL(X)$, let $H : CL(X) \times CL(X) \rightarrow [0, \infty]$ be defined by

$$H(B, C) = \begin{cases} \max\{\delta(B, C), \delta(C, B)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\delta(B, C) =: \sup_{a \in B} d_b(b, C)$. Such H is called the generalized Hausdorff–Pompiou metric induced by the metric d . The known result of Nadler [14] is as follows:

Definition 2 ([14]). Let (X, d) be a metric space. A map $f : X \rightarrow CB(X)$ is said to be a multi-valued contraction if there exists $0 \leq \lambda < 1$ such that

$$H(fx, fy) \leq \lambda d(x, y),$$

for all $x, y \in X$.

Bakhtin [18] and Czerwik [19] initiated the concept of a b -metric.

Definition 3 ([18,19]). Let X be a nonempty set and $s \geq 1$ be a given real number. Take $d_b : X \times X \rightarrow [0, \infty)$. Suppose that for all $\zeta, \eta, \xi \in X$, we have:

- (i) $d_b(\zeta, \eta) = 0$ if and only if $\zeta = \eta$;
- (ii) $d_b(\zeta, \eta) = d_b(\eta, \zeta)$;
- (iii) $d_b(\zeta, \xi) \leq s[d_b(\zeta, \eta) + d_b(\eta, \xi)]$.

Then, d_b is said a b -metric and the triplet (X, d_b, s) is called a b -metric space.

The authors in References [18,19] extended the Banach contraction principle to the class of b -metric spaces. For other results in the setting of b -metric spaces, see References [3,4,10,20–23].

Example 1 ([18,19]). Consider the set $X = [0, 1]$ endowed with the function $d_b : X \times X \rightarrow [0, \infty)$ defined by $d_b(\zeta, \eta) = |\zeta - \eta|^2$ for all $\zeta, \eta \in X$. Here, $(X, d_b, 2)$ is a b -metric space, but it is not a metric space.

Let (X, d_b, s) be a b -metric space. Let $\{\theta_n\}$ be a sequence in X .

- (i) $\{\theta_n\} \subseteq X$ converges to a point $x \in X$ if $\lim_{n \rightarrow \infty} d_b(\theta_n, x) = 0$,
- (ii) $\{\theta_n\} \subseteq X$ is Cauchy if, for each $\epsilon > 0$, there is some $n(\epsilon) \in \mathbb{N}$ such that $d_b(\theta_n, \theta_m) < \epsilon$ for all $m, n \geq n(\epsilon)$,
- (iii) (X, d_b, s) is said complete if any Cauchy sequence is convergent in X .

Lemma 1 ([2]). Let (X, d_b, s) be a b -metric space with $s \geq 1$. Suppose that $\{\theta_n\}$ and $\{\eta_n\}$ are b -convergent sequences to θ and η , respectively. Then,

$$\frac{1}{s} d_b(\theta, \eta) \leq \liminf_{n \rightarrow \infty} d_b(\theta_n, \eta_n) \leq \limsup_{n \rightarrow \infty} d_b(\theta_n, \eta_n) \leq s^2 d_b(\theta, \eta).$$

In the case that $\theta = \eta$, we get $\lim_{n \rightarrow \infty} d_b(\theta_n, \eta_n) = 0$. Moreover, for each $\eta \in X$,

$$\frac{1}{s} d_b(\theta, \eta) \leq \liminf_{n \rightarrow \infty} d_b(\theta_n, \eta) \leq \limsup_{n \rightarrow \infty} d_b(\theta_n, \eta) \leq s^2 d_b(\theta, \eta).$$

Lemma 2 ([2]). Let (X, d_b, s) be a b -metric space with $s \geq 1$. Let $\{\theta_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} d_b(\theta_n, \theta_{n+1}) = 0.$$

If $\{\theta_n\}$ is not a b -Cauchy sequence, then there exist $\epsilon > 0$ and $\{\theta_{m(k)}\}$ and $\{\theta_{n(k)}\}$ two sequences of positive integers such that

$$\begin{aligned} \epsilon &\leq \liminf_{n \rightarrow \infty} d_b(\theta_{m(k)}, \theta_{n(k)}) \leq \limsup_{n \rightarrow \infty} d_b(\theta_{m(k)}, \theta_{n(k)}) \leq s\epsilon \\ \frac{\epsilon}{s} &\leq \liminf_{n \rightarrow \infty} d_b(\theta_{m(k)}, \theta_{n(k)+1}) \leq \limsup_{n \rightarrow \infty} d_b(\theta_{m(k)}, \theta_{n(k)+1}) \leq s^2\epsilon \\ \frac{\epsilon}{s} &\leq \liminf_{n \rightarrow \infty} d_b(\theta_{m(k)+1}, \theta_{n(k)}) \leq \limsup_{n \rightarrow \infty} d_b(\theta_{m(k)+1}, \theta_{n(k)}) \leq s^2\epsilon \\ \frac{\epsilon}{s^2} &\leq \liminf_{n \rightarrow \infty} d_b(\theta_{m(k)+1}, \theta_{n(k)+1}) \leq \limsup_{n \rightarrow \infty} d_b(\theta_{m(k)+1}, \theta_{n(k)+1}) \leq s^3\epsilon. \end{aligned}$$

Lemma 3 ([21]). Let (X, d_b, s) be a b -metric space. For all $P, Q, R \in CB(X)$, and $\xi, \theta \in X$, we have:

- (i) $d_b(\xi, Q) \leq d_b(\xi, \eta)$ for any $\eta \in Q$;
- (ii) $d_b(P, Q) \leq H(P, Q)$;
- (iii) $\delta(P, Q) \leq H(P, Q)$;
- (iv) $d_b(\xi, Q) \leq H(P, Q)$ for any $\xi \in P$;
- (iiv) $H(P, P) = 0$;
- (vi) $H(P, Q) = H(Q, P)$;
- (vii) $H(P, R) \leq s(H(P, Q) + H(Q, R))$;
- (viii) $d_b(\xi, P) \leq s(d_b(\xi, \theta) + d_b(\theta, P))$.

Remark 1 ([21]). Let (X, d_b, s) be a b -metric space. The function $H : CL(X) \times CL(X) \rightarrow [0, \infty)$ is a generalized Hausdorff b -metric, that is, $H(A, B) = +\infty$ if $\max\{\delta(A, B), \delta(B, A)\}$ does not exist.

Lemma 4 ([21]). Let (X, d_b, s) be a b -metric space. For $P \in CL(X)$ and $\xi \in X$, we have

$$d_b(\xi, P) = 0 \Leftrightarrow \xi \in \bar{P} = P, \tag{1}$$

where the closure of the set A is denoted by \bar{A} .

The concept of α -admissibility was first initiated by Samet et al. [24].

Definition 4 ([24]). Given $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. f is called α -admissible if $\alpha(\zeta, \eta) \geq 1$ implies that $\alpha(f\zeta, f\eta) \geq 1$.

This concept was generalized by Sintunavarat [25].

Definition 5 ([25]). Let X be a nonempty set. Let $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be two mappings. Let $s \geq 1$ be a given real number. We say that f is weak α -admissible of type S if for $x \in X$ and $\alpha(x, fx) \geq s$, then $\alpha(fx, ffx) \geq s$.

Definition 6 ([26]). Let X be a nonempty set. Given $f : X \rightarrow CL(X)$ and $\alpha : X \times X \rightarrow [0, \infty)$. Then, f is said to be α -admissible whenever, for each $\xi \in X$ and $\eta \in f\xi$ with $\alpha(\xi, \eta) \geq 1$, we have $\alpha(\eta, \theta) \geq 1$ for each $\theta \in f\eta$.

After that, many authors used the concept of α -admissible contractive type mappings to investigate the existence of fixed points (see References [10,23,27–31] and references cited therein).

In 2014, Ansari [32] defined the concept of C -class functions.

Definition 7 ([32]). A mapping $F : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is called a C -class function if it is continuous and for $s, t \in [0, \infty)$, F satisfies the following two conditions:

- (a) $F(r, t) \leq r$;

(b) $F(r, t) = r$ implies that either $r = 0$ or $t = 0$.

The family of C -class functions is denoted by \mathcal{C} .

Example 2 ([32]). The following functions $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ are elements in \mathcal{C} :

- (a) $F(r, t) = r - t$ for all $r, t \in [0, \infty)$;
- (b) $F(r, t) = kr$ for all $r, t \in [0, \infty)$, where $0 < k < 1$;
- (c) $F(r, t) = r\beta(r)$ for all $r, t \in [0, \infty)$, where $\beta : [0, \infty) \rightarrow [0, 1)$ is continuous;
- (d) $F(r, t) = r - \varphi(r)$ for all $r, t \in [0, \infty)$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;
- (e) $F(s, t) = \frac{s}{(1+t)^a}$ with $a \in (0, \infty)$, for all $r, t \in [0, \infty)$;
- (f) $F(s, t) = \ln(t + k^s) / (1 + t)$ with $k > 1$, for all $r, t \in [0, \infty)$.

The generalization of the well-known Banach fixed point theorem with rational expressions was considered by Reference [33]. On the other hand, there are many works on fixed point theory using control functions (see References [24,28,34–37]). Let Φ be the set of lower semi-continuous functions $\phi : [0, \infty) \rightarrow [0, \infty)$ such that $\phi(t) = 0 \Leftrightarrow t = 0$. Let Ψ be the set of nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that

$$\sum_{n=1}^{+\infty} s^n \psi^n(t) < +\infty \text{ for each } t > 0, \tag{2}$$

where ψ^n is the n^{th} iterate of ψ and $s \geq 1$ is the coefficient of the b -metric space (X, d) .

Note that if $\psi \in \Psi$, we have $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$, $\psi(t) < \frac{t}{s} \leq t$ for each $t > 0$ and $\psi(0) = 0$.

Let X be a b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow 2^X$ be a multi-valued map. The graph of f is given as

$$Gr(f) = \{(\xi, \eta) \in X^2, \eta \in f\xi\}.$$

Such mapping f is called upper semi-continuous if the inverse image of closed sets is closed.

In this paper, inserting C -class functions we present some fixed point results in the setting of a b -metric space for α -admissible multi-valued type mappings of type S . An example and application are given.

2. Main Results

First, we introduce the notion of α -admissible multi-valued mappings of type S .

Definition 8. Let X be a nonempty set. Given $f : X \rightarrow CL(X)$ and $\alpha : X \times X \rightarrow [0, +\infty)$. Let $s \geq 1$ be a given real number. Such that f is said to be α -admissible of type S if for each $\xi \in X$ and $\eta \in f\xi$ with $\alpha(\xi, \eta) \geq s$, we have $\alpha(\eta, \theta) \geq s$ for each $\theta \in f\eta$.

Example 3. Let $X = [-1, 1]$ be endowed with the b -metric $d_b(\xi, \eta) = |\xi - \eta|^2$ with $s = 2$ for all $\xi, \eta \in X$. Define $\alpha : X \times X \rightarrow [0, \infty)$ and $f : X \rightarrow CL(X)$ by

$$f\xi = \begin{cases} \{0, 1\} & \text{if } \xi = -1, \\ \{1\} & \text{if } \xi = 0, \\ \{-\xi\} & \text{if } \xi \notin \{-1, 0\}. \end{cases}$$

$$\alpha(\xi, \eta) = \begin{cases} 2e^{\xi+\eta} & \text{if } \xi = \eta, \\ 4 & \text{if } \xi \neq \eta. \end{cases}$$

It is easy show that f is α -admissible of type S .

Example 4. Let $X = \mathbb{N} \cup \{0\}$ be endowed with the b -metric $d_b(\xi, \eta) = |\xi - \eta|^2$ for all $\xi, \eta \in X$ with $s = 2$. Define the mappings $\alpha : X \times X \rightarrow [0, \infty)$ and $f : X \rightarrow CL(X)$ by

$$f\xi = \begin{cases} \{0, 1\} & \text{if } \xi = 0, 1, \\ \{\xi, \xi + 1\} & \text{if } \xi > 1. \end{cases}$$

$$\alpha(\xi, \eta) = \begin{cases} 4 & \text{if } \xi, \eta \in \{0, 1\}, \\ e^{\xi+\eta} & \text{if } \xi, \eta > 1, \\ 0 & \text{otherwise.} \end{cases}$$

The mapping f is α -admissible of type S .

Example 5. Let $X = [0, \infty)$ be endowed with the b -metric $d_b(\xi, \eta) = |\xi - \eta|^2$ (with coefficient $s = 2$) for all $\xi, \eta \in X$. Define $f : X \rightarrow CL(X)$ and $\alpha : X \times X \rightarrow [0, \infty)$ by

$$f\xi = \begin{cases} [2, 2 + \frac{\xi}{3}] & \text{if } \xi \in [0, 1], \\ [5, 5 + 5\xi] & \text{if } \xi \in (1, \infty) \end{cases}$$

$$\alpha(\xi, \eta) = \begin{cases} 4 & \text{if } \xi, \eta \in [0, 1], \\ e^{\xi+\eta} & \text{otherwise.} \end{cases}$$

Clearly f is α -admissible of type S .

Definition 9. Let (X, d_b, s) be a b -metric space with constant $s \geq 1$ and $f : X \rightarrow CL(X)$. Such that f is an (α, F) -multi-valued mapping if there exist $\alpha : X \times X \rightarrow [0, \infty)$, $F \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Rightarrow H(fx, fy) \leq F(\psi(M(x, y)), \phi(M(x, y))), \tag{3}$$

where

$$M(x, y) = \max\{d_b(x, y), \frac{d_b(x, fx)}{1 + d_b(x, fx)}, \frac{d_b(y, fy)}{1 + d_b(y, fy)}, \frac{1}{2s}[d_b(x, fy) + d_b(y, fx)]\}. \tag{4}$$

Theorem 1. Let (X, d_b, s) be a b -metric space with constant $s \geq 1$ and $f : X \rightarrow CL(X)$ be an (α, F) -admissible multi-valued mapping. Assume that:

- (i) f is α -admissible of type S ;
- (ii) there exist $\theta_0 \in X$ and $\theta_1 \in f\theta_0$ such that $\alpha(\theta_0, \theta_1) \geq s$;
- (iii) $Gr(f)$ is a closed subset of X^2 .

Then, f has a fixed point.

Proof. By Condition (ii), $\theta_0 \in X$ and $\theta_1 \in f\theta_0$ such that $\alpha(\theta_0, \theta_1) \geq s$. If $\theta_0 = \theta_1$ or $\theta_1 \in f\theta_1$, we deduce that θ_1 is a fixed point of f and so the proof is done. Now, we assume that $\theta_0 \neq \theta_1$ and $\theta_1 \notin f\theta_1$. By Lemma 4, $d_b(\theta_1, f\theta_1) > 0$. Applying Equation (3), we have

$$0 < d_b(\theta_1, f\theta_1) \leq H(f\theta_0, f\theta_1) \leq F(\psi(M(\theta_0, \theta_1)), \phi(M(\theta_0, \theta_1))), \tag{5}$$

where

$$\begin{aligned}
 M(\theta_0, \theta_1) &= \max\{d_b(\theta_0, \theta_1), \frac{d_b(\theta_0, f\theta_0)}{1 + d_b(\theta_0, f\theta_0)}, \frac{d_b(\theta_1, f\theta_1)}{1 + d_b(\theta_1, f\theta_1)}, \frac{1}{2s} [d_b(\theta_0, f\theta_1) + d_b(\theta_1, f\theta_0)]\} \\
 &\leq \max\{d_b(\theta_0, \theta_1), d_b(\theta_0, f\theta_0), d_b(\theta_1, f\theta_1), \frac{1}{2s} d_b(\theta_0, f\theta_1)\} \\
 &\leq \max\{d_b(\theta_0, \theta_1), d_b(\theta_0, \theta_1), d_b(\theta_1, f\theta_1), \frac{1}{2} [d_b(\theta_0, \theta_1) + d_b(\theta_1, f\theta_1)]\} \\
 &= \max\{d_b(\theta_0, \theta_1), d_b(\theta_1, f\theta_1)\}.
 \end{aligned}
 \tag{6}$$

If $\max\{d_b(\theta_0, \theta_1), d_b(\theta_1, f\theta_1)\} = d_b(\theta_1, f\theta_1)$, then by (5) and having in mind that $\psi(t) < t$ for each $t > 0$, we get

$$0 < d_b(\theta_1, f\theta_1) \leq F(\psi(d_b(\theta_1, f\theta_1)), \phi(d_b(\theta_1, f\theta_1))) \leq \psi(d_b(\theta_1, f\theta_1)) < d_b(\theta_1, f\theta_1),$$

which is a contradiction. Thus, $\max\{d_b(\theta_0, \theta_1), d_b(\theta_1, f\theta_1)\} = d_b(\theta_0, \theta_1)$. Using again (5) and the fact that ψ is nondecreasing, we get that

$$0 < d_b(\theta_1, f\theta_1) \leq \psi(d_b(\theta_0, \theta_1)).$$

This implies that there exists $\theta_2 \in f\theta_1$ (of course, $\theta_2 \neq \theta_1$) such that

$$0 < d_b(\theta_1, \theta_2) < \psi(d_b(\theta_0, \theta_1)).$$

Since $\alpha(\theta_0, \theta_1) \geq s$, $\theta_1 \in f\theta_0$ and $\theta_2 \in f\theta_1$, using the fact that f is α -admissible, we get $\alpha(\theta_1, \theta_2) \geq s$. If $\theta_2 \in f\theta_2$, θ_2 is a fixed point of f . Otherwise, $\theta_2 \notin f\theta_2$, so we have $d_b(\theta_2, f\theta_2) > 0$. Applying again (3), we have

$$\begin{aligned}
 0 < d_b(\theta_2, f\theta_2) &\leq H(f\theta_1, f\theta_2) \\
 &\leq F(\psi(M(\theta_1, \theta_2)), \phi(M(\theta_1, \theta_2))),
 \end{aligned}
 \tag{7}$$

where

$$\begin{aligned}
 M(\theta_1, \theta_2) &= \max\{d_b(\theta_1, \theta_2), \frac{d_b(\theta_1, f\theta_1)}{1 + d_b(\theta_1, f\theta_1)}, \frac{d_b(\theta_2, f\theta_2)}{1 + d_b(\theta_2, f\theta_2)}, \frac{1}{2s} [d_b(\theta_1, f\theta_2) + d_b(\theta_2, f\theta_1)]\} \\
 &\leq \max\{d_b(\theta_1, \theta_2), d_b(\theta_1, \theta_2), d_b(\theta_2, f\theta_2), \frac{1}{2s} d_b(\theta_1, f\theta_2)\} \\
 &\leq \max\{d_b(\theta_1, \theta_2), d_b(\theta_1, \theta_2), d_b(\theta_2, f\theta_2), \frac{1}{2s} [d_b(\theta_1, \theta_2) + d_b(\theta_2, f\theta_2)]\} \\
 &= \max\{d_b(\theta_1, \theta_2), d_b(\theta_2, f\theta_2)\}.
 \end{aligned}
 \tag{8}$$

Proceeding similarly as above, we get that $\max\{d_b(\theta_1, \theta_2), d_b(\theta_2, f\theta_2)\} = d_b(\theta_1, \theta_2)$. From (7) and (8),

$$0 < d_b(\theta_2, f\theta_2) \leq \psi(d_b(\theta_1, \theta_2)) < \psi^2(d_b(\theta_0, \theta_1)).$$

This implies again that there exists $\theta_3 \in f\theta_2$ (of course, $\theta_3 \neq \theta_2$) such that

$$0 < d_b(\theta_2, \theta_3) < \psi^2(d_b(\theta_0, \theta_1)).$$

Since $\alpha(\theta_1, \theta_2) \geq s$, $\theta_2 \in f\theta_1$ and $\theta_3 \in f\theta_2$, using the fact that f is α -admissible, we get $\alpha(\theta_2, \theta_3) \geq s$. If $\theta_3 \in f\theta_3$, θ_3 is a fixed point of f . Otherwise, $\theta_3 \notin f\theta_3$, so we have $d_b(\theta_3, f\theta_3) > 0$. Similarly, we get

$$0 < d_b(\theta_3, f\theta_3) \leq \psi(d_b(\theta_2, \theta_3)) < \psi^3(d_b(\theta_0, \theta_1)).$$

By continuing this process, we can construct a sequence $\{\theta_n\}$ in X such that $\theta_n \notin f\theta_n, \theta_{n+1} \in f\theta_n, \alpha(\theta_n, \theta_{n+1}) \geq s$ and

$$0 < d_b(\theta_n, f\theta_n) \leq d_b(\theta_n, \theta_{n+1}) \leq \psi^n(d_b(\theta_0, \theta_1)), \forall n \in \mathbb{N}. \tag{9}$$

Let $m, n \in \mathbb{N}$ be such that $m > n$. Then,

$$d_b(\theta_n, \theta_m) \leq \sum_{i=n}^{m-1} s^{i-n+1} d_b(\theta_i, \theta_{i+1}) \leq \sum_{i=n}^{\infty} s^i \psi^i(d_b(\theta_0, \theta_1)).$$

Since $\psi \in \Psi, \{\theta_n\}$ is a Cauchy sequence in the complete b -metric space (X, d_b) . Hence, there exists $z \in X$ such that $z_n \rightarrow z$. Since $\theta_{n+1} \in f\theta_n$, we have $(\theta_n, \theta_{n+1}) \in Gr(f)$. The graph is closed, so as $n \rightarrow \infty$, we get that $(\theta_n, \theta_{n+1}) \rightarrow (z, z)$, with $(z, z) \in Gr(f)$. We deduce that $z \in fz$, that is, z is a fixed point of f . \square

Now, we state the following known lemma.

Lemma 5 ([38]). Assume that $fx \in CL(X)$ for each $x \in X$. If f is upper semi-continuous then $Gr(f)$ is closed in X^2 .

Following Lemma 5, we may provide the following result, even the upper semi-continuity condition is a strong hypothesis. Indeed, we may have a multivalued mapping such that its graph is closed, but it is not upper semi-continuous (see Reference [39]).

Theorem 2. Let (X, d_b, s) be a b -metric space with a constant $s \geq 1$ and $f : X \rightarrow CL(X)$ be an (α, F) -admissible multi-valued mapping. Assume that:

- (i) f is α -admissible of type S ;
- (ii) there exist $\theta_0 \in X$ and $\theta_1 \in f\theta_0$ such that $\alpha(\theta_0, \theta_1) \geq s$;
- (iii) f is upper semi-continuous.

Then, f has a fixed point.

It is also possible to omit the closedness of the graph of f (or the upper semi-continuity of the mapping f) by introducing a regularity condition and adding the continuity of ψ .

Theorem 3. Let (X, d_b, s) be a b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow CL(X)$ be an (α, F) -multi-valued mapping. Assume that:

- (i) f is α -admissible of type S and ψ is continuous;
- (ii) there exist $\theta_0 \in X$ and $\theta_1 \in f\theta_0$ such that $\alpha(\theta_0, \theta_1) \geq s$;
- (iii) if $\{\theta_n\}$ is a sequence in X with $\theta_n \rightarrow u \in X$ and $\alpha(\theta_n, \theta_{n+1}) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\theta_n, u) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Following the proof of Theorem 1, there exists a sequence $\{\theta_n\}$ such that

$$\theta_{n+1} \in f\theta_n, \theta_n \notin f\theta_n \text{ and } \alpha(\theta_n, \theta_{n+1}) \geq s \text{ for all } n \in \mathbb{N} \cup \{0\}, \tag{10}$$

Furthermore, $\{\theta_n\}$, which is a Cauchy sequence in X , converges to some u as $n \rightarrow \infty$. By Condition (iii), we get

$$\alpha(\theta_n, u) \geq s \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{11}$$

If $u \in fu$, the proof is completed. We assume that $d_b(u, fu) > 0$. We have

$$\begin{aligned}
 0 < d_b(u, fu) &\leq s[d_b(u, \theta_{n+1}) + d_b(\theta_{n+1}, fu)] \\
 &\leq sd_b(u, \theta_{n+1}) + sH(f\theta_n, fu) \\
 &\leq sd_b(u, \theta_{n+1}) + sF(\psi(M(\theta_n, u)), \phi(M(\theta_n, u))) \\
 &\leq sd_b(u, \theta_{n+1}) + s\psi(M(\theta_n, u))
 \end{aligned}
 \tag{12}$$

where

$$\begin{aligned}
 M(\theta_n, u) &= \max\{d_b(\theta_n, u), \frac{d_b(\theta_n, f\theta_n)}{1 + d_b(\theta_n, f\theta_n)}, \frac{d_b(u, fu)}{1 + d_b(u, fu)}, \frac{1}{2s}[d_b(\theta_n, fu) + d_b(u, f\theta_n)]\} \\
 &\leq \max\{d_b(\theta_n, u), d_b(\theta_n, \theta_{n+1}), d_b(u, fu), \frac{1}{2}[d_b(\theta_n, fu) + d_b(u, \theta_{n+1})]\} \\
 &\leq \max\{d_b(\theta_n, u), d_b(\theta_n, \theta_{n+1}), d_b(u, fu), \frac{1}{2}[d_b(\theta_n, fu) + d_b(u, \theta_{n+1})]\}.
 \end{aligned}
 \tag{13}$$

Letting $n \rightarrow \infty$, we have $\limsup_{n \rightarrow \infty} M(\theta_n, u) \leq d_b(u, fu)$. Using the continuity of ψ , we get $\limsup_{n \rightarrow \infty} \psi(M(\theta_n, u)) \leq \psi(d_b(u, fu))$. Letting $n \rightarrow \infty$ in (12), we obtain

$$0 < d_b(u, fu) \leq s\psi(d_b(u, fu)) < d_b(u, fu),$$

which is a contradiction. Therefore, $u \in fu$ and hence f has a fixed point. \square

Corollary 1. Let (X, d_b, s) be a b -metric space with constant $s \geq 1$ and $f : X \rightarrow CL(X)$ be an (α, F) -multivalued mapping. Suppose there exist $\alpha : X \rightarrow [0, \infty)$, $\psi \in \Psi$, and $\phi \in \Phi$ such that

$$\alpha(x, y)H(fx, fy) \leq F(\psi(M(x, y)), \phi(M(x, y))), \tag{14}$$

for all $x, y \in X$. Assume that:

- (i) f is α -admissible of type S ;
- (ii) there exist $\theta_0 \in X$ and $\theta_1 \in f\theta_0$ such that $\alpha(\theta_0, \theta_1) \geq s$;
- (iii) the graph of f is closed.

Then, f has a fixed point.

Proof. By (14), the contraction (3) holds for all $x, y \in X$ with $\alpha(x, y) \geq s$. Thus, f is an (α, F) -multi-valued mapping. By Theorem 1, the multi-valued mapping f has a fixed point. \square

Corollary 2. Let (X, d_b, s) be a b -metric space with a constant $s \geq 1$ and $f : X \rightarrow CL(X)$ be an (α, F) -multi-valued mapping. Assume that there exist $\alpha : X \rightarrow [0, \infty)$, $\psi \in \Psi$, and $\phi \in \Phi$ such that

$$\alpha(x, y)H(fx, fy) \leq F(\psi(M(x, y)), \phi(M(x, y))), \tag{15}$$

for all $x, y \in X$. Assume that:

- (i) f is α -admissible of type S and ψ is continuous;
- (ii) there exist $\theta_0 \in X$ and $\theta_1 \in f\theta_0$ such that $\alpha(\theta_0, \theta_1) \geq s$;
- (iii) $\{\theta_n\}$ is a sequence in X with $\theta_n \rightarrow x \in X$ and $\alpha(\theta_n, \theta_{n+1}) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\theta_n, x) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Again, by (15), the contraction (3) holds for all $x, y \in X$ such that $\alpha(x, y) \geq s$. Thus, f is an (α, F) -multi-valued mapping. By Theorem 3, f has a fixed point. \square

By considering singlevalued mappings in Theorem 1 and Theorem 3, respectively, we have the following:

Corollary 3. Let (X, d_b, s) be a b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$. Suppose there exist $\alpha : X \times X \rightarrow [0, +\infty)$, $F \in \mathcal{C}$, $\psi \in \Psi$, and $\phi \in \Phi$ such that

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Rightarrow d_b(fx, fy) \leq F(\psi(M(x, y)), \phi(M(x, y))), \tag{16}$$

where $M(x, y)$ was defined by (4). Assume that:

- (i) $u, v \in X, \alpha(u, v) \geq s$ implies $\alpha(fu, fv) \geq s$;
- (ii) there exists $\theta_0 \in X$ such that $\alpha(\theta_0, f\theta_0) \geq s$;
- (iii) the graph of f is closed.

Then, f has a fixed point.

Corollary 4. Let (X, d_b, s) be a b -metric space with coefficient $s \geq 1$ and $f : X \rightarrow X$. Suppose there exist $\alpha : X \times X \rightarrow [0, +\infty)$, $F \in \mathcal{C}$, $\psi \in \Psi$, and $\phi \in \Phi$ such that the Condition (16) holds. Assume that:

- (i) $u, v \in X, \alpha(u, v) \geq s$ implies $\alpha(fu, fv) \geq s$. Also, ψ is continuous;
- (ii) there exists $\theta_0 \in X$ such that $\alpha(\theta_0, f\theta_0) \geq s$;
- (iii) if for $\xi_0 \in X$, the sequence $\{\xi_n = f^n \xi_0\}$ in X is such that $f^n \xi_0 \rightarrow \xi \in X$ and $\alpha(f^n \xi_0, f^{n+1} \xi_0) \geq s$ for each integer $n \geq 0$, then $\alpha(f^n \xi_0, \xi) \geq s$ for each $n \geq 0$.

Then, f has a fixed point.

Example 6. Let $X = [0, \infty)$ be endowed with the b -metric $d_b(x, y) = (x - y)^2$ with $s = 2$ for all $x, y \in X$. Define $f : X \rightarrow CL(X)$ and $\alpha : X \rightarrow [0, \infty)$ by

$$fx = \begin{cases} [0, \frac{x}{4}] & \text{if } x \in [0, 1], \\ [x, x^2] & \text{if } x \in (1, \infty) \end{cases}$$

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, define the functions $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t, \phi(t) = \frac{15}{16}t$. Take $F(r, t) = r - t$ for all $r, t \in [0, \infty)$.

Firstly, we show that f is α -admissible of type S. Let $x \in X$ and $y \in fx$ with $\alpha(x, y) \geq s = 2$. Then, $x, y \in [0, 1]$. Let $z \in fy$, then $z \in [0, \frac{y}{4}] \subset [0, \frac{x}{16}] \subset [0, 1]$. Then,

$$\alpha(y, z) = 2 = s.$$

Hence, f is α -admissible of type S. For $\theta_0 = \frac{1}{2}$ and $\theta_1 = \frac{1}{16} \in f\theta_0$, we have $\alpha(\theta_0, \theta_1) = 2 = s$. For any sequence $\{\theta_n = f^n \theta_0\} \subseteq X$ (where $\theta_0 \in X$ is arbitrary) such that $\theta_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(\theta_n, \theta_{n+1}) = 2 = s$ for each $n \in \mathbb{N}$, we have $x_n, x \in [0, 1]$, so $\alpha(\theta_n, x) = 2 = s$ for each $n \in \mathbb{N}$.

Now, we will show that the conditions of Theorem 3 are fulfilled for all $x, y \in X$ with $\alpha(x, y) \geq s$, that is, $x, y \in [0, 1]$. Without loss of generality, take $x \neq y$. Then,

$$H(fx, fy) = \frac{(x - y)^2}{16}.$$

Thus,

$$H(fx, fy) = \frac{(x - y)^2}{16} = \frac{1}{16}d_b(x, y) \leq \frac{1}{16}M(x, y) = F(\psi(M(x, y)), \phi(M(x, y))).$$

All hypotheses of Theorem 3 are satisfied and f has a fixed point. Here, 0 and each $x > 1$ are fixed points of f .

Note also that the graph of f is closed, so we may also apply Theorem 1.

3. Application

The aim of this section is to give an application on the existence of a solution for an integral equation by applying Corollary 4.

First, let X be the set of continuous functions specified on the closed interval $[a, b]$. We endow X by the standard b -metric $d_b : X \times X \rightarrow [0, \infty)$:

$$d_b(x, y) = \left(\sup_{t \in [a, b]} |x(t) - y(t)| \right)^2,$$

for all $x, y \in X$. Then, (X, d_b) is a complete b -metric space with coefficient $s = 2$.

We consider the following integral equation:

$$y(r) = y_0 + \int_a^b P(r, t)g(t, y(t))dt \tag{17}$$

where $y_0 \in \mathbb{R}$ and $P : [a, b] \times [a, b] \rightarrow [0, \infty)$, $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Define $f : X \rightarrow X$ as

$$fy(r) = y_0 + \int_a^b P(r, t)g(t, y(t))dt. \tag{18}$$

Then, a solution of Equation (17) is equivalent to stating that the map f has a fixed point.

Now, we will prove the following result applying Corollary 4.

Theorem 4. Assume that the following conditions are satisfied:

- (i) there exists $\alpha : X \times X \rightarrow [0, \infty)$ such that if $\alpha(u, v) \geq s = 2$ for $u, v \in X$, we have for each $t \geq 0$,

$$|g(t, u(t)) - g(t, v(t))| \leq \sqrt{\ln(1 + (|u(t) - v(t)|)^2)},$$

and

$$\sup_{r \in [a, b]} \int_a^b P(r, t)dt \leq 1;$$

- (ii) $u, v \in X$, $\alpha(u, v) \geq 1$ implies $\alpha(fu, fv) \geq 1$;
- (iii) there exists $u_0 \in X$ such that $\alpha(u_0, fu_0) \geq s = 2$;
- (iv) if $\{\xi_n =: f^n \xi_0\}$ (where ξ_0 is arbitrary in X) is a sequence in X with $\xi_n \rightarrow \xi \in X$ and $\alpha(\xi_n, \xi_{n+1}) \geq s$ for each integer $n \geq 0$, then $\alpha(\xi_n, \xi) \geq s$ for each $n \geq 0$.

Then, the integral Equation (17) has a solution in X .

Proof. For all $u, v \in X$, we have

$$d_b(fu, fv) = \sup_{r \in [a, b]} |fu(r) - fv(r)|^2.$$

We also have for $r \in [a, b]$,

$$\begin{aligned}
 & (|fu(r) - fv(r)|)^2 \\
 = & (|\int_a^b P(r,t) |g(t,u(t)) - g(t,v(t))| dt |)^2 \\
 \leq & (\int_a^b P(r,t) |g(t,u(t)) - g(t,v(t))| dt)^2 \\
 \leq & (\int_a^b P(r,t) \sqrt{\ln(1 + (|u(t) - v(t)|)^2)} dt)^2 \\
 \leq & (\int_a^b P(r,t) \sqrt{\ln(1 + d_b(u,v))} dt)^2 \\
 \leq & (\int_a^b P(r,t) dt)^2 \ln(1 + d_b(u,v)) \\
 \leq & \ln(1 + d_b(u,v)).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 H(fu, fv) & \leq \ln(1 + d_b(u,v)) \leq \ln(1 + M(u,v)) \\
 & = M(u,v) - (M(u,v) - \ln(1 + M(u,v))) \\
 & = F(\psi(M(u,v)), \phi(M(u,v))),
 \end{aligned}$$

where $\psi(t) = t$ (it is continuous), $\phi(t) = t - \ln(1 + t)$, and $F(s, t) = s - t$. Since (iii) and (iv) hold, all hypotheses of Corollary 4 hold. Thus, f has a fixed point, that is, the integral Equation (17) has a solution in X . \square

4. Conclusions

Many researchers proved the existence and uniqueness of some fixed point results for self-mappings in b -metric spaces. In this paper, we have introduced the existence and uniqueness of fixed point results for (α, ψ, ϕ) -admissible multi-valued mappings in the setting of b -metric spaces by defining a weak contractive condition using C -functions and α -admissible multi-valued mappings of type S . We have provided examples and an application of integral equations to show the superiority of our results and given concepts. Furthermore, as perspectives, we suggest extending more results to find (common) fixed point results in generalized metric spaces as extended b -metric spaces, b -metric-like spaces. Otherwise, we recommend utilizing our main results for non-self-mappings to establish the existence of an optimal approximate solution.

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Abbreviations

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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